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Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under (ψ,φ)-contractivity conditions

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Abstract

In this paper we present some (unidimensional and) multidimensional fixed point results under (ψ,φ)-contractivity conditions in the framework of G^*-metric spaces, which are spaces that result from G-metric spaces (in the sense of Mustafa and Sims) omitting one of their axioms. We prove that these spaces let us consider easily the product of G^*-metrics. Our result clarifies and improves some recent results on this topic because, among other different reasons, we will not need a partial order on the underlying space. Furthermore, the way in which several contractivity conditions are proposed imply that our theorems cannot be reduced to metric spaces.

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1 Introduction

In the sixties, inspired by the mapping that associated the area of a triangle to its three vertices, Gähler [1, 2] introduced the concept of 2-*metric spaces*. Gähler believed that 2-metric spaces can be interpreted as a generalization of usual metric spaces. However, some authors demonstrated that there is no clear relationship between these notions. For instance, Ha *et al.* [3] showed that a 2-metric does not have to be a continuous function of its three variables. Later, inspired by the perimeter of a triangle rather than the area, Dhage [4] changed the axioms and presented the concept of *D-metric*. Different topological structures (see [5–7]) were considered in such spaces and, subsequently, several fixed point results were established. Unfortunately, most of their properties turned out to be false (see [8–10]). These considerations led to the concept of *G-metric space* introduced by Mustafa and Sims [11]. Since then, this theory has been expansively developed, paying a special attention to fixed point theorems (see, for instance, [12–28] and references therein).

The main aim of the present paper is to prove new unidimensional and multidimensional fixed point results in the framework of the G-metric spaces provided with a partial preorder (not necessarily a partial order). However, we need to overcome the well-known fact that the usual product of G-metrics is not necessarily a G-metric unless it comes from classical metrics (see [11], Section 4). Hence, we will omit one of the axioms that define a G-metric and we consider a new class of metrics, called G^*-*metrics*. As a consequence, our main results are valid in the context of G-metric spaces.

2 Preliminaries

Let n be a positive integer. Henceforth, X will denote a non-empty set and X^n will denote the product space $X \times X \times \overset{n}{\cdots} \times X$. Throughout this manuscript, m and k will denote non-negative integers and $i,j,s \in \{1,2,\ldots,n\}$. Unless otherwise stated, 'for all m' will mean 'for all $m \geq 0$' and 'for all i' will mean 'for all $i \in \{1,2,\ldots,n\}$'. Let $\mathbb{R}_0^+ = [0,\infty)$.

Definition 1 We will say that $\preccurlyeq$ is a partial preorder on X (or $(X,\preccurlyeq)$ is a preordered set or $(X,\preccurlyeq)$ is a partially preordered space) if the following properties hold.

- Reflexivity: $x \preccurlyeq x$ for all $x \in X$.
- Transitivity: If $x,y,z \in X$ verify $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$.

Henceforth, let $\{\mathsf{A},\mathsf{B}\}$ be a partition of $\Lambda_n = \{1,2,\ldots,n\}$, that is, $\mathsf{A} \cup \mathsf{B} = \Lambda_n$ and $\mathsf{A} \cap \mathsf{B} = \varnothing$ such that A and B are non-empty sets. In the sequel, we will denote

$$\Omega_{\mathsf{A},\mathsf{B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathsf{A}) \subseteq \mathsf{A} \text{ and } \sigma(\mathsf{B}) \subseteq \mathsf{B}\} \quad \text{and}$$
$$\Omega'_{\mathsf{A},\mathsf{B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathsf{A}) \subseteq \mathsf{B} \text{ and } \sigma(\mathsf{B}) \subseteq \mathsf{A}\}.$$

From now on, let $\Upsilon = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ be an n-tuple of mappings from $\{1,2,\ldots,n\}$ into itself verifying $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$.

If $(X,\preccurlyeq)$ is a partially preordered space, $x,y \in X$ and $i \in \Lambda_n$, we will use the following notation:

$$x \preccurlyeq_i y \quad \Leftrightarrow \quad \begin{cases} x \preccurlyeq y, & \text{if } i \in \mathsf{A}, \\ x \succcurlyeq y, & \text{if } i \in \mathsf{B}. \end{cases}$$

Consider on the product space X^n the following partial preorder: for $\mathsf{X} = (x_1,x_2,\ldots,x_n)$, $\mathsf{Y} = (y_1,y_2,\ldots,y_n) \in X^n$,

$$\mathsf{X} \sqsubseteq \mathsf{Y} \quad \Leftrightarrow \quad x_i \preccurlyeq_i y_i \quad \text{for all } i. \tag{1}$$

Notice that $\sqsubseteq$ depends on A and B. We say that two points X and Y are $\sqsubseteq$-*comparable* if $\mathsf{X} \sqsubseteq \mathsf{Y}$ or $\mathsf{X} \sqsupseteq \mathsf{Y}$.

Proposition 2 *If* $\mathsf{X} \sqsubseteq \mathsf{Y}$ *and* $\sigma \in \Omega_{\mathsf{A},\mathsf{B}} \cup \Omega'_{\mathsf{A},\mathsf{B}}$, *then* $(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)})$ *and* $(y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(n)})$ *are* $\sqsubseteq$-*comparable. In particular,*

$$(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}) \sqsubseteq (y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(n)}) \quad \textit{if } \sigma \in \Omega_{\mathsf{A},\mathsf{B}},$$
$$(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}) \sqsupseteq (y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(n)}) \quad \textit{if } \sigma \in \Omega'_{\mathsf{A},\mathsf{B}}.$$

Proof Suppose that $x_i \preccurlyeq_i y_i$ for all i. Hence $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ for all i. Fix $\sigma \in \Omega_{\mathsf{A},\mathsf{B}}$. If $i \in \mathsf{A}$, then $\sigma(i) \in \mathsf{A}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preccurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$. If $i \in \mathsf{B}$, then $\sigma(i) \in \mathsf{B}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succcurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$. In any case, if $\sigma \in \Omega_{\mathsf{A},\mathsf{B}}$, then $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$ for all i. It follows that $(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}) \sqsubseteq (y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(n)})$.

Now fix $\sigma \in \Omega'_{\mathsf{A},\mathsf{B}}$. If $i \in \mathsf{A}$, then $\sigma(i) \in \mathsf{B}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succcurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \succcurlyeq_i y_{\sigma(i)}$. If $i \in \mathsf{B}$, then $\sigma(i) \in \mathsf{A}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preccurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \succcurlyeq_i y_{\sigma(i)}$. $\square$

Let $F : X^n \to X$ be a mapping.

Definition 3 (Roldán *et al.* [20]) A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called an Υ*-fixed point of the mapping F* if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = x_i \quad \text{for all } i. \tag{2}$$

Definition 4 (Roldán *et al.* [20]) Let $(X, \preccurlyeq)$ be a partially preordered space. We say that F has the *mixed monotone property (w.r.t.* $\{\mathsf{A}, \mathsf{B}\}$*)* if F is monotone non-decreasing in the arguments of A and monotone non-increasing in the arguments of B, *i.e.*, for all $x_1, x_2, \ldots, x_n, y, z \in X$ and all i,

$$y \preccurlyeq z \quad \Rightarrow \quad F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preccurlyeq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

We will use the following results about real sequences in the proof of our main theorems.

Lemma 5 *Let* $\{a_m^1\}_{m\in\mathbb{N}}, \ldots, \{a_m^n\}_{m\in\mathbb{N}}$ *be n real lower bounded sequences such that* $\{\max(a_m^1, \ldots, a_m^n)\}_{m\in\mathbb{N}} \to \delta$. *Then there exist* $i_0 \in \{1, 2, \ldots, n\}$ *and a subsequence* $\{a_{m(k)}^{i_0}\}_{k\in\mathbb{N}}$ *such that* $\{a_{m(k)}^{i_0}\}_{k\in\mathbb{N}} \to \delta$.

Proof Let $b_m = \max(a_m^1, a_m^2, \ldots, a_m^n)$ for all m. As $\{b_m\}$ is convergent, it is bounded. As $a_m^i \le b_m$ for all m and i, then every $\{a_m^i\}$ is bounded. As $\{a_m^1\}_{m\in\mathbb{N}}$ is a real bounded sequence, it has a convergent subsequence $\{a_{\sigma_1(m)}^1\}_{m\in\mathbb{N}} \to a_1$. Consider the subsequences $\{a_{\sigma_1(m)}^2\}_{m\in\mathbb{N}}, \{a_{\sigma_1(m)}^3\}_{m\in\mathbb{N}}, \ldots, \{a_{\sigma_1(m)}^n\}_{m\in\mathbb{N}}$, that are $n-1$ real bounded sequences, and the sequence $\{b_{\sigma_1(m)}\}_{m\in\mathbb{N}}$ that also converges to δ. As $\{a_{\sigma_1(m)}^2\}_{m\in\mathbb{N}}$ is a real bounded sequence, it has a convergent subsequence $\{a_{\sigma_2\sigma_1(m)}^2\}_{m\in\mathbb{N}} \to a_2$. Then the sequences $\{a_{\sigma_2\sigma_1(m)}^3\}_{m\in\mathbb{N}}, \{a_{\sigma_2\sigma_1(m)}^4\}_{m\in\mathbb{N}}, \ldots, \{a_{\sigma_2\sigma_1(m)}^n\}_{m\in\mathbb{N}}$ also are $n-2$ real bounded sequences and $\{a_{\sigma_2\sigma_1(m)}^1\}_{m\in\mathbb{N}} \to a_1$ and $\{b_{\sigma_2\sigma_1(m)}\}_{m\in\mathbb{N}} \to \delta$. Repeating this process n times, we can find n subsequences $\{a_{\sigma(m)}^1\}_{m\in\mathbb{N}}, \{a_{\sigma(m)}^2\}_{m\in\mathbb{N}}, \ldots, \{a_{\sigma(m)}^n\}_{m\in\mathbb{N}}$ (where $\sigma = \sigma_n \cdots \sigma_1$) such that $\{a_{\sigma(m)}^i\}_{m\in\mathbb{N}} \to a_i$ for all i. And $\{b_{\sigma(m)}\}_{m\in\mathbb{N}} \to \delta$. But

$$\{b_{\sigma(m)}\}_{m\in\mathbb{N}} = \left\{\max\left(a_{\sigma(m)}^n, \ldots, a_{\sigma(m)}^n\right)\right\}_{m\in\mathbb{N}} \to \max(a_1, \ldots, a_n),$$

so $\delta = \max(a_1, \ldots, a_n)$ and there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $a_{i_0} = \delta$. Therefore, there exist $i_0 \in \{1, 2, \ldots, n\}$ and a subsequence $\{a_{\sigma(m)}^{i_0}\}_{m\in\mathbb{N}}$ such that $\{a_{\sigma(m)}^{i_0}\}_{m\in\mathbb{N}} \to a_{i_0} = \delta$. □

Lemma 6 *Let* $\{a_m\}_{m\in\mathbb{N}}$ *be a sequence of non-negative real numbers which has not any subsequence converging to zero. Then, for all* $\varepsilon > 0$, *there exist* $\delta \in]0, \varepsilon[$ *and* $m_0 \in \mathbb{N}$ *such that* $a_m \ge \delta$ *for all* $m \ge m_0$.

