Wangkeeree and Preechasilp *Fixed Point Theory and Applications* 2013, **2013**:160
http://www.fixedpointtheoryandapplications.com/content/2013/1/160

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Viscosity approximation methods for nonexpansive semigroups in CAT(0) spaces

Rabian Wangkeeree[1,2*] and Pakkapon Preechasilp[1]

*Correspondence: rabianw@nu.ac.th
[1]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand
[2]Centre of Excellence in Mathematics (CHE), Si Ayutthaya Road, Bangkok, 10400, Thailand

Abstract

In this paper, we study the strong convergence of Moudafi's viscosity approximation methods for approximating a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings in CAT(0) spaces. We prove that the proposed iterative scheme converges strongly to a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings which is also a unique solution of the variational inequality. The results presented in this paper extend and enrich the existing literature.

Keywords: viscosity approximation method; nonexpansive semigroup; variational inequality; CAT(0) space; common fixed point

1 Introduction

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or metric) *segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}_2}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for all $i,j \in \{1,2,3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x,y \in \triangle$ and all comparison points $\bar{x},\bar{y} \in \bar{\triangle}$,

$$d(x,y) \le d_{\mathbb{E}^2}(\bar{x},\bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{1.1}$$

This is the (CN)-inequality of Bruhat and Tits [1]. In fact (*cf.* [2], p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN)-inequality.

It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces, $\mathbb{R}$-trees (see [2]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [4]), and many others. Complete CAT(0) spaces are often called Hadamard spaces.

It is proved in [2] that a normed linear space satisfies the (CN)-inequality if and only if it satisfies the parallelogram identity, *i.e.*, is a pre-Hilbert space; hence it is not so unusual to have an inner product-like notion in Hadamard spaces. Berg and Nikolaev [5] introduced the concept of quasilinearization as follows.

Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then *quasilinearization* is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\bigl(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\bigr) \quad (a, b, c, d \in X). \tag{1.2}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \tag{1.3}$$

for all $a, b, c, d \in X$. It is known [5, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

In 2010, Kakavandi and Amini [6] introduced the concept of a dual space for CAT(0) spaces as follows. Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \tag{1.4}$$

where $C(X)$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with a Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$ for all $t \in \mathbb{R}$ and $a, b \in X$, where

$$L(f) = \sup\left\{\frac{f(x) - f(y)}{d(x, y)} : x, y \in X, x \neq y\right\}$$

is the Lipschitz semi-norm of the function $f : X \to \mathbb{R}$. Now, define the pseudometric D on $\mathbb{R} \times X \times X$ by

$$D\bigl((t, a, b), (s, c, d)\bigr) = L\bigl(\Theta(t, a, b) - \Theta(s, c, d)\bigr).$$

Lemma 1.1 [6, Lemma 2.1] $D((t, a, b), (s, c, d)) = 0$ *if and only if* $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ *for all* $x, y \in X$.

For a complete CAT(0) space (X,d), the pseudometric space $(\mathbb{R}\times X\times X,D)$ can be considered as a subspace of the pseudometric space $(\mathrm{Lip}(X,R),L)$ of all real-valued Lipschitz functions. Also, D defines an equivalence relation on $\mathbb{R}\times X\times X$, where the equivalence class of $t\overrightarrow{ab}:=(t,a,b)$ is

$$[t\overrightarrow{ab}]=\{s\overrightarrow{cd}:t\langle\overrightarrow{ab},\overrightarrow{xy}\rangle=s\langle\overrightarrow{cd},\overrightarrow{xy}\rangle\ \forall x,y\in X\}.$$

The set $X^*:=\{[t\overrightarrow{ab}]:(t,a,b)\in\mathbb{R}\times X\times X\}$ is a metric space with metric D, which is called the dual metric space of (X,d).

Recently, Dehghan and Rooin [7] introduced the duality mapping in CAT(0) spaces and studied its relation with subdifferential, by using the concept of quasilinearization. Then they presented a characterization of metric projection in CAT(0) spaces as follows.

Theorem 1.2 [7, Theorem 2.4] *Let C be a nonempty convex subset of a complete* CAT(0) *space X, $x\in X$ and $u\in C$. Then*

$$u=P_Cx\quad\text{if and only if}\quad\langle\overrightarrow{yu},\overrightarrow{ux}\rangle\geq 0\quad\text{for all } y\in C.$$

From now on, let $\mathbb{N}$ be the set of positive integers, let $\mathbb{R}$ be the set of real numbers, and let $\mathbb{R}^+$ be the set of nonnegative real numbers. Let C be a nonempty, closed and convex subset of a complete CAT(0) space X. A family $\mathcal{S}:=\{T(t):t\in\mathbb{R}^+\}$ of self-mappings of C is called a one-parameter continuous semigroup of nonexpansive mappings if the following conditions hold:

(i) for each $t\in\mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on C, *i.e.*,

$$d\big(T(t)x,T(t)y\big)\leq d(x,y),\quad\forall x,y\in C;$$

(ii) $T(s+t)=T(t)\circ T(s)$ for all $t,s\in\mathbb{R}^+$;

(iii) for each $x\in X$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

A family $\mathcal{S}:=\{T(t):t\in\mathbb{R}^+\}$ of mappings is called a one-parameter strongly continuous semigroup of nonexpansive mappings if conditions (i), (ii) and (iii) and the following condition are satisfied:

(iv) $T(0)x=x$ for all $x\in C$.

