Et et al. *Fixed Point Theory and Applications* 2013, **2013**:165
http://www.fixedpointtheoryandapplications.com/content/2013/1/165

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Some geometric properties of a new modular space defined by Zweier operator

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Dedicated to Professor Hari M Srivastava

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Abstract

In this paper, we define the modular space $\mathcal{Z}_\sigma(s,p)$ by using the Zweier operator and a modular. Then, we consider it equipped with the Luxemburg norm and also examine the uniform Opial property and property β. Finally, we show that this space has the fixed point property.

MSC: 40A05; 46A45; 46B20

Keywords: Zweier operator; Luxemburg norm; modular space; uniform Opial property; property (β)

1 Introduction

In literature, there are many papers about the geometrical properties of different sequence spaces such as [1–9]. Opial [10] introduced the Opial property and proved that the sequence spaces ℓ_p $(1 < p < \infty)$ have this property but $L_p[0, 2\pi]$ $(p \neq 2, 1 < p < \infty)$ does not have it. Franchetti [11] showed that any infinite dimensional Banach space has an equivalent norm that satisfies the Opial property. Later, Prus [12] introduced and investigated the uniform Opial property for Banach spaces. The Opial property is important because Banach spaces with this property have the weak fixed point property.

2 Definition and preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $S(X)$ (resp. $B(X)$) be the unit sphere (resp. the unit ball) of X. A Banach space X has the Opial property if for any weakly null sequence $\{x_n\}$ in X and any x in $X \setminus \{0\}$, the inequality $\lim_{n\to\infty} \inf \|x\| < \lim_{n\to\infty} \inf \|x_n + x\|$ holds. We say that X has the uniform Opial property if for any $\varepsilon > 0$ there exists $r > 0$ such that for any $x \in X$ with $\|x\| \geq \varepsilon$ and any weakly null sequence $\{x_n\}$ in the unit sphere of X, the inequality $1 + r \leq \lim_{n\to\infty} \inf \|x_n + x\|$ holds.

For a bounded set $A \subset X$, the ball-measure of noncompactness was defined by $\beta(A) = \inf\{\varepsilon > 0 : A$ can be covered by finitely many balls with diameter $\leq \varepsilon\}$. The function Δ defined by $\Delta(\varepsilon) = \inf\{1 - \inf(\|x\| : x \in A) : A$ is closed convex subset of $B(X)$ with $\beta(A) \leq \varepsilon\}$ is called the modulus of noncompact convexity. A Banach space X is said to have property (L), if $\lim_{\varepsilon\to 1^-} \Delta(\varepsilon) = 1$. This property is an important concept in the fixed point theory and a Banach space X possesses property (L) if and only if it is reflexive and has the uniform Opial property.

A Banach space X is said to satisfy the weak fixed point property if every nonempty weakly compact convex subset C and every nonexpansive mapping $T : C \to C(\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C)$ have a fixed point, that is, there exists $x \in C$ such that $T(x) = x$. Property (L) and the fixed point property were also studied by Goebel and Kirk [13], Toledano *et al.* [14], Benavides [15], Benavides and Phothi [16]. A Banach space X is said to have property (H) if every weakly convergent sequence on the unit sphere is convergent in norm. Clarkson [17] introduced the uniform convexity, and it is known that the uniform convexity implies the reflexivity of Banach spaces. Huff [18] introduced the concept of nearly uniform convexity of Banach spaces. A Banach space X is called uniformly convex (UC) if for each $\varepsilon > 0$, there is $\delta > 0$ such that for $x, y \in S(X)$, the inequality $\|x - y\| > \varepsilon$ implies that $\|\frac{1}{2}(x + y)\| < 1 - \delta$. For any $x \notin B(X)$, the drop determined by x is the set $D(x, B(X)) = \operatorname{conv}(\{x\} \cup B(X))$. A Banach space X has the drop property (D) if for every closed set C disjoint with $B(X)$, there exists an element $x \in C$ such that $D(x, B(X)) \cap C = \{x\}$. Rolewicz [19] showed that the Banach space X is reflexive if X has the drop property. Later, Montesinos [20] extended this result and proved that X has the drop property if and only if X is reflexive and has property (H). A sequence $\{x_n\}$ is said to be ε-separated sequence for some $\varepsilon > 0$ if

$$sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

A Banach space X is called nearly uniformly convex (NUC) if for every $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $sep(x_n) > \varepsilon$, we have $\operatorname{conv}(x_n) \cap ((1 - \delta)B(X)) \neq \emptyset$. Huff [18] proved that every (NUC) Banach space is reflexive and has property (H). A Banach space X has property (β) if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $sep(x_n) \geq \varepsilon$, there is an index k for which $\|\frac{x + x_k}{2}\| < 1 - \delta$.