Proof Suppose that the conclusion is not true. Then there exists $\varepsilon_0 > 0$ such that, for all $\delta \in]0, \varepsilon_0[$, there exists $m_0 \in \mathbb{N}$ verifying $a_{m_0} < \delta$. Let $k_0 \in \mathbb{N}$ be such that $1/k_0 < \varepsilon_0$. For all $k \in \mathbb{N}$, take $\delta_k = 1/(k + k_0) \in]0, \varepsilon_0[$. Then there exists $m(k) \in \mathbb{N}$ verifying $0 \le a_{m(k)} < \delta_k = 1/(k + k_0)$. Taking limit when $k \to \infty$, we deduce that $\lim_{k\to\infty} a_{m(k)} = 0$. Then $\{a_m\}$ has a subsequence converging to zero (maybe, reordering $\{a_{m(k)}\}$), but this is a contradiction. □

Let

$$\Psi = \{\phi : [0,\infty) \to [0,\infty) : \phi \text{ is continuous, non-decreasing and } \phi^{-1}(\{0\}) = \{0\}\}.$$

Lemma 7 *If $\psi \in \Psi$ and $\{a_m\} \subset [0,\infty)$ verifies $\{\psi(a_m)\} \to 0$, then $\{a_m\} \to 0$.*

Proof If the conclusion does not hold, there exists $\varepsilon_0 > 0$ such that, for all $m_0 \in \mathbb{N}$, there exists $m \geq m_0$ verifying $a_m \geq \varepsilon_0$. This means that $\{a_m\}$ has a partial subsequence $\{a_{m(k)}\}_k$ such that $a_{m(k)} \geq \varepsilon_0$. As ψ is non-decreasing, $\psi(\varepsilon_0) \leq \psi(a_{m(k)})$ for all $k \in \mathbb{N}$. Therefore, $\{\psi(a_m)\}_m$ has a subsequence $\{\psi(a_{m(k)})\}_k$ lower bounded by $\psi(\varepsilon_0) > 0$, but this is impossible since $\lim_{m\to\infty} \psi(a_m) = 0$. □

Lemma 8 *Let $\{a_m^1\}, \{a_m^2\}, \ldots, \{a_m^n\}, \{b_m^1\}, \{b_m^2\}, \ldots, \{b_m^n\} \subset [0,\infty)$ be $2n$ sequences of non-negative real numbers and suppose that there exist $\psi, \varphi \in \Psi$ such that*

$$\psi(a_{m+1}^i) \leq (\psi - \varphi)(b_m^i) \quad \text{for all } i \text{ and all } m, \quad \text{and}$$
$$\psi\left(\max_{1\leq i\leq n} b_m^i\right) \leq \psi\left(\max_{1\leq i\leq n} a_m^i\right) \quad \text{for all } m.$$

Then $\{a_m^i\} \to 0$ for all i.

Proof Let $c_m = \max_{1\leq i\leq n} a_m^i$ for all m. Then, for all m,

$$\begin{aligned}\psi(c_{m+1}) &= \psi\left(\max_{1\leq i\leq n} a_{m+1}^i\right) = \max_{1\leq i\leq n} \psi(a_{m+1}^i) \leq \max_{1\leq i\leq n}\left[(\psi - \varphi)(b_m^i)\right] \leq \max_{1\leq i\leq n} \psi(b_m^i) \\ &= \psi\left(\max_{1\leq i\leq n} b_m^i\right) \leq \psi\left(\max_{1\leq i\leq n} a_m^i\right) = \psi(c_m).\end{aligned}$$

Therefore, $\{\psi(c_m)\}$ is a non-increasing, bounded below sequence. Then it is convergent. Let $\Delta \geq 0$ be such that $\{\psi(c_m)\} \to \Delta$ and $\Delta \leq \psi(c_m)$. Let us show that $\Delta = 0$. Since

$$\left\{\max_{1\leq i\leq n} \psi(a_m^i)\right\} = \left\{\psi\left(\max_{1\leq i\leq n} a_m^i\right)\right\} = \{\psi(c_m)\} \to \Delta,$$

Lemma 5 guarantees that there exist $i_0 \in \{1,2,\ldots,n\}$ and a partial subsequence $\{a_{m(k)}^{i_0}\}_{k\in\mathbb{N}}$ such that $\{\psi(a_{m(k)}^{i_0})\} \to \Delta$. Moreover,

$$0 \leq \psi(a_{m(k)}^{i_0}) \leq (\psi - \varphi)(b_{m(k)-1}^{i_0}) \quad \text{for all } k. \tag{3}$$

Consider the sequence $\{b_{m(k)-1}^{i_0}\}_{k\in\mathbb{N}}$. If this sequence has a partial subsequence converging to zero, then we can take limit in (3) when $k \to 0$ using that partial subsequence, and we deduce $\Delta = 0$. On the contrary, if $\{b_{m(k)-1}^{i_0}\}_{k\in\mathbb{N}}$ has not any partial subsequence converging to zero, Lemma 6 assures us that there exist $\delta \in]0,1[$ and $k_0 \in \mathbb{N}$ such that $b_{m(k)-1}^{i_0} \geq \delta$ for all $k \geq k_0$. Since φ is non-decreasing, $-\varphi(b_{m(k)-1}^{i_0}) \leq -\varphi(\delta) < 0$. Then, by (3), for all $k \geq k_0$,

$$\begin{aligned}0 &\leq \psi(a_{m(k)}^{i_0}) \leq (\psi - \varphi)(b_{m(k)-1}^{i_0}) = \psi(b_{m(k)-1}^{i_0}) - \varphi(b_{m(k)-1}^{i_0}) \leq \psi(b_{m(k)-1}^{i_0}) - \varphi(\delta) \\ &\leq \psi\left(\max_{1\leq i\leq n} b_{m(k)-1}^i\right) - \varphi(\delta) \leq \psi\left(\max_{1\leq i\leq n} a_{m(k)-1}^i\right) - \varphi(\delta) = \psi(c_{m(k)-1}) - \varphi(\delta).\end{aligned}$$

Taking limit as $k \to \infty$, we deduce $\Delta \leq \Delta - \varphi(\delta)$, which is impossible. This proves that $\Delta = 0$. Since $\{\psi(c_m)\} \to \Delta = 0$, Lemma 7 implies that $\{c_m\} \to 0$, which is equivalent to $\{a_m^i\} \to 0$ for all i. □

Corollary 9 *If $\psi, \varphi \in \Psi$ and $\{a_m\}, \{b_m\} \subset [0,\infty)$ verify $\psi(a_{m+1}) \leq (\psi - \varphi)(b_m)$ and $\psi(b_m) \leq \psi(a_m)$ for all m, then $\{a_m\} \to 0$.*

Corollary 10 *If $\psi, \varphi \in \Psi$ and $\{a_m\} \subset [0,\infty)$ verifies $\psi(a_{m+1}) \leq \psi(a_m) - \varphi(a_m)$ for all m, then $\{a_m\} \to 0$.*

Definition 11 (Mustafa and Sims [11]) A generalized metric (or a G-metric) on X is a mapping $G : X^3 \to \mathbb{R}_0^+$ verifying, for all $x, y, z \in X$:

(G_1) $G(x,x,x) = 0$.
(G_2) $G(x,x,y) > 0$ if $x \neq y$.
(G_3) $G(x,x,y) \leq G(x,y,z)$ if $y \neq z$.
(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables).
(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ (*rectangle inequality*).

Let $\{(X_i, G_i)\}_{i=1}^n$ be a family of G-metric spaces, consider the product space $X = X_1 \times X_2 \times \cdots \times X_n$ and define G^m and G^s on X^3 by

$$G^m(\mathsf{X},\mathsf{Y},\mathsf{Z}) = \max_{1 \leq i \leq n} G_i(x_i, y_i, z_i) \quad \text{and} \quad G^s(\mathsf{X},\mathsf{Y},\mathsf{Z}) = \sum_{i=1}^n G_i(x_i, y_i, z_i)$$

for all $\mathsf{X} = (x_1, x_2, \ldots, x_n)$, $\mathsf{Y} = (y_1, y_2, \ldots, y_n)$, $\mathsf{Z} = (z_1, z_2, \ldots, z_n) \in X$.

A classical example of G-metric comes from a metric space (X,d), where $G(x,y,z) = d_{xy} + d_{yz} + d_{zx}$ measures the perimeter of a triangle. In this case, property (G_3) has an obvious geometric interpretation: the length of an edge of a triangle is less than or equal to its semiperimeter, that is, $2d_{xy} \leq d_{xy} + d_{yz} + d_{zx}$. However, property ($G_3$) implies that, in general, the major structures G^m and G^s are not necessarily G-metrics on $X_1 \times X_2 \times \cdots \times X_n$. Only when each G_i is *symmetric* (that is, $G(x,x,y) = G(y,y,x)$ for all x, y), the product is also a G-metric (see [11]). But in this case, symmetric G-metrics can be reduced to usual metrics, which limits the interest in this kind of spaces.

In order to prove our main results, that are also valid in G-metric spaces, we will not need property (G_3). Omitting this property, we consider a class of spaces for which G^m and G^s have the same initial metric structure. Then we present the following spaces.

3 G^*-metric spaces

Definition 12 A G^*-metric on X is a mapping $G : X^3 \to \mathbb{R}_0^+$ verifying (G_1), (G_2), (G_4) and (G_5).

The *open ball $B(x,r)$ of center $x \in X$ and radius $r > 0$* in a G^*-metric space (X,G) is

$$B(x,r) = \{y \in X : G(x,x,y) < r\}.$$

The following lemma is a characterization of the topology generated by a neighborhood system at each point.

Lemma 13 *Let X be a set and, for all $x \in X$, let β_x be a non-empty family of subsets of X verifying*:

1. $x \in N$ *for all* $N \in \beta_x$.
2. *For all* $N_1, N_2 \in \beta_x$, *there exists* $N_3 \in \beta_x$ *such that* $N_3 \subseteq N_1 \cap N_2$.
3. *For all* $N \in \beta_x$, *there exists* $N' \in \beta_x$ *such that for all* $y \in N'$, *there exists* $N'' \in \beta_y$ *verifying* $N'' \subseteq N$.

Then there exists a unique topology τ on X such that β_x is a neighborhood system at x.

Let (X, G) be a G^*-metric space and consider the family $\beta_x = \{B(x,r) : r > 0\}$. It is clear that $x \in B(x,r)$ (by (G_1), $G(x,x,x) = 0$) and $N_3 = B(x, \min(r,s)) \subseteq B(x,r) \cap B(x,s)$. Next, let $N = N' = B(x,r) \in \beta_x$ and let $y \in N' = B(x,r)$. We have to prove that there exists $s > 0$ such that $N'' = B(y,s) \subseteq B(x,r) = N$. Indeed, if $y = x$, then we can take $s = r > 0$. On the contrary, if $y \neq x$, then $0 < G(x,x,y) < r$ by (G_2). Let $r' \in]G(x,x,y), r[$ arbitrary and let $s = r - r' > 0$ (that is, $r' + s = r$). Now we prove that $B(y,s) \subseteq B(x,r)$. Let $z \in B(y,s)$. Then, using (G_4) and (G_5),

$$G(x,x,z) = G(z,x,x) \overset{a=y}{\leq} G(z,y,y) + G(y,x,x) = G(x,x,y) + G(y,y,z) < r' + s = r.$$

Then $z \in B(x,r)$ and, as a consequence, $B(y,s) \subseteq B(x,r)$. Lemma 13 guarantees that there exists a unique topology τ_G on X such that $\beta_x = \{B(x,r) : r > 0\}$ is a neighborhood system at each $x \in X$.