We shall denote by $\mathcal{F}$ the common fixed point set of $\mathcal{S}$, that is,

$$\mathcal{F}:=F(\mathcal{S})=\{x\in C:T(t)x=x,t\in\mathbb{R}^+\}=\bigcap_{t\in\mathbb{R}^+}F\big(T(t)\big).$$

One classical way to study nonexpansive mappings is to use contractions to approximate nonexpansive mappings. More precisely, take $t\in(0,1)$ and define a contraction $T_t:C\to C$ by

$$T_t=tu+(1-t)Tx,\quad\forall x\in C,$$

where $u\in C$ is an arbitrary fixed element. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of x_t is as $t\to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [8] proved that x_t converges strongly to a fixed point of T that is nearest to u in the framework of Hilbert spaces. Reich [9] extended Browder's result to the setting of Banach spaces and proved, in a uniformly smooth Banach space, that x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Halpern [10] introduced the following explicit iterative scheme (1.5) for a nonexpansive mapping T on a subset C of a Hilbert space by taking any points $u, x_1 \in C$ and defined the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n. \tag{1.5}$$

He proved that the sequence $\{x_n\}$ generated by (1.5) converges to a fixed point of T.

It is an interesting problem to extend the above (Browder's [8] and Halpern's [10]) results to the nonexpansive semigroup case. In [11], Shioji and Takahashi introduced the following implicit iteration in a Hilbert space:

$$x_n = \alpha_n u + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds, \tag{1.6}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $u \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ is a sequence of positive real numbers divergent to ∞. Under suitable conditions, they proved strong convergence of $\{x_n\}$ to a member of $\mathcal{F}$.

Later, Suzuki [12] was the first to introduce in a Hilbert space the following iteration process:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \tag{1.7}$$

where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on C such that $\mathcal{F} \neq \emptyset$ and $\{\alpha_n\}$ and $\{t_n\}$ are appropriate sequences of real numbers. He proved that $\{x_n\}$ generated by (1.7) converges strongly to the element of $\mathcal{F}$ nearest to u. Using Moudafi's viscosity approximation methods, Song and Xu [13] introduced the following iteration process:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \tag{1.8}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1. \tag{1.9}$$

They proved that $\{x_n\}$ converges to the same point of $\mathcal{F}$ in a reflexive strictly Banach space with a uniformly Gâteaux differentiable norm.

In the similar way, Dhompongsa *et al.* [14] extended Browder's iteration to a strongly continuous semigroup of nonexpansive mappings $\{T(t) : t \geq 0\}$ in a complete CAT(0) space X as follows:

$$x_n = \alpha_n x_0 \oplus T(t_n)x_n, \quad \forall n \geq 1,$$

where C is a nonempty closed convex subset of a complete CAT(0) space X, $x_0 \in C$, $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$, and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. The proved that $\mathcal{F} \neq \emptyset$ and $\{x_n\}$ converges to the element of $\mathcal{F}$ nearest to u. For other related results, see [15, 16].

In 2012, Shi and Chen [17], studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping T: for a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be a unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$; *i.e.*,

$$x_t = tf(x_t) \oplus (1-t)Tx_t, \tag{1.10}$$

and $x_0 \in C$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.11}$$

where $\{\alpha_n\} \subset (0,1)$. They proved $\{x_t\}$ defined by (1.10) converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ in the framework of CAT(0) space satisfying property $\mathcal{P}$, *i.e.*, if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x,y_1]}u)d(x,y_1) \leq d(x, P_{[x,y_2]}u)d(x,y_2) + d(x,u)d(y_1,y_2).$$

Furthermore, they also obtained that $\{x_n\}$ defined by (1.11) converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ under certain appropriate conditions imposed on $\{\alpha_n\}$.

By using the concept of quasilinearization, Wangkeeree and Preechasilp [18] improved Shi and Chen's results. In fact, they proved the strong convergence theorems for two given iterative schemes (1.10) and (1.11) in a complete CAT(0) space without the property $\mathcal{P}$.