For a real vector space X, a function $\rho : X \to [0, \infty]$ is called a modular if it satisfies the following conditions:

(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ρ is called convex if

(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ρ on X, the space

$$X_\rho = \{x \in X : \rho(\sigma x) < \infty \text{ for some } \sigma > 0\}$$

is called a modular space. In general, the modular is not subadditive and thus it does not behave as a norm or a distance. But we can associate the modular with an F-norm. A functional $\|\cdot\| : X \to [0, \infty]$ defines an F-norm if and only if

(i) $\|x\| = 0 \Leftrightarrow x = 0$,
(ii) $\|\alpha x\| = \|x\|$ whenever $|\alpha| = 1$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$,
(iv) if $\alpha_n \to \alpha$ and $\|x_n - x\| \to 0$, then $\|\alpha_n x_n - \alpha x\| \to 0$.

F-norm defines a distance on X by $d(x,y) = \|x - y\|$. If the linear metric space (X,d) is complete, then it is called an F-space. The modular space X_ρ can be equipped with the following F-norm:

$$\|x\| = \inf\left\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \leq \alpha\right\}.$$

If the modular ρ is convex, then the equality $\|x\| = \inf\{\alpha > 0 : \rho(\frac{x}{\alpha}) \leq 1\}$ defines a norm which is called the Luxemburg norm.

A modular ρ is said to satisfy the δ_2-condition if for any $\varepsilon > 0$, there exist constants $K \geq 2$, $a > 0$ such that $\rho(2u) \leq K\rho(u) + \varepsilon$ for all $u \in X_\rho$ with $\rho(u) \leq a$. If ρ provides the δ_2-condition for any $a > 0$ with $K \geq 2$ dependent on a, then ρ provides the strong δ_2-condition (briefly $\rho \in \delta_2^s$).

Let us denote by ℓ^0 the space of all real sequences. The Cesàro sequence spaces

$$Ces_p = \left\{x \in \ell^0 : \sum_{n=1}^{\infty}\left(n^{-1}\sum_{i=1}^{n}|x_i|\right)^p < \infty\right\}, \quad 1 \leq p < \infty,$$

and

$$Ces_\infty = \left\{x \in \ell^0 : \sup_n n^{-1}\sum_{i=1}^{n}|x_i| < \infty\right\},$$

were introduced by Shiue [21]. Jagers [22] determined the Köthe duals of the sequence space Ces_p $(1 < p < \infty)$. It can be shown that the inclusion $\ell_p \subset Ces_p$ is strict for $1 < p < \infty$ although it does not hold for $p = 1$. Also, Suantai [23] defined the generalized Cesàro sequence space by

$$ces(p) = \left\{x \in \ell^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\right\},$$

where $\rho(x) = \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{i=1}^{n}|x(i)|)^{p_n}$. If $p = (p_n)$ is bounded, then

$$ces(p) = \left\{x = (x_k) : \sum_{n=1}^{\infty}\left(n^{-1}\sum_{i=1}^{n}\left|x(i)\right|\right)^{p_n} < \infty\right\}.$$

The sequence space $C(s,p)$ was defined by Bilgin [24] as follows:

$$C(s,p) = \left\{x = (x_k) : \sum_{r=0}^{\infty}\left(2^{-r}\sum_{r}k^{-s}|x_k|\right)^{p_r} < \infty, s \geq 0\right\}$$

for $p = (p_r)$ with $\inf p_r > 0$, where $\sum_r$ denotes a sum over the ranges $2^r \leq k < 2^{r+1}$. The special case of $C(s,p)$ for $s = 0$ is the space

$$Ces(p) = \left\{x = (x_k) : \sum_{r=0}^{\infty}\left(2^{-r}\sum_{r}|x_k|\right)^{p_r} < \infty\right\}$$

which was introduced by Lim [25]. Also, the inclusion $Ces(p) \subseteq C(s,p)$ holds. A paranorm on $C(s,p)$ is given by

$$\rho(x) = \left(\sum_{r=0}^{\infty}\left(2^{-r}\sum_{r} k^{-s}|x_k|\right)^{p_r}\right)^{1/M}$$

for $M = \max(1,H)$ and $H = \sup p_r < \infty$.