Next, let us show that τ_G is Hausdorff. Let $x, y \in X$ be two points such that $x \neq y$. By (G_2), $r = G(x,x,y) > 0$. We claim that $B(x, r/4) \cap B(y, r/4) = \varnothing$. We reason by contradiction. Let $z \in B(x, r/4) \cap B(y, r/4)$, that is, $G(x,x,z) < r/4$ and $G(y,y,z) < r/4$. Using (G_4) and (G_5) twice

$$\begin{aligned} 0 < r = G(x,x,y) &= G(y,x,x) \leq G(y,z,z) + G(z,x,x) = G(z,z,y) + G(x,x,z) \\ &\leq G(z,y,y) + G(y,z,y) + G(x,x,z) = G(y,y,z) + G(y,y,z) + G(x,x,z) \\ &< \frac{r}{4} + \frac{r}{4} + \frac{r}{4} = \frac{3r}{4} < r, \end{aligned}$$

which is impossible. Then $B(x, r/4) \cap B(y, r/4) = \varnothing$ and τ_G is Hausdorff.

A subset $A \subseteq X$ is G-open if for all $x \in A$ there exists $r > 0$ such that $B(x,r) \subseteq A$. Following classic techniques, it is possible to prove that there exists a unique topology τ_G on X such that $\beta_x = \{B(x,r) : r > 0\}$ is a neighborhood system at each $x \in X$. Furthermore, τ_G is a Hausdorff topology. In this topology, we characterize the notions of convergent sequence and Cauchy sequence in the following way. Let (X, G) be a G^*-metric space, let $\{x_m\} \subseteq X$ be a sequence and let $x \in X$.

- $\{x_m\}$ *G-converges to x*, and we will write $\{x_m\} \xrightarrow{G} x$ if $\lim_{m,m' \to \infty} G(x_m, x_{m'}, x) = 0$, that is, for all $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ verifying that $G(x_m, x_{m'}, x) < \varepsilon$ for all $m, m' \in \mathbb{N}$ such that $m, m' \geq m_0$.
- $\{x_m\}$ is *G-Cauchy* if $\lim_{m,m',m'' \to \infty} G(x_m, x_{m'}, x_{m''}) = 0$, that is, for all $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ verifying that $G(x_m, x_{m'}, x_{m''}) < \varepsilon$ for all $m, m', m'' \in \mathbb{N}$ such that $m, m', m'' \geq m_0$.

Lemma 14 *Let (X, G) be a G^*-metric space, let $\{x_m\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.*

(a) $\{x_m\}$ *G-converges to* x.
(b) $\lim_{m\to\infty} G(x,x,x_m) = 0$.
(c) $\lim_{m\to\infty} G(x_m,x_m,x) = 0$.
(d) $\lim_{m\to\infty} G(x_m,x_m,x) = 0$ *and* $\lim_{m\to\infty} G(x_m,x_{m+1},x) = 0$.
(e) $\lim_{m\to\infty} G(x,x,x_m) = 0$ *and* $\lim_{m\to\infty} G(x_m,x_{m+1},x) = 0$.

Notice that the condition $\lim_{m\to\infty} G(x_m,x_{m+1},x) = 0$ is not strong enough to prove that $\{x_m\}$ G-converges to x.

Proposition 15 *The limit of a G-convergent sequence in a G^*-metric space is unique.*

Lemma 16 *If (X,G) is a G^*-metric space and $\{x_m\} \subseteq X$ is a sequence, then the following conditions are equivalent.*
(a) $\{x_m\}$ *is G-Cauchy.*
(b) $\lim_{m,m'\to\infty} G(x_m,x_{m'},x_{m'}) = 0$.
(c) $\lim_{m,m'\to\infty} G(x_m,x_{m+1},x_{m'}) = 0$.

Remark 17 As a consequence, a sequence $\{x_m\} \subseteq X$ is not G-Cauchy if and only if there exist $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}_{k\in\mathbb{N}}$ and $\{x_{m(k)}\}_{k\in\mathbb{N}}$ such that $k < n(k) < m(k) < n(k+1)$, $G(x_{n(k)},x_{n(k)+1},x_{m(k)}) \geq \varepsilon_0$ and $G(x_{n(k)},x_{n(k)+1},x_{m(k)-1}) < \varepsilon_0$ for all k.

Definition 18 Let (X,G) be a G^*-metric space and let $\preccurlyeq$ be a preorder on X. We will say that $(X,G,\preccurlyeq)$ is *regular non-decreasing* (respectively, *regular non-increasing*) if for all $\preccurlyeq$-monotone non-decreasing (respectively, non-increasing) sequence $\{x_m\}$ such that $\{x_m\} \stackrel{G}{\to} z_0$, we have that $x_m \preccurlyeq z_0$ (respectively, $x_m \succcurlyeq z_0$) for all m. We will say that $(X,G,\preccurlyeq)$ is *regular* if it is both regular non-decreasing and regular non-increasing.

Some authors said that $(X,G,\preccurlyeq)$ verifies the *sequential monotone property* if $(X,G,\preccurlyeq)$ is regular (see [20]). The notion of *G-continuous mapping* $F: X^n \to X$ follows considering on X the topology τ_G and in X^n the product topology.

Definition 19 If (X,G) is a G^*-metric space, we will say that a mapping $F: X^n \to X$ is *G-continuous* if for all n sequences $\{a_m^1\},\{a_m^2\},\dots,\{a_m^n\} \subseteq X$ such that $\{a_m^i\} \stackrel{G}{\to} a_i \in X$ for all i, we have that $\{F(a_m^1,a_m^2,\dots,a_m^n)\} \stackrel{G}{\to} F(a_1,a_2,\dots,a_n)$.

In this topology, the notion of *convergence* is the following.

$$\begin{aligned}\{x_m\} \stackrel{G}{\to} x &\Leftrightarrow \bigl[\forall B(x,r), \exists m_0 \in \mathbb{N} : \bigl(m \geq m_0 \Rightarrow x_m \in B(x,r)\bigr)\bigr] \\ &\Leftrightarrow \bigl[\forall \varepsilon > 0, \exists m_0 \in \mathbb{N} : \bigl(m \geq m_0 \Rightarrow G(x,x,x_m) < \varepsilon\bigr)\bigr] \\ &\Leftrightarrow \Bigl[\lim_{m\to\infty} G(x,x,x_m) = 0\Bigr].\end{aligned}$$

This property can be characterized as follows.

Lemma 20 *Let (X,G) be a G^*-metric space, let $\{x_m\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.*
(a) $\{x_m\}$ *G-converges to* x (*that is,* $\lim_{m,m'\to\infty} G(x_m,x_{m'},x) = 0$, *which means that for all* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $G(x_m,x_{m'},x)$ *for all* $m,m' \geq m_0$).

(b) $\lim_{m\to\infty} G(x,x,x_m) = 0$.
(c) $\lim_{m\to\infty} G(x_m,x_m,x) = 0$.
(d) $\lim_{m\to\infty} G(x_m,x_m,x) = 0$ *and* $\lim_{m\to\infty} G(x_m,x_{m+1},x) = 0$.
(e) $\lim_{m\to\infty} G(x,x,x_m) = 0$ *and* $\lim_{m\to\infty} G(x_m,x_{m+1},x) = 0$.

Proof [(a) ⇒ (c)] It is apparent using $m = m'$.
[(c) ⇒ (b)] Using (G_5), $G(x,x,x_m) \leq G(x,x_m,x_m) + G(x_m,x,x_m) = 2G(x_m,x_m,x)$.
[(b) ⇒ (a)] Using (G_4) and (G_5),

$$G(x_m,x_{m'},x) \leq G(x_m,x,x) + G(x,x_{m'},x) \leq 2\max\bigl(G(x,x,x_m),G(x,x,x_{m'})\bigr).$$

[(a) ⇒ (d),(e)] It is apparent using $m' = m$ and $m' = m+1$.
[(d) ⇒ (c)] It is evident.
[(e) ⇒ (b)] It is evident. □

Corollary 21 *If* (X,G) *is a G-metric space, then* $\{x_m\} \xrightarrow{G} x$ *if and only if* $\lim_{m\to\infty} G(x_m, x_{m+1},x) = 0$.

Proof We only need to prove that the condition is sufficient. Suppose that $\lim_{m\to\infty} G(x_m, x_{m+1},x) = 0$. In a G-metric space, the following property holds (see [11]):

$$G(x,y,z) \leq G(x,a,z) + G(a,y,z) \quad \text{for all } x,y,z,a \in X.$$

Then, using $a = x_{m+1}$,

$$G(x,x,x_m) = G(x,x_{m+1},x_m) + G(x_{m+1},x,x_m) = 2G(x_m,x_{m+1},x).$$

This proves (b) in the previous lemma. □

Proposition 22 *The limit of a G-convergent sequence in a* G^**-metric space is unique*.

Proof Suppose that $\{x_m\} \xrightarrow{G} x$ and $\{x_m\} \xrightarrow{G} y$. Then

$$G(x,x,y) = G(y,x,x) \leq G(y,x_m,x_m) + G(x_m,x,x).$$

By items (a) and (c) of Lemma 20, we deduce that $G(x,x,y) = 0$, which means that $x = y$ by (G_2). □

In the topology τ_G, the notion of *Cauchy sequence* is the following.

$$\{x_m\} \text{ is } G\text{-Cauchy} \quad \Leftrightarrow \quad \bigl[\forall\varepsilon > 0, \exists m_0 \in \mathbb{N} : \bigl(m,m',m'' \geq m_0 \Rightarrow G(x_m,x_{m'},x_{m''}) < \varepsilon\bigr)\bigr].$$

This definition can be characterized as follows.

Lemma 23 *If* (X,G) *is a* G^**-metric space and* $\{x_m\} \subseteq X$ *is a sequence, then the following conditions are equivalent*.
(a) $\{x_m\}$ *is G-Cauchy*.

(b) $\lim_{m,m'\to\infty} G(x_m, x_{m'}, x_{m'}) = 0$.
(c) $\lim_{m,m'\to\infty} G(x_m, x_{m+1}, x_{m'}) = 0$.

Proof [(b) ⇒ (a)] Using (G_5), $G(x_m, x_{m'}, x_{m''}) \leq G(x_m, x_{m'}, x_{m'}) + G(x_{m'}, x_{m'}, x_{m''})$.

[(a) ⇒ (c)] It is apparent using $m'' = m + 1$.

[(c) ⇒ (b)] Let $\varepsilon > 0$ and let $m_0 \in \mathbb{N}$ be such that $G(x_m, x_{m+1}, x_{m'}) < \varepsilon/2$ for all $m, m' \geq m_0$. Then

$$m', m \geq m_0 \Rightarrow G(x_{m'}, x_{m'+1}, x_m) < \varepsilon/2,$$

$$m', m' + 1 \geq m_0 \Rightarrow G(x_{m'}, x_{m'+1}, x_{m'+1}) < \varepsilon/2.$$

Therefore, using (G_4) and (G_5),

$$\begin{aligned} G(x_m, x_{m'}, x_{m'}) &= G(x_{m'}, x_{m'}, x_m) \leq G(x_{m'}, x_{m'+1}, x_{m'+1}) + G(x_{m'+1}, x_{m'}, x_m) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, $\lim_{m,m'\to\infty} G(x_m, x_{m'}, x_{m'}) = 0$. □

4 Product of G^*-metric spaces

Lemma 24 *Let $\{(X_i, G_i)\}_{i=1}^n$ be a family of G^*-metric spaces, consider the product space $\mathbb{X} = X_1 \times X_2 \times \cdots \times X_n$ and define $G_n^{\max}$ and G_n^{sum} on $\mathbb{X}^3$ by*

$$G_n^{\max}(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \max_{1\leq i\leq n} G_i(x_i, y_i, z_i) \quad \textit{and} \quad G_n^{\text{sum}}(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \sum_{i=1}^{n} G_i(x_i, y_i, z_i)$$

for all $\mathsf{X} = (x_1, x_2, \ldots, x_n), \mathsf{Y} = (y_1, y_2, \ldots, y_n), \mathsf{Z} = (z_1, z_2, \ldots, z_n) \in \mathbb{X}$. Then the following statements hold.