Motivated and inspired by Song and Xu [13], Dhompongsa *et al.* [14], and Wangkeeree and Preechasilp [18], in this paper we aim to study the strong convergence theorems of Moudafi's viscosity approximation methods for a one-parameter continuous semigroup of nonexpansive mappings $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces. Let C be a nonempty, closed and convex subset of a CAT(0) space X. For a given contraction f on C and $\alpha_n \in (0,1)$, let $x_n \in C$ be a unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x$; *i.e.*,

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad n \geq 0, \tag{1.12}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad n \geq 0. \tag{1.13}$$

We prove that the iterative schemes $\{x_n\}$ defined by (1.12) and $\{x_n\}$ defined by (1.13) converge strongly to the same point $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}f(\tilde{x})$, which is the unique solution of the variational inequality

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F},$$

where $\mathcal{F}$ is the common fixed point set of $\mathcal{S}$, that is,

$$\mathcal{F} := F(\mathcal{S}) = \{x \in C : T(t)x = x, t \in \mathbb{R}^+\} = \bigcap_{t\in\mathbb{R}^+} F(T(t)).$$

2 Preliminaries

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y) \quad \text{and} \quad d(z,y) = (1-t)d(x,y).$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$. A subset C of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x, y \in C$.

The following lemmas play an important role in our paper.

Lemma 2.1 [2, Proposition 2.2] *Let X be a* CAT(0) *space, $p,q,r,s \in X$ and $\lambda \in [0,1]$. Then*

$$d\bigl(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s\bigr) \le \lambda d(p,r) + (1-\lambda)d(q,s).$$

Lemma 2.2 [19, Lemma 2.4] *Let X be a* CAT(0) *space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then*

$$d\bigl(\lambda x \oplus (1-\lambda)y, z\bigr) \le \lambda d(x,z) + (1-\lambda)d(y,z).$$

Lemma 2.3 [19, Lemma 2.5] *Let X be a* CAT(0) *space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then*

$$d^2\bigl(\lambda x \oplus (1-\lambda)y, z\bigr) \le \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y).$$

The concept of Δ-convergence introduced by Lim [20] in 1976 was shown by Kirk and Panyanak [21] in CAT(0) spaces to be very similar to the weak convergence in Banach space setting. Next, we give the concept of Δ-convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r\bigl(x, \{x_n\}\bigr) = r\bigl(\{x_n\}\bigr)\bigr\}.$$

It is known from Proposition 7 of [22] that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. The uniqueness of an asymptotic center implies that a CAT(0) space X satisfies Opial's property, *i.e.*, for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that '$I - T$ is demiclosed at zero' if the conditions $\{x_n\} \subseteq C$ Δ-converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in F(T)$.

Lemma 2.4 [21] *Every bounded sequence in a complete* CAT(0) *space always has a* Δ-*convergent subsequence*.

Lemma 2.5 [23] *If C is a closed convex subset of a complete* CAT(0) *space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.6 [23] *If C is a closed convex subset of X and $T: C \to X$ is a nonexpansive mapping, then the conditions $\{x_n\}$ Δ-converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in C$ and $Tx = x$.*

Having the notion of quasilinearization, Kakavandi and Amini [6] introduced the following notion of convergence.

A sequence $\{x_n\}$ in the complete CAT(0) space (X, d) w-converges to $x \in X$ if

$$\lim_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0,$$

i.e., $\lim_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0$ for all $y \in X$.

It is obvious that convergence in the metric implies w-convergence, and it is easy to check that w-convergence implies Δ-convergence [6, Proposition 2.5], but it is showed in [24, Example 4.7] that the converse is not valid. However, the following lemma shows another characterization of Δ-convergence as well as, more explicitly, a relation between w-convergence and Δ-convergence.

Lemma 2.7 [24, Theorem 2.6] *Let X be a complete* CAT(0) *space*, *$\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if* $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \le 0$ *for all $y \in X$.*

Lemma 2.8 [25, Lemma 2.1] *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the property*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\beta_n, \quad n \ge 0,$$

where $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\{a_n\}$ converges to zero as $n \to \infty$.

3 Viscosity approximation methods

In this section, we present the strong convergence theorems of Moudafi's viscosity approximation methods for a one-parameter continuous semigroup of nonexpansive mappings $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces. Before proving main results, we need the following two vital lemmas.

Lemma 3.1 *Let X be a complete* CAT(0) *space. Then, for all $u, x, y \in X$, the following inequality holds*:

$$d^2(x, u) \le d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Proof Using (1.2), we have that

$$\begin{aligned} d^2(y,u) - d^2(x,u) - 2\langle\overrightarrow{yx},\overrightarrow{xu}\rangle &= d^2(y,u) - d^2(x,u) - 2\langle\overrightarrow{yu},\overrightarrow{xu}\rangle - 2\langle\overrightarrow{ux},\overrightarrow{xu}\rangle \\ &= d^2(y,u) - d^2(x,u) - 2\langle\overrightarrow{yu},\overrightarrow{xu}\rangle + 2d^2(x,u) \\ &= d^2(y,u) + d^2(x,u) - 2\langle\overrightarrow{yu},\overrightarrow{xu}\rangle \\ &\geq d^2(y,u) + d^2(x,u) - 2d(y,u)d(x,u) \\ &= \bigl(d^2(y,u) - d^2(x,u)\bigr)^2 \geq 0. \end{aligned}$$

Therefore we obtain that

$$d^2(x,u) \leq d^2(y,u) + 2\langle\overrightarrow{xy},\overrightarrow{xu}\rangle,$$

which is the desired result. □

Lemma 3.2 *Let X be a* CAT(0) *space. For any $t \in [0,1]$ and $u,v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x,y \in X$,*

(i) $\langle\overrightarrow{u_t x},\overrightarrow{u_t y}\rangle \leq t\langle\overrightarrow{ux},\overrightarrow{u_t y}\rangle + (1-t)\langle\overrightarrow{vx},\overrightarrow{u_t y}\rangle$;

(ii) $\langle\overrightarrow{u_t x},\overrightarrow{uy}\rangle \leq t\langle\overrightarrow{ux},\overrightarrow{uy}\rangle + (1-t)\langle\overrightarrow{vx},\overrightarrow{uy}\rangle$ *and* $\langle\overrightarrow{u_t x},\overrightarrow{vy}\rangle \leq t\langle\overrightarrow{ux},\overrightarrow{vy}\rangle + (1-t)\langle\overrightarrow{vx},\overrightarrow{vy}\rangle$.

Proof (i) It follows from (CN)-inequality (1.1) that

$$\begin{aligned} 2\langle\overrightarrow{u_t x},\overrightarrow{u_t y}\rangle &= d^2(u_t,y) + d^2(x,u_t) - d^2(x,y) \\ &\leq td^2(u,y) + (1-t)d^2(v,y) - t(1-t)d^2(u,v) + d^2(x,u_t) - d^2(x,y) \\ &= td^2(u,y) + td^2(x,u_t) - td^2(u,u_t) - td^2(x,y) \\ &\quad + (1-t)d^2(v,y) + (1-t)d^2(x,u_t) - (1-t)d^2(v,u_t) - (1-t)d^2(x,y) \\ &\quad + td^2(u,u_t) + (1-t)d^2(v,u_t) - t(1-t)d^2(u,v) \\ &= t\bigl[d^2(u,y) + d^2(x,u_t) - d^2(u,u_t) - d^2(x,y)\bigr] \\ &\quad + (1-t)\bigl[d^2(v,y) + d^2(x,u_t) - d^2(v,u_t) - d^2(x,y)\bigr] \\ &\quad + t(1-t)^2 d^2(u,v) + (1-t)t^2 d^2(u,v) - t(1-t)d^2(u,v) \\ &= t\langle\overrightarrow{ux},\overrightarrow{u_t y}\rangle + (1-t)\langle\overrightarrow{vx},\overrightarrow{u_t y}\rangle. \end{aligned}$$

(ii) The proof is similar to (i). □

For any $\alpha_n \in (0,1)$, $t_n \in [0,\infty)$ and a contraction f with coefficient $\alpha \in (0,1)$, define the mapping $G_n : C \to C$ by

$$G_n(x) = \alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \quad \forall x \in C. \tag{3.1}$$

It is not hard to see that G_n is a contraction on C. Indeed, for $x,y \in C$, we have

$$\begin{aligned} d\bigl(G_n(x),G_n(y)\bigr) &= d\bigl(\alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)y\bigr) \\ &\leq d\bigl(\alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)x\bigr) \end{aligned}$$

$$
\begin{aligned}
&+ d\big(\alpha_n f(y) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)y\big) \\
&\le \alpha_n d\big(f(x), f(y)\big) + (1-\alpha_n) d\big(T(t_n)x, T(t_n)y\big) \\
&\le \alpha_n \alpha d(x,y) + (1-\alpha_n) d(x,y) \\
&= \big(1-\alpha_n(1-\alpha)\big) d(x,y).
\end{aligned}
$$

Therefore we have that G_n is a contraction mapping. Let $x_n \in C$ be the unique fixed point of G_n; that is,

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n \quad \text{for all } n \ge 0. \tag{3.2}$$

Now we are in a position to state and prove our main results.

Theorem 3.3 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular* (*in short, u.a.r.*) *on C, that is, for all $h \ge 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t)x\big), T(t)x\big) = 0.$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose that $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and let $\{x_n\}$ be given by (3.2). *Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to the following variational inequality*:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \ge 0, \quad \forall x \in \mathcal{F}. \tag{3.3}$$

Proof We first show that $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$
\begin{aligned}
d(x_n, p) &= d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, p\big) \le \alpha_n d\big(f(x_n), p\big) + (1-\alpha_n) d\big(T(t_n)x_n, p\big) \\
&\le \alpha_n d\big(f(x_n), p\big) + (1-\alpha_n) d(x_n, p).
\end{aligned}
$$

Then

$$d(x_n, p) \le d\big(f(x_n), p\big) \le d\big(f(x_n), f(p)\big) + d\big(f(p), p\big) \le \alpha d(x_n, p) + d\big(f(p), p\big).$$