1. *$G_n^{\max}$ and G_n^{sum} are G^*-metrics on $\mathbb{X}$.*
2. *If $A_m = (a_m^1, a_m^2, \ldots, a_m^n) \in \mathbb{X}$ for all m and $A = (a_1, a_2, \ldots, a_n) \in \mathbb{X}$, then $\{A_m\}$ $G_n^{\max}$-converges (respectively, G_n^{sum}-converges) to A if and only if each $\{a_m^i\}$ G_i-converges to a_i.*
3. *$\{A_m\}$ is $G_n^{\max}$-Cauchy if and only if each $\{a_m^i\}$ is G_i-Cauchy.*
4. *$(\mathbb{X}, G_n^{\max})$ (respectively, $(\mathbb{X}, G_n^{\text{sum}})$) is complete if and only if every (X_i, G_i) is complete.*
5. *For all i, let $\preceq_i$ be a preorder on X_i and define $\mathsf{X} \preceq \mathsf{Y}$ if and only if $x_i \preceq_i y_i$ for all i. Then $(X, G_n^{\max}, \preceq)$ is regular (respectively, regular non-decreasing, regular non-increasing) if and only if each factor (X_i, G_i) is also regular (respectively, regular non-decreasing, regular non-increasing).*

Proof Let us denote $G = G_n^{\max}$. Taking into account that $G_n^{\max} \leq G_n^{\text{sum}} \leq nG_n^{\max}$, we will only develop the proof using G.

(1) It is a straightforward exercise to prove the following statements.

- $G(\mathsf{X}, \mathsf{X}, \mathsf{X}) = \max_{1\leq i\leq n} G_i(x_i, x_i, x_i) = \max_{1\leq i\leq n} 0 = 0$.
- If $\mathsf{Y} \neq \mathsf{Z}$, there exists $j \in \{1, 2, \ldots, n\}$ such that $y_j \neq z_j$. Then $G(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \max_{1\leq i\leq n} G_i(x_i, y_i, z_i) \geq G_j(x_j, y_j, z_j) > 0$.
- Symmetry in all three variables of G follows from symmetry in all three variables of each G_i.

- We have that

$$G(\mathsf{X},\mathsf{Y},\mathsf{Z}) = \max_{1\le i\le n} G_i(x_i,y_i,z_i) \le \max_{1\le i\le n}\bigl[G_i(x_i,a_i,a_i)+G_i(a_i,y_i,z_i)\bigr]$$
$$\le \max_{1\le i\le n} G_i(x_i,a_i,a_i) + \max_{1\le i\le n} G_i(a_i,y_i,z_i) = G(\mathsf{X},\mathsf{A},\mathsf{A}) + G(\mathsf{A},\mathsf{Y},\mathsf{Z}).$$

Then G is a G^*-metric on $\mathbb{X}$.

(2) We use Lemma 20. Suppose that $\{A_m\}$ G-converges to A and let $\varepsilon > 0$. Then, for all $j\in\{1,2,\dots,n\}$ and all m,

$$G_j\bigl(a_j,a_j,a_m^j\bigr) \le \max_{1\le i\le n} G_i\bigl(a_i,a_i,a_m^i\bigr) = G(A,A,A_m).$$

Therefore, $\{a_m^j\}$ G_j-converges to a_j. Conversely, assume that each $\{a_m^i\}$ G_i-converges to a_i. Let $\varepsilon > 0$ and let $m_i\in\mathbb{N}$ be such that if $m\ge m_i$, then $G_i(a_i,a_i,a_m^i) < \varepsilon$. If $m_0 = \max(m_1,m_2,\dots,m_n)$ and $m,m'\ge m_0$, then $G(A,A,A_m) = \max_{1\le i\le n} G_i(a_i,a_i,a_m^i) < \varepsilon$, so $\{A_m\}$ G-converges to A.

(3) We use Lemma 23. Suppose that $\{A_m\}$ is G-Cauchy and let $\varepsilon > 0$. Then, for all $j\in\{1,2,\dots,n\}$ and all m, m',

$$G_j\bigl(a_m^j,a_m^j,a_{m'}^j\bigr) \le \max_{1\le i\le n} G_i\bigl(a_m^i,a_m^i,a_{m'}^i\bigr) = G(A_m,A_m,A_{m'}).$$

Therefore, $\{a_m^j\}$ is G_j-Cauchy. Conversely, assume that each $\{a_m^i\}$ is G_i-Cauchy. Let $\varepsilon > 0$ and let $m_i\in\mathbb{N}$ be such that if $m,m'\ge m_i$, then $G_i(a_m^j,a_m^j,a_{m'}^j) < \varepsilon$. If $m_0 = \max(m_1,m_2,\dots,m_n)$ and $m,m'\ge m_0$, then $G(A_m,A_m,A_{m'}) = \max_{1\le i\le n} G_i(a_m^i,a_m^i,a_{m'}^i) < \varepsilon$, so $\{A_m\}$ is G-Cauchy.

(4) It is an easy consequence of items 2 and 3 since

$$\{A_m\}G\text{-Cauchy} \Leftrightarrow \text{each }\bigl\{a_m^i\bigr\}G\text{-Cauchy} \Leftrightarrow \text{each }\bigl\{a_m^i\bigr\}G\text{-convergent}$$
$$\Leftrightarrow \{A_m\}G\text{-convergent}.$$

(5) A sequence $\{A_m\}$ on $\mathbb{X}$ is $\preceq$-monotone non-decreasing if and only if each sequence $\{a_m^i\}$ is $\preceq$-monotone non-decreasing. Moreover, $\{A_m\}$ G-converges to $A = (a_1,a_2,\dots,a_n)\in\mathbb{X}$ if and only if each $\{a_m^i\}$ G_i-converges to a_i. Finally, $A_m\preceq A$ if and only if $a_m^i\preceq_i a_i$ for all i. Therefore, $(X,G_n^{\max},\preceq)$ is regular non-decreasing if and only if each factor (X_i,G_i) is also regular non-decreasing. Other statements may be proved similarly. □

Taking $(X_i,G_i) = (X,G)$ for all i, we derive the following result.

Corollary 25 *Let (X,G) be a G^*-metric space and consider on the product space X^n the mappings G_n and G_n' defined by*

$$G_n(\mathsf{X},\mathsf{Y},\mathsf{Z}) = \max_{1\le i\le n} G(x_i,y_i,z_i) \quad\textit{and}\quad G_n'(\mathsf{X},\mathsf{Y},\mathsf{Z}) = \sum_{i=1}^{n} G(x_i,y_i,z_i)$$

for all $\mathsf{X} = (x_1,x_2,\dots,x_n), \mathsf{Y} = (y_1,y_2,\dots,y_n), \mathsf{Z} = (z_1,z_2,\dots,z_n)\in X^n$.

1. *G_n and G'_n are G^*-metrics on X^n.*
2. *If $A_m = (a_m^1, a_m^2, \ldots, a_m^n) \in X^n$ for all m and $A = (a_1, a_2, \ldots, a_n) \in X^n$, then $\{A_m\}$ G_n-converges (respectively, G'_n-converges) to A if and only if each $\{a_m^i\}$ G-converges to a_i.*
3. *$\{A_m\}$ is G_n-Cauchy (respectively, G'_n-Cauchy) if and only if each $\{a_m^i\}$ is G-Cauchy.*
4. *(X, G_n) (respectively, (X^n, G'_n)) is complete if and only if (X, G) is complete.*
5. *If (X, G) is $\preccurlyeq$-regular, then (X^n, G_n) is $\sqsubseteq$-regular.*

5 Unidimensional fixed point result in partially preordered G^*-metric spaces

Theorem 26 *Let $(X, \preccurlyeq)$ be a preordered set endowed with a G^*-metric G and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a) *(X, G) is complete.*
(b) *T is non-decreasing (w.r.t. $\preccurlyeq$).*
(c) *Either T is G-continuous or $(X, G, \preccurlyeq)$ is regular non-decreasing.*
(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*
(e) *There exist two mappings $\psi, \varphi \in \Psi$ such that, for all $x, y \in X$ with $x \preccurlyeq y$,*

$$\psi\big(G\big(Tx, Ty, T^2x\big)\big) \leq \psi\big(G(x, y, Tx)\big) - \varphi\big(G(x, y, Tx)\big).$$

Then T has a fixed point. Furthermore, if for all $z_1, z_2 \in X$ fixed points of T there exists $z \in X$ such that $z_1 \preccurlyeq z$ and $z_2 \preccurlyeq z$, we obtain uniqueness of the fixed point.

Proof Define $x_m = T^m x_0$ for all $m \geq 1$. Since T is non-decreasing (w.r.t. $\preccurlyeq$), then $x_m \preccurlyeq x_{m+1}$ for all $m \geq 0$. Then

$$\begin{aligned}\psi\big(G(x_{m+1}, x_{m+2}, x_{m+2})\big) &= \psi\big(G\big(Tx_m, Tx_{m+1}, T^2x_m\big)\big)\\ &\leq \psi\big(G(x_m, x_{m+1}, Tx_m)\big) - \varphi\big(G(x_m, x_{m+1}, Tx_m)\big)\\ &= \psi\big(G(x_m, x_{m+1}, x_{m+1})\big) - \varphi\big(G(x_m, x_{m+1}, x_{m+1})\big).\end{aligned}$$

Applying Lemma 10, $\{G(x_m, x_{m+1}, x_{m+1})\} \to 0$. Let us show that $\{x_m\}$ is G-Cauchy. Reasoning by contradiction, if $\{x_m\}$ is not G-Cauchy, by Remark 17, there exist $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ verifying $k < n(k) < m(k) < n(k+1)$,

$$G(x_{n(k)}, x_{m(k)}, x_{n(k)+1}) > \varepsilon_0 \quad \text{and} \quad G(x_{n(k)}, x_{m(k)-1}, x_{n(k)+1}) \leq \varepsilon_0 \quad \text{for all } k \geq 1. \tag{4}$$

Therefore

$$\begin{aligned}0 &< \psi(\varepsilon_0) \leq \psi\big(G(x_{n(k)}, x_{m(k)}, x_{n(k)+1})\big) = \psi\big(G\big(Tx_{n(k)-1}, Tx_{m(k)-1}, T^2x_{n(k)-1}\big)\big)\\ &\leq \psi\big(G(x_{n(k)-1}, x_{m(k)-1}, Tx_{n(k)-1})\big) - \varphi\big(G(x_{n(k)-1}, x_{m(k)-1}, Tx_{n(k)-1})\big)\\ &= \psi\big(G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)})\big) - \varphi\big(G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)})\big).\end{aligned} \tag{5}$$

Consider the sequence of non-negative real numbers $\{G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)})\}$. If this sequence has a partial subsequence converging to zero, then we can take the limit in (5) using this partial subsequence and we would deduce $0 < \psi(\varepsilon_0) \leq 0$, which is impossible. Then $\{G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)})\}$ cannot have a partial subsequence converging to zero. This means that there exist $\delta > 0$ and $k_0 \in \mathbb{N}$ such that

$$G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)}) \geq \delta \quad \text{for all } k \geq k_0.$$

Since φ is non-decreasing, $-\varphi(G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)}) \leq -\varphi(\delta) < 0$. By ($G_5$) and (4),

$$\begin{aligned}
&G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)}) \\
&\quad = G(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}) \quad [x = x_{n(k)-1}, y = x_{n(k)}, z = x_{m(k)-1}, a = x_{n(k)+1}] \\
&\quad \leq G(x_{n(k)-1}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)}, x_{m(k)-1}) \\
&\quad = G(x_{n(k)-1}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}) \\
&\qquad [x = x_{n(k)-1}, y = z = x_{n(k)+1}, a = x_{n(k)}] \\
&\quad \leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)}, x_{n(k)+1}, x_{m(k)-1}) \\
&\quad \leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + \varepsilon_0.
\end{aligned}$$

Since ψ is non-decreasing, it follows from (5) that

$$\begin{aligned}
0 &< \psi(\varepsilon_0) \leq \psi\bigl(G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)})\bigr) - \varphi\bigl(G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)})\bigr) \\
&\leq \psi\bigl(G(x_{n(k)-1}, x_{m(k)-1}, x_{n(k)})\bigr) - \varphi(\delta) \\
&\leq \psi\bigl(G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + \varepsilon_0\bigr) - \varphi(\delta).
\end{aligned}$$

Taking limit when $k \to \infty$, we deduce that $0 < \psi(\varepsilon_0) \leq \psi(\varepsilon_0) - \varphi(\delta) < \psi(\varepsilon_0)$, which is impossible. This contradiction finally proves that $\{x_m\}$ is G-Cauchy. Since (X, G) is complete, there exists $z_0 \in X$ such that $\{x_m\} \xrightarrow{G} z_0$.