This implies that

$$d(x_n, p) \le \frac{1}{1-\alpha} d\big(f(p), p\big).$$

Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. We get that

$$
\begin{aligned}
d\big(x_n, T(t_n)x_n\big) &= d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, T(t_n)x_n\big) \\
&\le \alpha_n d\big(f(x_n), T(t_n)x_n\big) + (1-\alpha_n) d\big(T(t_n)x_n, T(t_n)x_n\big) \\
&\le \alpha_n d\big(f(x_n), T(t_n)x_n\big) \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, then for all $h > 0$,

$$\lim_{n\to\infty} d\bigl(T(h)\bigl(T(t_n)x_n\bigr), T(t_n)x_n\bigr) \leq \lim_{n\to\infty} \sup_{x\in B} d\bigl(T(h)\bigl(T(t_n)x\bigr), T(t_n)x\bigr) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} d\bigl(x_n, T(h)x_n\bigr) &\leq d\bigl(x_n, T(t_n)x_n\bigr) + d\bigl(T(t_n)x_n, T(h)\bigl(T(t_n)x_n\bigr)\bigr) \\ &\quad + d\bigl(T(h)\bigl(T(t_n)x_n\bigr), T(h)x_n\bigr) \\ &\leq 2d\bigl(x_n, T(t_n)x_n\bigr) + d\bigl(T(t_n)x_n, T(h)\bigl(T(t_n)x_n\bigr)\bigr) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.4}$$

We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F}. \tag{3.5}$$

Since $\{x_n\}$ is bounded, by Lemma 2.4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which Δ-converges to a point $\tilde{x}$, denoted by $\{x_j\}$. We claim that $\tilde{x} \in \mathcal{F}$. Since every CAT(0) space has Opial's property, for any $h \geq 0$, if $T(h)\tilde{x} \neq \tilde{x}$, we have

$$\begin{aligned} \limsup_{j\to\infty} d\bigl(x_j, T(h)\tilde{x}\bigr) &\leq \limsup_{j\to\infty} \bigl\{d\bigl(x_j, T(h)x_j\bigr) + d\bigl(T(h)x_j, T(h)\tilde{x}\bigr)\bigr\} \\ &\leq \limsup_{j\to\infty} \bigl\{d\bigl(x_j, T(h)x_j\bigr) + d(x_j, \tilde{x})\bigr\} \\ &= \limsup_{j\to\infty} d(x_j, \tilde{x}) \\ &< \limsup_{j\to\infty} d\bigl(x_j, T(h)\tilde{x}\bigr). \end{aligned}$$

This is a contradiction, and hence $\tilde{x} \in \mathcal{F}$. So we have the claim. It follows from Lemma 3.2(i) that

$$\begin{aligned} d^2(x_j, \tilde{x}) &= \langle \overrightarrow{x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq \alpha_j \bigl\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle + (1-\alpha_j) \bigl\langle \overrightarrow{T(t_j)x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle \\ &\leq \alpha_j \bigl\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle + (1-\alpha_j) d\bigl(T(t_j)x_j, \tilde{x}\bigr) d(x_j, \tilde{x}) \\ &\leq \alpha_j \bigl\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle + (1-\alpha_j) d^2(x_j, \tilde{x}). \end{aligned}$$

It follows that

$$\begin{aligned} d^2(x_j, \tilde{x}) &\leq \bigl\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle \\ &= \bigl\langle \overrightarrow{f(x_j)f(\tilde{x})}, \overrightarrow{x_j\tilde{x}} \bigr\rangle + \bigl\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle \\ &\leq d\bigl(f(x_j), f(\tilde{x})\bigr) d(x_j, \tilde{x}) + \bigl\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle \\ &\leq \alpha d^2(x_j, \tilde{x}) + \bigl\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \bigr\rangle, \end{aligned}$$

and thus

$$d^2(x_j,\tilde{x}) \le \frac{1}{1-\alpha}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\rangle. \tag{3.6}$$

Since $\{x_j\}$ Δ-converges to $\tilde{x}$, by Lemma 2.7, we have

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\rangle \le 0.$$

It follows from (3.6) that $\{x_j\}$ converges strongly to $\tilde{x}$. Next, we show that $\tilde{x}$ solves the variational inequality (3.3). Applying Lemma 2.3, for any $q \in \mathcal{F}$,

$$\begin{aligned} d^2(x_j,q) &= d^2\big(\alpha_j f(x_j) \oplus (1-\alpha_j)T(t_j)x_j, q\big) \\ &\le \alpha_j d^2\big(f(x_j),q\big) + (1-\alpha_j)d^2\big(T(t_j)x_j,q\big) - \alpha_j(1-\alpha_j)d^2\big(f(x_j),T(t_j)x_j\big) \\ &\le \alpha_j d^2\big(f(x_j),q\big) + (1-\alpha_j)d^2(x_j,q) - \alpha_j(1-\alpha_j)d^2\big(f(x_j),T(t_j)x_j\big). \end{aligned}$$