Now suppose that T is G-continuous. Then $\{x_{m+1}\} = \{Tx_m\} \xrightarrow{G} Tz_0$. By the unicity of the limit, $Tz_0 = z_0$ and z_0 is a fixed point of T.

On the contrary, suppose that $(X, G, \preccurlyeq)$ is regular non-decreasing. Since $\{x_m\} \xrightarrow{G} z_0$ and $\{x_m\}$ is monotone non-decreasing (w.r.t. $\preccurlyeq$), it follows that $x_m \preccurlyeq z_0$ for all m. Hence

$$\begin{aligned}
\psi\bigl(G(x_{m+1}, Tz_0, x_{m+2})\bigr) &= \psi\bigl(G\bigl(Tx_m, Tz_0, T^2 x_m\bigr)\bigr) \\
&\leq \psi\bigl(G(x_m, z_0, Tx_m)\bigr) - \varphi\bigl(G(x_m, z_0, Tx_m)\bigr) \\
&= \psi\bigl(G(x_m, x_{m+1}, z_0)\bigr) - \varphi\bigl(G(x_m, x_{m+1}, z_0)\bigr).
\end{aligned}$$

Since $\{x_m\} \xrightarrow{G} z_0$, then $\{G(x_m, x_{m+1}, z_0)\} \to 0$. Taking limit when $k \to \infty$, we deduce that $\{\psi(G(x_{m+1}, Tz_0, x_{m+2}))\} \to 0$. By Lemma 7, $\{G(x_{m+1}, x_{m+2}, Tz_0)\} \to 0$, so $\{x_m\} \xrightarrow{G} Tz_0$ and we also conclude that z_0 is a fixed point of T.

To prove the uniqueness, let $z_1, z_2 \in X$ be two fixed points of T. By hypothesis, there exists $z \in X$ such that $z_1 \preccurlyeq z$ and $z_2 \preccurlyeq z$. Let us show that $\{T^m z\} \xrightarrow{G} z_1$. Indeed,

$$\begin{aligned}
\psi\bigl(G\bigl(z_1, z_1, T^{m+1} z\bigr)\bigr) &= \psi\bigl(G\bigl(Tz_1, TT^m z, T^2 z_1\bigr)\bigr) \\
&\leq \psi\bigl(G\bigl(z_1, T^m z, Tz_1\bigr)\bigr) - \varphi\bigl(G\bigl(z_1, T^m z, Tz_1\bigr)\bigr) \\
&= \psi\bigl(G\bigl(z_1, z_1, T^m z\bigr)\bigr) - \varphi\bigl(G\bigl(z_1, z_1, T^m z\bigr)\bigr).
\end{aligned}$$

By Lemma 10, we deduce $\{G(z_1, z_1, T^m z)\} \to 0$, that is, $\{T^m z\} \xrightarrow{G} z_1$. The same reasoning proves that $\{T^m z\} \xrightarrow{G} z_2$, so $z_1 = z_2$. $\square$

We particularize the previous theorem in two cases. If take $\psi(t) = t$ in Theorem 26, then we get the following results.

Corollary 27 *Let $(X, \preccurlyeq)$ be a preordered set endowed with a G^*-metric G and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold*:

(a) *(X, G) is complete.*
(b) *T is non-decreasing (w.r.t. $\preccurlyeq$).*
(c) *Either T is G-continuous or $(X, G, \preccurlyeq)$ is regular non-decreasing.*
(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*
(e) *There exists a mapping $\varphi \in \Psi$ such that, for all $x, y \in X$ with $x \preccurlyeq y$,*

$$G\bigl(Tx, Ty, T^2 x\bigr) \le G(x, y, Tx) - \varphi\bigl(G(x, y, Tx)\bigr).$$

Then T has a fixed point. Furthermore, if for all $z_1, z_2 \in X$ fixed points of T there exists $z \in X$ such that $z_1 \preccurlyeq z$ and $z_2 \preccurlyeq z$, we obtain uniqueness of the fixed point.

If take $\varphi(t) = (1 - k)t$ with $k \in [0, 1)$ in Corollary 27, then we derive the following result.

Corollary 28 *Let $(X, \preccurlyeq)$ be a preordered set endowed with a G^*-metric G and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold*:

(a) *(X, G) is complete.*
(b) *T is non-decreasing (w.r.t. $\preccurlyeq$).*
(c) *Either T is G-continuous or $(X, G, \preccurlyeq)$ is regular non-decreasing.*
(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*
(e) *There exists a constant $k \in [0, 1)$ such that, for all $x, y \in X$ with $x \preccurlyeq y$,*

$$G\bigl(Tx, Ty, T^2 x\bigr) \le kG(x, y, Tx).$$

Then T has a fixed point. Furthermore, if for all $z_1, z_2 \in X$ fixed points of T there exists $z \in X$ such that $z_1 \preccurlyeq z$ and $z_2 \preccurlyeq z$, we obtain uniqueness of the fixed point.

6 Multidimensional Υ-fixed point results in partially preordered G^*-metric spaces

In this section we extend Theorem 26 to an arbitrary number of variables. To do that, it is necessary to introduce the following notation. Given a mapping $F : X^n \to X$, we define $\mathbb{F}_\Upsilon : X^n \to X^n$ by

$$\begin{aligned}&\mathbb{F}_\Upsilon(x_1, x_2, \ldots, x_n)\\ &\quad= \bigl(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})\bigr),\end{aligned}$$

and $F^2_\Upsilon = F \circ \mathbb{F}_\Upsilon : X^n \to X$ will be

$$\begin{aligned}&F^2_\Upsilon(x_1, x_2, \ldots, x_n)\\ &\quad= F\bigl(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})\bigr)\end{aligned}$$

for all $\mathsf{X} = (x_1, x_2, \ldots, x_n) \in X^n$.

Lemma 29

1. $\mathsf{Z} \in X^n$ *is a* Υ*-fixed point of* F *if and only if* Z *is a fixed point of* $\mathbb{F}_\Upsilon$ *(that is,* $\mathbb{F}_\Upsilon \mathsf{Z} = \mathsf{Z}$*).*
2. *If* F *has the mixed monotone property, then* $\mathbb{F}_\Upsilon$ *is* $\sqsubseteq$*-monotone non-decreasing on* X^n*.*
3. *If* (X, G) *is a* G^**-metric space and* F *is* G*-continuous, then* $\mathbb{F}_\Upsilon : X^n \to X^n$ *is* G_n*-continuous and* $F^2_\Upsilon = F \circ \mathbb{F}_\Upsilon : X^n \to X$ *is* G*-continuous.*

6.1 A first multidimensional contractivity result

In this subsection we apply Theorem 26 considering $T = \mathbb{F}_\Upsilon$ defined on $(X^n, G_n, \sqsubseteq)$. In order to do that, we notice that joining some of the previous results, we obtain the following consequences.

- If (X, G) is complete, it follows from Corollary 25 that (X^n, G_n) is also complete.
- By item 2 of Lemma 29, if F has the mixed monotone property, then $\mathbb{F}_\Upsilon$ is $\sqsubseteq$-monotone non-decreasing on X^n.
- By item 3 of Lemma 29, if F is G-continuous, then $\mathbb{F}_\Upsilon : X^n \to X^n$ is G_n-continuous and $F^2_\Upsilon = F \circ \mathbb{F}_\Upsilon : X^n \to X$ is G-continuous.
- If $(X, G, \preccurlyeq)$ is regular, it follows from Corollary 25 that $(X^n, G_n, \sqsubseteq)$ is also regular.
- If $x_0^1, x_0^2, \ldots, x_0^n \in X$ are such that $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i, then $\mathsf{X}_0 = (x_0^1, x_0^2, \ldots, x_0^n) \in X^n$ verifies $\mathsf{X}_0 \sqsubseteq \mathbb{F}_\Upsilon(\mathsf{X}_0)$.

We study how the contractivity condition

$$\psi\bigl(G_n\bigl(\mathbb{F}_\Upsilon \mathsf{X}, \mathbb{F}_\Upsilon \mathsf{Y}, \mathbb{F}^2_\Upsilon \mathsf{X}\bigr)\bigr) \le (\psi - \varphi)\bigl(G_n(\mathsf{X}, \mathsf{Y}, \mathbb{F}_\Upsilon \mathsf{X})\bigr) \quad \text{for all } \mathsf{X}, \mathsf{Y} \in X^n \text{ such that } \mathsf{X} \sqsubseteq \mathsf{Y}$$

may be equivalently established. Let $\mathsf{X} = (x_1, x_2, \ldots, x_n) \in X^n$ and let $z_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) \in X$ for all i. Then

$$\begin{aligned}
\mathbb{F}^2_\Upsilon \mathsf{X} &= \mathbb{F}_\Upsilon\bigl(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \ldots, \\
&\qquad F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})\bigr) \\
&= \mathbb{F}_\Upsilon(z_1, z_2, \ldots, z_n) \\
&= \bigl(F(z_{\sigma_1(1)}, z_{\sigma_1(2)}, \ldots, z_{\sigma_1(n)}), F(z_{\sigma_2(1)}, z_{\sigma_2(2)}, \ldots, z_{\sigma_2(n)}), \ldots, F(z_{\sigma_n(1)}, z_{\sigma_n(2)}, \ldots, z_{\sigma_n(n)})\bigr) \\
&= \bigl(F\bigl(F(x_{\sigma_{\sigma_1(1)}(1)}, \ldots, x_{\sigma_{\sigma_1(1)}(n)}), F(x_{\sigma_{\sigma_1(2)}(1)}, \ldots, x_{\sigma_{\sigma_1(2)}(n)}), \ldots, F(x_{\sigma_{\sigma_1(n)}(1)}, \ldots, x_{\sigma_{\sigma_1(n)}(n)})\bigr), \\
&F\bigl(F(x_{\sigma_{\sigma_2(1)}(1)}, \ldots, x_{\sigma_{\sigma_2(1)}(n)}), F(x_{\sigma_{\sigma_2(2)}(1)}, \ldots, x_{\sigma_{\sigma_2(2)}(n)}), \ldots, F(x_{\sigma_{\sigma_2(n)}(1)}, \ldots, x_{\sigma_{\sigma_n(n)}(n)})\bigr), \ldots, \\
&F\bigl(F(x_{\sigma_{\sigma_n(1)}(1)}, \ldots, x_{\sigma_{\sigma_n(1)}(n)}), F(x_{\sigma_{\sigma_n(2)}(1)}, \ldots, x_{\sigma_{\sigma_n(2)}(n)}), \ldots, F(x_{\sigma_{\sigma_n(n)}(1)}, \ldots, x_{\sigma_{\sigma_n(n)}(n)})\bigr)\bigr) \\
&= \bigl(F^2_\Upsilon(x_{\sigma_1(1)}, x_{\sigma_1(2)} \ldots, x_{\sigma_1(n)}), F^2_\Upsilon(x_{\sigma_2(1)}, x_{\sigma_2(2)} \ldots, x_{\sigma_2(n)}), \ldots, \\
&\qquad F^2_\Upsilon(x_{\sigma_n(1)}, x_{\sigma_n(2)} \ldots, x_{\sigma_n(n)})\bigr).
\end{aligned}$$

It follows that

$$\begin{aligned}
&G_n(\mathsf{X}, \mathsf{Y}, \mathbb{F}_\Upsilon \mathsf{X}) = \max_{1 \le i \le n} G\bigl(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})\bigr) \quad \text{and} \\
&G_n\bigl(\mathbb{F}_\Upsilon \mathsf{X}, \mathbb{F}_\Upsilon \mathsf{Y}, \mathbb{F}^2_\Upsilon \mathsf{X}\bigr) = \max_{1 \le i \le n} G\bigl(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)}), \\
&\qquad\qquad F^2_\Upsilon(x_{\sigma_i(1)}, x_{\sigma_i(2)} \ldots, x_{\sigma_i(n)})\bigr).
\end{aligned}$$

Therefore, a possible version of Theorem 26 applied to $(X^n, G_n, \sqsubseteq)$ taking $T = \mathbb{F}_\Upsilon$ is the following.

Theorem 30 *Let (X, G) be a complete G^*-metric space and let $\preccurlyeq$ be a partial preorder on X. Let $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be an n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself verifying $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$. Let $F : X^n \to X$ be a mapping verifying the mixed monotone property on X. Assume that there exist $\psi, \varphi \in \Psi$ such that*

$$\begin{aligned} &\max_{1\le i\le n} \psi\big(G\big(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)}), F^2_\Upsilon(x_{\sigma_i(1)}, x_{\sigma_i(2)} \ldots, x_{\sigma_i(n)})\big)\big) \\ &\quad \le (\psi - \varphi)\Big(\max_{1\le i\le n} G\big(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})\big)\Big) \end{aligned} \tag{6}$$

for which $x_i \preccurlyeq_i y_i$ for all i. Suppose either F is continuous or $(X, G, \preccurlyeq)$ is regular. If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i, then F has, at least, one Υ-fixed point.

6.2 A second multidimensional contractivity result

In this section we introduce a slightly different contractivity condition that cannot be directly deduced applying Theorem 26 to $(X, G_n, \sqsubseteq)$ taking $T = \mathbb{F}_\Upsilon$, because the contractivity condition is weaker. Then we need to show a classical proof.

Theorem 31 *Let (X, G) be a complete G^*-metric space and let $\preccurlyeq$ be a partial preorder on X. Let $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be an n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself verifying $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$. Let $F : X^n \to X$ be a mapping verifying the mixed monotone property on X. Assume that there exist $\psi, \varphi \in \Psi$ such that*

$$\begin{aligned} &\psi\big(G\big(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F^2_\Upsilon(x_1, x_2, \ldots, x_n)\big)\big) \\ &\quad \le (\psi - \varphi)\Big(\max_{1\le i\le n} G\big(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})\big)\Big) \end{aligned} \tag{7}$$

for which $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are $\sqsubseteq$-comparable. Suppose either F is continuous or $(X, G, \preccurlyeq)$ is regular. If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i, then F has, at least, one Υ-fixed point.

Notice that (6) and (7) are very different contractivity conditions. For instance, (6) would be simpler if the image of all σ_i are sets with a few points.

Proof Define $\mathsf{X}_0 = (x_0^1, x_0^2, \ldots, x_0^n)$ and let $x_1^i = F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i. If $\mathsf{X}_1 = (x_1^1, x_1^2, \ldots, x_1^n)$, then $x_0^i \preccurlyeq_i x_1^i$ for all i is equivalent to $\mathsf{X}_0 \sqsubseteq \mathsf{X}_1 = \mathbb{F}_\Upsilon(\mathsf{X}_0)$. By recurrence, define $x_{m+1}^i = F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})$ for all i and all m, and we have that $\mathsf{X}_m \sqsubseteq \mathsf{X}_{m+1} = \mathbb{F}_\Upsilon(\mathsf{X}_m)$. This means that the sequence $\{\mathsf{X}_{m+1} = \mathbb{F}_\Upsilon(\mathsf{X}_m)\}$ is $\sqsubseteq$-monotone non-decreasing. Since $(X^n, G_n, \sqsubseteq)$ is complete, it is only necessary to prove that $\{\mathsf{X}_m\}$ is G_n-Cauchy in order to deduce that it is G_n-convergent. By item 3 of Lemma 24, it will be sufficient to prove that each sequence $\{x_m^i\}$ is G-Cauchy. Firstly, notice that $\mathsf{X}_{m+1} = \mathbb{F}_\Upsilon(\mathsf{X}_m)$ means that

$$x_{m+1}^i = F\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big) \quad \text{for all } i \text{ and all } m.$$

Hence

$$\begin{aligned} x_{m+2}^{i} &= F\big(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}\big) \\ &= F\big(F\big(x_m^{\sigma_{\sigma_i(1)}(1)}, x_m^{\sigma_{\sigma_i(1)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(1)}(n)}\big), F\big(x_m^{\sigma_{\sigma_i(2)}(1)}, x_m^{\sigma_{\sigma_i(2)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(2)}(n)}\big), \ldots, \\ &\quad F\big(x_m^{\sigma_{\sigma_i(n)}(1)}, x_m^{\sigma_{\sigma_i(n)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(n)}(n)}\big)\big) = F_\Upsilon^2\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big). \end{aligned}$$

Furthermore, for all m,

$$\begin{aligned} F_\Upsilon^2(\mathsf{X}_m) &= F_\Upsilon^2\big(x_m^1, x_m^2, \ldots, x_m^n\big) \\ &= F\big(F\big(x_m^{\sigma_1(1)}, x_m^{\sigma_1(2)}, \ldots, x_m^{\sigma_1(n)}\big), F\big(x_m^{\sigma_2(1)}, x_m^{\sigma_2(2)}, \ldots, x_m^{\sigma_2(n)}\big), \ldots, \\ &\quad F\big(x_m^{\sigma_n(1)}, x_m^{\sigma_n(2)}, \ldots, x_m^{\sigma_n(n)}\big)\big) \\ &= F\big(x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^n\big) = F(\mathsf{X}_{m+1}). \end{aligned} \tag{8}$$

Therefore, for all i and all m,

$$\begin{aligned} &\psi\big(G\big(x_{m+1}^i, x_{m+2}^i, x_{m+2}^i\big)\big) \\ &\quad = \psi\big(G\big(F\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big), F\big(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}\big), F_\Upsilon^2\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big)\big) \\ &\quad \le (\psi - \varphi)\Big(\max_{1\le j\le n} G\big(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, F\big(x_m^{\sigma_{\sigma_i(j)}(1)}, x_m^{\sigma_{\sigma_i(j)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(j)}(n)}\big)\big)\Big) \\ &\quad = (\psi - \varphi)\Big(\max_{1\le j\le n} G\big(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}\big)\Big). \end{aligned}$$

Since ψ is non-decreasing, for all i and all m,

$$\psi\Big(\max_{1\le j\le n} G\big(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}\big)\Big) \le \psi\Big(\max_{1\le j\le n} G\big(x_m^j, x_{m+1}^j, x_{m+1}^j\big)\Big).$$

Applying Lemma 8 using

$$a_m^i = G\big(x_m^i, x_{m+1}^i, x_{m+1}^i\big) \quad \text{and} \quad b_m^i = \max_{1\le j\le n} G\big(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}\big)$$

for all i and all m, we deduce that

$$\big\{G\big(x_m^i, x_{m+1}^i, x_{m+1}^i\big)\big\} \to 0 \quad \text{for all } i, \quad \text{that is,} \quad \big\{G_n(\mathsf{X}_m, \mathsf{X}_{m+1}, \mathsf{X}_{m+1})\big\} \to 0. \tag{9}$$

Next, we prove that every sequence $\{x_m^i\}$ is G-Cauchy reasoning by contradiction. Suppose that $\{x_m^{i_1}\}_{m\ge0}, \ldots, \{x_m^{i_s}\}_{m\ge0}$ are not G-Cauchy ($s \ge 1$) and $\{x_m^{i_{s+1}}\}_{m\ge0}, \ldots, \{x_m^{i_n}\}_{m\ge0}$ are G-Cauchy, being $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. From Proposition 2, for all $r \in \{1, 2, \ldots, s\}$, there exist $\varepsilon_r > 0$ and subsequences $\{x_{n_r(k)}^{i_r}\}_{k\in\mathbb{N}}$ and $\{x_{m_r(k)}^{i_r}\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &k < n_r(k) < m_r(k) < n_r(k+1), \qquad G\big(x_{n_r(k)}^{i_r}, x_{n_r(k)+1}^{i_r}, x_{m_r(k)}^{i_r}\big) \ge \varepsilon_r, \\ &G\big(x_{n_r(k)}^{i_r}, x_{n_r(k)+1}^{i_r}, x_{m_r(k)-1}^{i_r}\big) < \varepsilon_r. \end{aligned}$$

Now, let $\varepsilon_0 = \max(\varepsilon_1, \ldots, \varepsilon_s) > 0$ and $\varepsilon'_0 = \min(\varepsilon_1, \ldots, \varepsilon_s) > 0$. Since $\{x_m^{i_{s+1}}\}_{m\geq 0}, \ldots, \{x_m^{i_n}\}_{m\geq 0}$ are G-Cauchy, for all $j \in \{i_{s+1}, \ldots, i_n\}$, there exists $n^j \in \mathbb{N}$ such that if $m, m' \geq n^j$, then $G(x_m^j, x_{m+1}^j, x_{m'}^j) < \varepsilon'_0/8$. Define $n_0 = \max_{j\in\{i_{s+1},\ldots,i_n\}}(n^j)$. Therefore, we have proved that there exists $n_0 \in \mathbb{N}$ such that if $m, m' \geq n_0$ then

$$G\big(x_m^j, x_{m+1}^j, x_{m'}^j\big) < \varepsilon'_0/4 \quad \text{for all } j \in \{i_{s+1}, \ldots, i_n\}. \tag{10}$$

Next, let $q \in \{1, 2, \ldots, s\}$ be such that $\varepsilon_q = \varepsilon_0 = \max(\varepsilon_1, \ldots, \varepsilon_s)$. Let $k_1 \in \mathbb{N}$ be such that $n_0 < n_q(k_1)$ and define $n(1) = n_q(k_1)$. Consider the numbers $n(1)+1, n(1)+2, \ldots, m_q(k_1)$ until finding the first positive integer $m(1) > n(1)$ verifying

$$\max_{1\leq r\leq s} G\big(x_{n(1)}^{i_r}, x_{n(1)+1}^{i_r}, x_{m(1)}^{i_r}\big) \geq \varepsilon_0, \qquad G\big(x_{n(1)}^{i_j}, x_{n(1)+1}^{i_j}, x_{m(1)-1}^{i_j}\big) < \varepsilon_0 \quad \text{for all } j \in \{1, 2, \ldots, s\}.$$