It implies that

$$d^2(x_j,q) \le d^2\big(f(x_j),q\big) - (1-\alpha_j)d^2\big(f(x_j),T(t_j)x_j\big).$$

Taking the limit through $j \to \infty$, we can get that

$$d^2(\tilde{x},q) \le d^2\big(f(\tilde{x}),q\big) - d^2\big(f(\tilde{x}),\tilde{x}\big).$$

Hence

$$0 \le \frac{1}{2}\big[d^2(\tilde{x},\tilde{x}) + d^2\big(f(\tilde{x}),q\big) - d^2(\tilde{x},q) - d^2\big(f(\tilde{x}),\tilde{x}\big)\big] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}}\rangle, \quad \forall q \in \mathcal{F}.$$

That is, $\tilde{x}$ solves the inequality (3.3). Finally, we show that the sequence $\{x_n\}$ converges to $\tilde{x}$. Assume that $x_{n_i} \to \hat{x}$, where $i \to \infty$. By the same argument, we get that $\hat{x} \in \mathcal{F}$ and solves the variational inequality (3.3), *i.e.*,

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle \le 0, \tag{3.7}$$

and

$$\langle \overrightarrow{\hat{x}f\hat{x}}, \overrightarrow{\hat{x}\tilde{x}}\rangle \le 0. \tag{3.8}$$

Adding up (3.7) and (3.8), we get that

$$\begin{aligned} 0 &\ge \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\ &= \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\ &= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\ &\ge \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - d\big(f(\hat{x}),f(\tilde{x})\big)d(\hat{x},\tilde{x}) \end{aligned}$$

$$\begin{aligned}&\geq d^2(\tilde{x},\hat{x}) - \alpha d(\hat{x},\tilde{x})d(\hat{x},\tilde{x})\\&\geq d^2(\tilde{x},\hat{x}) - \alpha d^2(\hat{x},\tilde{x})\\&\geq (1-\alpha)d^2(\tilde{x},\hat{x}).\end{aligned}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x},\hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence the sequence x_n converges strongly to $\tilde{x}$, which is the unique solution to the variational inequality (3.3). This completes the proof. □

If $f \equiv u$, then the following result can be obtained directly from Theorem 3.3.

Corollary 3.4 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C, that is, for all $h \geq 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x\in B} d\bigl(T(h)\bigl(T(t)x\bigr), T(t)x\bigr) = 0.$$

Let u be any element in C. Suppose $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let $\{x_n\}$ be given by

$$x_n = \alpha_n u \oplus (1-\alpha_n)T(t_n)x_n.$$

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}\tilde{x}$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}u}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F}. \tag{3.9}$$

Theorem 3.5 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C, that is, for all $h \geq 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x\in B} d\bigl(T(h)\bigl(T(t)x\bigr), T(t)x\bigr) = 0.$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose that $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$, $x_0 \in C$, and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 0, \tag{3.10}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and*
(iii) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to the variational inequality (3.3).

Proof We first show that the sequence $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned} d(x_{n+1},p) &= d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, p\big) \\ &\le \alpha_n d\big(f(x_n),p\big) + (1-\alpha_n) d\big(T(t_n)x_n,p\big) \\ &\le \alpha_n \big(d\big(f(x_n),f(p)\big) + d\big(f(p),p\big)\big) + (1-\alpha_n) d\big(T(t_n)x_n,p\big) \\ &\le \max\left\{d(x_n,p), \frac{1}{1-\alpha} d\big(f(p),p\big)\right\}. \end{aligned}$$

By induction, we have

$$d(x_n,p) \le \max\left\{d(x_0,p), \frac{1}{1-\alpha} d\big(f(p),p\big)\right\}$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. Using the assumption that $\lim_{n\to\infty} \alpha_n = 0$, we get that

$$d\big(x_{n+1}, T(t_n)x_n\big) \le \alpha_n d\big(f(x_n), T(t_n)x_n\big) \to 0 \quad \text{as } n \to \infty.$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, then for all $h \ge 0$,

$$\lim_{n\to\infty} d\big(T(h)\big(T(t_n)x_n\big), T(t_n)x_n\big) \le \lim_{n\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t_n)x\big), T(t_n)x\big) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} & d\big(x_{n+1}, T(h)x_{n+1}\big) \\ &\quad \le d\big(x_{n+1}, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \\ &\qquad + d\big(T(h)\big(T(t_n)x_n\big), T(h)x_{n+1}\big) \\ &\quad \le 2d\big(x_{n+1}, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.11}$$

Let $\{z_m\}$ be a sequence in C such that

$$z_m = \alpha_m f(z_m) \oplus (1-\alpha_m)T(t_m)z_m.$$

It follows from Theorem 3.3 that $\{z_m\}$ converges strongly as $m \to \infty$ to a fixed point $\tilde{x} \in \mathcal{F}$, which solves the variational inequality (3.3). Now, we claim that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \le 0.$$