Now let $k_2 \in \mathbb{N}$ be such that $m(1) < n_q(k_2)$ and define $n(2) = n_q(k_2)$. Consider the numbers $n(2)+1, n(2)+2, \ldots, m_q(k_2)$ until finding the first positive integer $m(2) > n(2)$ verifying

$$\begin{aligned} &\max_{1\leq r\leq s} G\big(x_{n(2)}^{i_r}, x_{n(2)+1}^{i_r}, x_{m(2)}^{i_r}\big) \geq \varepsilon_0, \\ &G\big(x_{n(2)}^{i_j}, x_{n(2)+1}^{i_j}, x_{m(2)-1}^{i_j}\big) < \varepsilon_0 \quad \text{for all } j \in \{1, 2, \ldots, s\}. \end{aligned}$$

Repeating this process, we can find sequences such that, for all $k \geq 1$,

$$\begin{aligned} &n_0 < n(k) < m(k) < n(k+1), \qquad \max_{1\leq r\leq s} G\big(x_{n(k)}^{i_r}, x_{n(k)+1}^{i_r}, x_{m(k)}^{i_r}\big) \geq \varepsilon_0, \\ &G\big(x_{n(k)}^{i_j}, x_{n(k)+1}^{i_j}, x_{m(k)-1}^{i_j}\big) < \varepsilon_0 \quad \text{for all } j \in \{1, 2, \ldots, s\}. \end{aligned}$$

Note that by (10), $G(x_{n(k)}^{i_r}, x_{n(k)+1}^{i_r}, x_{m(k)}^{i_r}), G(x_{n(k)}^{i_r}, x_{n(k)+1}^{i_r}, x_{m(k)-1}^{i_r}) < \varepsilon'_0/4 < \varepsilon_0/2$ for all $r \in \{s+1, s+2, \ldots, n\}$, so

$$\begin{aligned} &\max_{1\leq j\leq n} G\big(x_{n(k)}^j, x_{n(k)+1}^j, x_{m(k)}^j\big) = \max_{1\leq r\leq s} G\big(x_{n(k)}^{i_r}, x_{n(k)+1}^{i_r}, x_{m(k)}^{i_r}\big) \geq \varepsilon_0 \quad \text{and} \\ &G\big(x_{n(k)}^i, x_{n(k)+1}^i, gx_{m(k)-1}^i\big) < \varepsilon_0 \end{aligned} \tag{11}$$

for all $i \in \{1, 2, \ldots, n\}$ and all $k \geq 1$. Next, for all k, let $i(k) \in \{1, 2, \ldots, s\}$ be an index such that

$$G\big(x_{n(k)}^{i(k)}, x_{n(k)+1}^{i(k)}, x_{m(k)}^{i(k)}\big) = \max_{1\leq r\leq s} G\big(x_{n(k)}^{i_r}, x_{n(k)+1}^{i_r}, x_{m(k)}^{i_r}\big) = \max_{1\leq j\leq n} G\big(x_{n(k)}^j, x_{n(k)+1}^j, x_{m(k)}^j\big) \geq \varepsilon_0.$$

Notice that, applying (G_5) twice and (11), for all k and all j,

$$\begin{aligned} G\big(x_{n(k)-1}^j, x_{n(k)}^j, x_{m(k)-1}^j\big) &\leq G\big(x_{n(k)-1}^j, x_{n(k)}^j, x_{n(k)}^j\big) + G\big(x_{n(k)}^j, x_{n(k)}^j, x_{m(k)-1}^j\big) \\ &\leq G\big(x_{n(k)-1}^j, x_{n(k)}^j, x_{n(k)}^j\big) + G\big(x_{n(k)}^j, x_{n(k)+1}^j, x_{n(k)+1}^j\big) \\ &\quad + G\big(x_{n(k)+1}^j, x_{n(k)}^j, x_{m(k)-1}^j\big) \\ &\leq G\big(x_{n(k)-1}^j, x_{n(k)}^j, x_{n(k)}^j\big) + G\big(x_{n(k)}^j, x_{n(k)+1}^j, x_{n(k)+1}^j\big) + \varepsilon_0. \end{aligned} \tag{12}$$

Applying Proposition 2 to guarantee that the following points are $\sqsubseteq$-comparable, the contractivity condition (7) assures us for all k

$$\begin{aligned}
0 < \psi(\varepsilon_0) &\leq \psi\left(G\left(x_{n(k)}^{i(k)}, x_{n(k)+1}^{i(k)}, x_{m(k)}^{i(k)}\right)\right) = \psi\left(G\left(x_{n(k)}^{i(k)}, x_{m(k)}^{i(k)}, x_{n(k)+1}^{i(k)}\right)\right) \\
&= \psi\Big(G\Big(F\left(x_{n(k)-1}^{\sigma_{i(k)}(1)}, x_{n(k)-1}^{\sigma_{i(k)}(2)}, \ldots, x_{n(k)-1}^{\sigma_{i(k)}(n)}\right), F\left(x_{m(k)-1}^{\sigma_{i(k)}(1)}, x_{m(k)-1}^{\sigma_{i(k)}(2)}, \ldots, x_{m(k)-1}^{\sigma_{i(k)}(n)}\right), \\
&\quad F_{\Upsilon}^{2}\left(x_{n(k)-1}^{\sigma_{i(k)}(1)}, x_{n(k)-1}^{\sigma_{i(k)}(2)}, \ldots, x_{n(k)-1}^{\sigma_{i(k)}(n)}\right)\Big)\Big) \\
&\leq (\psi - \varphi)\left(\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}, F\left(x_{n(k)-1}^{\sigma_{i(k)}(1)}, x_{n(k)-1}^{\sigma_{i(k)}(2)}, \ldots, x_{n(k)-1}^{\sigma_{i(k)}(n)}\right)\right)\right) \\
&= (\psi - \varphi)\left(\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}\right)\right) \\
&= (\psi - \varphi)\left(\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}\right)\right).
\end{aligned} \tag{13}$$

Consider the sequence

$$\left\{\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}\right)\right\}_{k\geq 1}. \tag{14}$$

If this sequence has a subsequence that converges to zero, then we can take limit when $k \to \infty$ in (13) using this subsequence, so that we would have $0 < \psi(\varepsilon_0) \leq \psi(0) - \varphi(0) = 0$, which is impossible since $\varepsilon_0 > 0$. Therefore, the sequence (14) has no subsequence converging to zero. In this case, taking $\varepsilon_0 > 0$ in Lemma 6, there exist $\delta \in]0, \varepsilon_0[$ and $k_0 \in \mathbb{N}$ such that $\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}) \geq \delta$ for all $k \geq k_0$. It follows that, for all $k \geq k_0$, $-\varphi(\max_{1\leq j\leq n} G(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)})) \leq -\varphi(\delta)$. Thus, by (13) and (12),

$$\begin{aligned}
0 < \psi(\varepsilon_0) &\leq \psi\left(\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}\right)\right) - \varphi\left(\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}\right)\right) \\
&\leq \psi\left(\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{\sigma_{i(k)}(j)}, x_{n(k)}^{\sigma_{i(k)}(j)}, x_{m(k)-1}^{\sigma_{i(k)}(j)}\right)\right) - \varphi(\delta) \\
&\leq \psi\left(\max_{1\leq j\leq n} G\left(x_{n(k)-1}^{j}, x_{n(k)}^{j}, x_{m(k)-1}^{j}\right)\right) - \varphi(\delta) \\
&\leq \psi\left(\max_{1\leq j\leq n}\left(G\left(x_{n(k)-1}^{j}, x_{n(k)}^{j}, x_{n(k)}^{j}\right) + G\left(x_{n(k)}^{j}, x_{n(k)+1}^{j}, x_{n(k)+1}^{j}\right)\right) + \varepsilon_0\right) - \varphi(\delta).
\end{aligned} \tag{15}$$

Taking limit in (15) as $k \to \infty$ and taking into account (9), we deduce that $0 < \psi(\varepsilon_0) \leq \psi(\varepsilon_0) - \varphi(\delta)$, which is impossible. The previous reasoning proves that every sequence $\{x_m^i\}$ is G-Cauchy.

Corollary 25 guarantees that the sequence $\{\mathbb{F}_{\Upsilon}^{m}(\mathsf{X}_0) = \mathsf{X}_m = (x_m^1, x_m^2, \ldots, x_m^n)\}$ is G_n-Cauchy. Since (X^n, G_n) is complete (again by Corollary 25), there exists $\mathsf{Z} \in X^n$ such that $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$, that is, if $\mathsf{Z} = (z_1, z_2, \ldots, z_n)$ then

$$\left\{G\left(x_m^i, x_{m+1}^i, z_i\right)\right\} \to 0 \quad \text{for all } i. \tag{16}$$

Suppose that F is G-continuous. In this case, item 3 of Lemma 29 implies that $\mathbb{F}_{\Upsilon}$ is G_n-continuous, so $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$ and $\{\mathsf{X}_{m+1} = \mathbb{F}_{\Upsilon}(\mathsf{X}_m)\} \xrightarrow{G_n} \mathbb{F}_{\Upsilon}(\mathsf{Z})$. By the unicity of the G_n-limit, $\mathbb{F}_{\Upsilon}(\mathsf{Z}) = \mathsf{Z}$, which means that Z is a Υ-fixed point of F.

Suppose that $(X, G, \preccurlyeq)$ *is regular*. In this case, by Corollary 25, $(X^n, G_n, \sqsubseteq)$ is also regular. Then, taking into account that $\{\mathsf{X}_m = \mathbb{F}_\Upsilon^m(\mathsf{X}_0)\}$ is a $\sqsubseteq$-monotone non-decreasing sequence such that $\{\mathsf{X}_m\} \xrightarrow{G_n} \mathsf{Z}$, we deduce that $\mathsf{X}_m \sqsubseteq \mathsf{Z}$ for all m. From Proposition 2, since $(x_m^1, x_m^2, \ldots, x_m^n) = \mathsf{X}_m \sqsubseteq \mathsf{Z} = (z_1, z_2, \ldots, z_n)$, then $(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})$ and $(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)})$ are $\sqsubseteq$-comparable for all i and all m. Notice that for all i and all m,

$$\begin{aligned}F\big(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}\big) &= F\big(F\big(x_m^{\sigma_{\sigma_i(1)}(1)}, x_m^{\sigma_{\sigma_i(1)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(1)}(n)}\big), \ldots, \\ &\quad F\big(x_m^{\sigma_{\sigma_i(n)}(1)}, x_m^{\sigma_{\sigma_i(n)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(n)}(n)}\big)\big) \\ &= F_\Upsilon^2\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big).\end{aligned}$$

It follows from condition (7) and (8) that, for all i,

$$\begin{aligned}&\psi\big(G\big(F\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big), F\big(x_{m+1}^{\sigma_i(1)}, x_{m+1}^{\sigma_i(2)}, \ldots, x_{m+1}^{\sigma_i(n)}\big), F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)})\big)\big) \\ &\quad = \psi\big(G\big(F\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big), F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)}), F_\Upsilon^2\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big)\big)\big) \\ &\quad \le (\psi - \varphi)\Big(\max_{1\le j\le n} G\big(x_m^{\sigma_i(j)}, z_{\sigma_i(j)}, F\big(x_m^{\sigma_{\sigma_i(j)}(1)}, x_m^{\sigma_{\sigma_i(j)}(2)}, \ldots, x_m^{\sigma_{\sigma_i(j)}(n)}\big)\big)\Big) \\ &\quad = (\psi - \varphi)\Big(\max_{1\le j\le n} G\big(x_m^{\sigma_i(j)}, z_{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}\big)\Big) \le \psi\Big(\max_{1\le j\le n} G\big(x_m^{\sigma_i(j)}, x_{m+1}^{\sigma_i(j)}, z_{\sigma_i(j)}\big)\Big) \\ &\quad \le \psi\Big(\max_{1\le j\le n} G\big(x_m^j, x_{m+1}^j, z_j\big)\Big).\end{aligned}$$

By (16) we deduce that

$$\big\{F\big(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\big)\big\} \to F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)}) \quad \text{for all } i,$$

which means that

$$\begin{aligned}&\big\{\mathbb{F}_\Upsilon \mathsf{X}_m = \big(F\big(x_m^{\sigma_1(1)}, x_m^{\sigma_1(2)}, \ldots, x_m^{\sigma_1(n)}\big), \ldots, F\big(x_m^{\sigma_n(1)}, x_m^{\sigma_n(2)}, \ldots, x_m^{\sigma_n(n)}\big)\big)\big\} \\ &\quad \xrightarrow{G_n} \big(F(z_{\sigma_1(1)}, z_{\sigma_1(2)}, \ldots, z_{\sigma_1(n)}), \ldots, F(z_{\sigma_n(1)}, z_{\sigma_n(2)}, \ldots, z_{\sigma_n(n)})\big) = \mathbb{F}_\Upsilon \mathsf{Z}.\end{aligned}$$

Since $\{\mathbb{F}_\Upsilon \mathsf{X}_m = \mathsf{X}_{m+1}\} \xrightarrow{G_n} \mathsf{Z}$, we conclude that $\mathbb{F}_\Upsilon \mathsf{Z} = \mathsf{Z}$, that is, Z is a Υ-fixed point of F. □

If we take $\psi(t) = t$ in Theorem 26, then we get the following results.

Corollary 32 *Let* (X, G) *be a complete* G^**-metric space and let* $\preccurlyeq$ *be a partial preorder on* X. *Let* $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *be an* n*-tuple of mappings from* $\{1, 2, \ldots, n\}$ *into itself verifying* $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{A}$ *and* $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{B}$. *Let* $F : X^n \to X$ *be a mapping verifying the mixed monotone property on* X. *Assume that there exists* $\varphi \in \Psi$ *such that*

$$\begin{aligned}&G\big(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F_\Upsilon^2(x_1, x_2, \ldots, x_n)\big) \\ &\quad \le \max_{1\le i\le n} G\big(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})\big) - \varphi\Big(\max_{1\le i\le n} G\big(x_i, y_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})\big)\Big)\end{aligned}$$

for which $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ *are* $\sqsubseteq$*-comparable. Suppose either* F *is continuous or* $(X, G, \preccurlyeq)$ *is regular. If there exist* $x_0^1, x_0^2, \ldots, x_0^n \in X$ *verifying* $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ *for all* i, *then* F *has, at least, one* Υ*-fixed point.*

If we take $\varphi(t) = (1-k)t$ for all $t \geq 0$, with $k \in [0,1)$, in Corollary 32, then we derive the following result.

Corollary 33 *Let (X,G) be a complete G^*-metric space and let $\preccurlyeq$ be a partial preorder on X. Let $\Upsilon = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ be an n-tuple of mappings from $\{1,2,\ldots,n\}$ into itself verifying $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$. Let $F : X^n \to X$ be a mapping verifying the mixed monotone property on X. Assume that there exists $k \in [0,1)$ such that*

$$\begin{aligned} &G\big(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n),F^2_\Upsilon(x_1,x_2,\ldots,x_n)\big) \\ &\quad \leq k \max_{1\leq i\leq n} G\big(x_i,y_i,F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)})\big) \end{aligned} \tag{17}$$

for which $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n) \in X^n$ are $\sqsubseteq$-comparable. Suppose either F is continuous or $(X,G,\preccurlyeq)$ is regular. If there exist $x_0^1,x_0^2,\ldots,x_0^n \in X$ verifying $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$ for all i, then F has, at least, one Υ-fixed point.

Example 34 Let $X = \{0,1,2,3,4\}$ and let G be the G-metric on X given, for all $x,y,z \in X$, by $G(x,y,z) = \max(|x-y|,|x-z|,|y-z|)$. Then (X,G) is complete and G generates the discrete topology on X. Consider on X the following partial order:

$$x,y \in X, \qquad x \preccurlyeq y \quad \Leftrightarrow \quad x = y \quad \text{or} \quad (x,y) = (0,2).$$

Define $F : X^n \to X$ by

$$F(x_1,x_2,\ldots,x_n) = \begin{cases} 0, & \text{if } x_1,x_2,\ldots,x_n \in \{0,1,2\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then the following statements hold.

1. F is a G-continuous mapping.
2. If $y,z \in X$ verify $y \preccurlyeq z$, then either $y,z \in \{0,1,2\}$ or $y,z \in \{3,4\}$. In particular, $F(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) = F(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)$ and F has the mixed monotone property on X.
3. If $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n) \in X^n$ are $\sqsubseteq$-comparable, then $F(x_1,x_2,\ldots,x_n) = F(y_1,y_2,\ldots,y_n)$. In particular, (17) holds for $k = 1/2$.
   For simplicity, henceforth, suppose that n is even and let A (respectively, B) be the set of all odd (respectively, even) numbers in $\{1,2,\ldots,n\}$.
4. For a mapping $\sigma : \Lambda_n \to \Lambda_n$, we use the notation $\sigma \equiv (\sigma(1),\sigma(2),\ldots,\sigma(n))$ and consider

   $$\sigma_i \equiv (i,i+1,\ldots,n-1,n,1,2,\ldots,i-1) \quad \text{for all } i.$$

   Then $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if i is odd and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if i is even. Let $\Upsilon = (\sigma_1,\sigma_2,\ldots,\sigma_n)$.
5. Take $x_0^i = 0$ if i is odd and $x_0^i = 2$ if i is even. Then $x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$ for all i.

Therefore, we can apply Corollary 33 to conclude that F has, at least, one Υ-fixed point. To finish, we prove the previous statements.

If $\{x_m\} \xrightarrow{G} x$, then there exists $m_0 \in \mathbb{N}$ such that $|x_m - x| = G(x,x,x_m) < 1/2$ for all $m \geq m_0$. Since X is discrete, then $x_m = x$ for all $m \geq m_0$. This proves that τ_G is the discrete topology on X.

1. If $\{a_m^1\}, \{a_m^2\}, \ldots, \{a_m^n\} \subseteq X$ are n sequences such that $\{a_m^i\} \xrightarrow{G} a_i \in X$ for all i, then there exists $m_0 \in \mathbb{N}$ such that $a_m^i = a_i$ for all $m \geq m_0$ and all i. Then $\{F(a_m^1, a_m^2, \ldots, a_m^n)\} \xrightarrow{G} F(a_1, a_2, \ldots, a_n)$ and F is G-continuous.
2. If $y, z \in X$ verify $y \preccurlyeq z$, the either $y = z$ (in this case, there is nothing to prove) or $(y,z) = (0,2)$. Then either $y, z \in \{0,1,2\}$ or $y, z \in \{3,4\}$. In particular,

$$F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) = \left\{ \begin{array}{ll} 0 & \text{if } x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n \in \{0,1,2\}, \\ 1, & \text{otherwise} \end{array} \right\}$$
$$= F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

   Hence F has the mixed monotone property on X.
3. Suppose that $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are $\sqsubseteq$-comparable, and we claim that $F(x_1, x_2, \ldots, x_n) = F(y_1, y_2, \ldots, y_n)$. Indeed, assume, for instance, that $x_i \preccurlyeq_i y_i$ for all i. By item 2, for all i, either $x_i, y_i \in \{0,1,2\}$ or $x_i, y_i \in \{3,4\}$. Then

$$\begin{aligned} F(x_1, x_2, \ldots, x_n) &= \left\{ \begin{array}{ll} 0 & \text{if } x_1, x_2, \ldots, x_n \in \{0,1,2\}, \\ 1, & \text{otherwise} \end{array} \right\} \\ &= \left\{ \begin{array}{ll} 0 & \text{if } y_1, y_2, \ldots, y_n \in \{0,1,2\}, \\ 1, & \text{otherwise} \end{array} \right\} \\ &= F(y_1, y_2, \ldots, y_n). \end{aligned}$$

   If $x_i \succcurlyeq_i y_i$ for all i, the proof is similar. Next, we prove that (17) holds using $k = 1/4$. If $(x_1, x_2, \ldots, x_n) \in X^n$, then $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) \in \{0,1\} \subset \{0,1,2\}$. Therefore

$$\begin{aligned} &F_\Upsilon^2(x_1, x_2, \ldots, x_n) \\ &\quad = F\big(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \ldots, \\ &\quad\quad F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})\big) \\ &\quad = 0. \end{aligned}$$

   Suppose that $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are $\sqsubseteq$-comparable. It follows that

$$\begin{aligned} &G\big(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n), F_\Upsilon^2(x_1, x_2, \ldots, x_n)\big) \\ &\quad = \max\big(\big|F(x_1, x_2, \ldots, x_n) - F(y_1, y_2, \ldots, y_n)\big|, \\ &\quad\quad \big|F(x_1, x_2, \ldots, x_n) - 0\big|, \big|F(y_1, y_2, \ldots, y_n) - 0\big|\big) \\ &\quad = \max\big(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)\big) \\ &\quad = \begin{cases} 0 & \text{if } F(x_1, x_2, \ldots, x_n) = F(y_1, y_2, \ldots, y_n) = 0, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

It is clear that (17) holds if the previous number is 0. On the contrary, suppose that

$$G\big(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n),F_\Upsilon^2(x_1,x_2,\ldots,x_n)\big)=1.$$

Then $F(x_1,x_2,\ldots,x_n)=1$ or $F(y_1,y_2,\ldots,y_n)=1$ (both cases are similar). Assume, for instance, that $F(x_1,x_2,\ldots,x_n)=1$. Then there exists $i_0\in\{1,2,\ldots,n\}$ such that $x_{i_0}\in\{3,4\}$. In particular

$$\big|x_{i_0}-F(x_{\sigma_{i_0}(1)},x_{\sigma_{i_0}(2)},\ldots,x_{\sigma_{i_0}(n)})\big|\geq 3-1=2.$$

Therefore

$$\begin{aligned}\max_{1\leq i\leq n}G\big(x_i,y_i,F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)})\big)&\geq G\big(x_{i_0},y_{i_0},F(x_{\sigma_{i_0}(1)},x_{\sigma_{i_0}(2)},\ldots,x_{\sigma_{i_0}(n)})\big)\\&\geq\big|x_{i_0}-F(x_{\sigma_{i_0}(1)},x_{\sigma_{i_0}(2)},\ldots,x_{\sigma_{i_0}(n)})\big|\geq 2.\end{aligned}$$

This means that

$$\begin{aligned}&G\big(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n),F_\Upsilon^2(x_1,x_2,\ldots,x_n)\big)\\&\quad=1=\frac{1}{2}2\leq\frac{1}{2}\max_{1\leq i\leq n}G\big(x_i,y_i,F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)})\big).\end{aligned}$$

Therefore, in this case, (17) also holds.

4. It is evident.
5. Since $x_0^i\in\{0,1,2\}$ for all i, then $F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})=0$ for all i. If i is odd, then $x_0^i=0\preccurlyeq_i 0=F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$. If i is even, then $x_0^i=2\succcurlyeq 0=F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$, so $x_0^i\preccurlyeq_i F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)})$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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