It follows from Lemma 3.2(i) that

$$\begin{aligned} d^2(z_m, x_{n+1}) &= \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\ &\le \alpha_m \langle \overrightarrow{f(z_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + (1-\alpha_m) \langle \overrightarrow{T(t_m)z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\ &= \alpha_m \langle \overrightarrow{f(z_m)f(\tilde{x})}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m \langle \overrightarrow{\tilde{x} z_m}, \overrightarrow{z_m x_{n+1}} \rangle \end{aligned}$$

$$\begin{aligned}
&+ \alpha_m \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + (1-\alpha_m)\langle \overrightarrow{T(t_m)z_m T(t_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
&+ (1-\alpha_m)\langle \overrightarrow{T(t_m)x_{n+1}x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
\le{}& \alpha_m \alpha d(z_m,\tilde{x})d(z_m,x_{n+1}) + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m d(\tilde{x},z_m)d(z_m,x_{n+1}) \\
&+ \alpha_m d^2(z_m,x_{n+1}) + (1-\alpha_m)d^2(z_m,x_{n+1}) \\
&+ (1-\alpha_m)d\bigl(T(t_m)x_{n+1},x_{n+1}\bigr)d(z_m,x_{n+1}) \\
\le{}& \alpha_m \alpha d(z_m,\tilde{x})M + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m d(\tilde{x},z_m)M + \alpha_m d^2(z_m,x_{n+1}) \\
&+ (1-\alpha_m)d^2(z_m,x_{n+1}) + (1-\alpha_m)d\bigl(T(t_m)x_{n+1},x_{n+1}\bigr)M \\
\le{}& d^2(z_m,x_{n+1}) + \alpha_m \alpha d(z_m,\tilde{x})M + \alpha_m d(\tilde{x},z_m)M + d\bigl(T(t_m)x_{n+1},x_{n+1}\bigr)M \\
&+ \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle,
\end{aligned}$$

where $M \ge \sup_{m,n\ge 1}\{d(z_m,x_n)\}$. This implies that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \rangle \le (1+\alpha)d(z_m,\tilde{x})M + \frac{d(T(t_m)x_{n+1},x_{n+1})}{\alpha_m}M. \tag{3.12}$$

Taking the upper limit as $n \to \infty$ first, and then $m \to \infty$, inequality (3.12) yields that

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \rangle \le 0. \tag{3.13}$$

Since

$$\begin{aligned}
\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle &= \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m\tilde{x}} \rangle \\
&\le \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \rangle + d\bigl(f(\tilde{x}),\tilde{x}\bigr)d(z_m,\tilde{x}).
\end{aligned}$$

Thus, by taking the upper limit as $n \to \infty$ first, and then $m \to \infty$ the last inequality, it follows from $z_m \to \tilde{x}$ and (3.13) that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \le 0.$$

Finally, we prove that $x_n \to \tilde{x}$ as $n \to \infty$. For any $n \in \mathbb{N}$, we set $y_n = \alpha_n\tilde{x} \oplus (1-\alpha_n)T(t_n)x_n$. It follows from Lemma 3.1 and Lemma 3.2(i), (ii) that

$$\begin{aligned}
d^2(x_{n+1},\tilde{x}) \le{}& d^2(y_n,\tilde{x}) + 2\langle \overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
\le{}& \bigl(\alpha_n d(\tilde{x},\tilde{x}) + (1-\alpha_n)d\bigl(T(t_n)x_n,\tilde{x}\bigr)\bigr)^2 \\
&+ 2\bigl[\alpha_n \langle \overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + (1-\alpha_n)\langle \overrightarrow{T(t_n)x_n y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\bigr] \\
\le{}& (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\bigl[\alpha_n\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)T(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&+ (1-\alpha_n)\alpha_n \langle \overrightarrow{T(t_n)x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + (1-\alpha_n)(1-\alpha_n)\langle \overrightarrow{T(t_n)x_n T(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\bigr] \\
\le{}& (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\bigl[\alpha_n\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)T(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&+ (1-\alpha_n)\alpha_n \langle \overrightarrow{T(t_n)x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + (1-\alpha_n)^2 d\bigl(T(t_n)x_n, T(t_n)x_n\bigr)d(x_{n+1}\tilde{x})\bigr]
\end{aligned}$$

$$\begin{aligned}
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\bigl[\alpha_n^2\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\bigr]\\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_{n+1}\tilde{x}}\rangle + 2\alpha_n\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\alpha d(x_n,\tilde{x})d(x_{n+1},\tilde{x}) + 2\alpha_n\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + \alpha_n\alpha\bigl(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\bigr) + 2\alpha_n\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
d^2(x_{n+1},\tilde{x}) &\le \frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n^2 M,
\end{aligned}$$

where $M \ge \sup_{n\ge 0}\{d^2(x_n,\tilde{x})\}$. It then follows that

$$d^2(x_{n+1},\tilde{x}) \le \bigl(1-\alpha_n'\bigr)d^2(x_n,\tilde{x}) + \alpha_n'\beta_n',$$

where

$$\alpha_n' = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \quad\text{and}\quad \beta_n' = \frac{(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)}M + \frac{1}{(1-\alpha)}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle.$$

Applying Lemma 2.8, we can conclude that $x_n \to \tilde{x}$. This completes the proof. □

If $f \equiv u$, then the following corollary can be obtained directly from Theorem 3.5.

Corollary 3.6 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular* (*in short, u.a.r.*) *on C, that is, for all $h \ge 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty}\sup_{x\in B} d\bigl(T(h)\bigl(T(t)x\bigr),T(t)x\bigr) = 0.$$

Suppose that $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$, $x_0 \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T(t_n)x_n, \quad \forall n \ge 0, \tag{3.14}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and*
(iii) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}\tilde{x}$, which is equivalent to the variational inequality (3.9).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

Acknowledgements
The first author is supported by the Centre of Excellence in Mathematics under the Commission on Higher Education, Ministry of Education, Thailand.

Received: 22 December 2012 Accepted: 14 May 2013 Published: 19 June 2013

References

1. Bruhat, F, Tits, J: Groupes réductifs sur un corps local. I. Donêes radicielles valuées. Publ. Math. IHES **41**, 5-251 (1972)
2. Bridson, M, Haefliger, A: Metric Spaces of Nonpositive Curvature. Springer, Berlin (1999)
3. Brown, KS: Buildings. Springer, New York (1989)
4. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 83. Dekker, New York (1984)
5. Berg, ID, Nikolaev, IG: Quasilinearization and curvature of Alexandrov spaces. Geom. Dedic. **133**, 195-218 (2008)
6. Ahmadi Kakavandi, B, Amini, M: Duality and subdifferential for convex functions on complete CAT(0) metric spaces. Nonlinear Anal. **73**, 3450-3455 (2010)
7. Dehghan, H, Rooin, J: A characterization of metric projection in CAT(0) spaces. In: International Conference on Functional Equation, Geometric Functions and Applications (ICFGA 2012) 10-12th May 2012, pp. 41-43. Payame Noor University, Tabriz (2012)
8. Browder, FE: Fixed point theorems for noncompact mappings in Hilbert spaces. Proc. Natl. Acad. Sci. USA **53**, 1272-1276 (1965)
9. Reich, S: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287-292 (1980)
10. Halpern, B: Fixed points of nonexpansive maps. Bull. Am. Math. Soc. **73**, 957-961 (1967)
11. Shioji, N, Takahashi, W: Strong convergence theorems for asymptotically nonexpansive semi-groups in Hilbert spaces. Nonlinear Anal. **34**, 87-99 (1998)
12. Suzuki, T: On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces. Proc. Am. Math. Soc. **131**, 2133-2136 (2003)
13. Song, Y, Xu, S: Strong convergence theorems for nonexpansive semigroup in Banach spaces. J. Math. Anal. Appl. **338**, 152-161 (2008)
14. Dhompongsa, S, Fupinwong, W, Kaewkhao, A: Common fixed points of a nonexpansive semigroup and a convergence theorem for Mann iterations in geodesic metric spaces. Nonlinear Anal. **70**, 4268-4273 (2009)
15. Cho, YJ, Ćirić, L, Wang, SH: Convergence theorems for nonexpansive semigroups in CAT(0) spaces. Nonlinear Anal. **74**, 6050-6059 (2011)
16. Lin, L-J, Chuang, C, Yu, ZT: Convergence theorems for some semigroups of nonexpansive mappings in complete CAT(0) spaces. Fixed Point Theory Appl. **2012**, Article ID 155 (2012). doi:10.1186/1687-1812-2012-155
17. Shi, LY, Chen, RD: Strong convergence of viscosity approximation methods for nonexpansive mappings in CAT(0) spaces. J. Appl. Math. **2012**, Article ID 421050 (2012). doi:10.1155/2012/421050
18. Wangkeeree, R, Preechasilp, P: Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces. J. Inequal. Appl. **2013**, Article ID 93 (2013). doi:10.1186/1029-242X-2013-93
19. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**, 2572-2579 (2008)
20. Lim, TC: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179-182 (1976)
21. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689-3696 (2008)
22. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762-772 (2006)
23. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**, 35-45 (2007)
24. Ahmadi Kakavandi, B: Weak topologies in complete CAT(0) metric spaces, Proc. Am. Math. Soc. **141**, 1029-1039 (2013) S 0002-9939(2012)11743-5
25. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**, 659-678 (2003)

doi:10.1186/1687-1812-2013-160
Cite this article as: Wangkeeree and Preechasilp: **Viscosity approximation methods for nonexpansive semigroups in CAT(0) spaces.** *Fixed Point Theory and Applications* 2013 **2013**:160.