The Z-transform of a sequence $x = (x_k)$ is defined by $(Zx)_n = y_n = \alpha x_n + (1-\alpha)x_{n-1}$ by using the Zweier operator

$$Z = (z_{nk}) = \begin{cases} \alpha, & k = n, \\ 1-\alpha, & k = n-1, \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } n,k \in \mathbb{N} \text{ and } \alpha \in \mathcal{F}\backslash\{0\},$$

where $\mathcal{F}$ is the field of all complex or real numbers. The Zweier operator was studied by Şengönül and Kayaduman [26].

Now we introduce a new modular sequence space $\mathcal{Z}_\sigma(s,p)$ by

$$\mathcal{Z}_\sigma(s,p) = \{x \in \ell^0 : \sigma(tx) < \infty, \text{ for some } t > 0\},$$

where $\sigma(x) = \sum_{r=0}^{\infty}(2^{-r}\sum_r k^{-s}|\alpha x_k + (1-\alpha)x_{k-1}|)^{p_r} < \infty$ and $s \geq 0$. If we take $\alpha = 1$, then $\mathcal{Z}_\sigma(s,p) = C(s,p)$; if $\alpha = 1$ and $s = 0$, then $\mathcal{Z}_\sigma(s,p) = Ces(p)$. It can be easily seen that $\sigma : \mathcal{Z}_\sigma(s,p) \to [0,\infty]$ is a modular on $\mathcal{Z}_\sigma(s,p)$. We define the Luxemburg norm on the sequence space $\mathcal{Z}_\sigma(s,p)$ as follows:

$$\|x\| = \inf\left\{t > 0 : \sigma\left(\frac{x}{t}\right) \leq 1\right\}, \quad \forall x \in \mathcal{Z}_\sigma(s,p).$$

It is easy to see that the space $\mathcal{Z}_\sigma(s,p)$ is a Banach space with respect to the Luxemburg norm.

Throughout the paper, suppose that $p = (p_r)$ is bounded with $p_r > 1$ for all $r \in \mathbb{N}$ and

$$\begin{aligned} e_i &= (\overbrace{0,0,\ldots,0}^{i-1},1,0,0,0,\ldots), \\ x_{|_i} &= (x(1),x(2),x(3),\ldots,x(i),0,0,0,\ldots), \\ x_{|_{\mathbb{N}-i}} &= (0,0,0,\ldots,x(i+1),x(i+2),\ldots), \end{aligned}$$

for $i \in \mathbb{N}$ and $x \in \ell^0$. In addition, we will require the following inequalities:

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}), \qquad |a_k + b_k|^{t_k} \leq |a_k|^{t_k} + |b_k|^{t_k},$$

where $t_k = \frac{p_k}{M} \leq 1$ and $C = \max\{1, 2^{H-1}\}$ with $H = \sup p_k$.

3 Main results

Since ℓ_p is reflexive and convex, $\ell(p)$-type spaces have many useful applications, and it is natural to consider a geometric structure of these spaces. From this point of view, we generalized the space $C(s,p)$ by using the Zweier operator and then obtained the equality $\mathcal{Z}_\sigma(s,p) = Ces(p)$, that is, it was seen that the structure of the space $Ces(p)$ was preserved. In this section, our goal is to investigate a geometric structure of the modular space $\mathcal{Z}_\sigma(s,p)$ related to the fixed point theory. For this, we will examine property (β) and the uniform Opial property for $\mathcal{Z}_\sigma(s,p)$. Finally, we will give some fixed point results. To do this, we need some results which are important in our opinion.

Lemma 3.1 [2] *If $\sigma \in \delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left|\sigma(u+v) - \sigma(u)\right| < \varepsilon,$$

where $u, v \in X_\sigma$ with $\sigma(u) \leq L$ and $\sigma(v) \leq \delta$.

Lemma 3.2 [2] *If $\sigma \in \delta_2^s$, convergence in norm and in modular are equivalent in X_σ.*

Lemma 3.3 [2] *If $\sigma \in \delta_2^s$, then for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geq 1 + \delta$ implies $\sigma(x) \geq 1 + \varepsilon$.*

Now we give the following two lemmas without proof.

Lemma 3.4 *If $\|x\|_L < 1$ for any $x \in \mathcal{Z}_\sigma(s,p)$, then $\sigma(x) \leq \|x\|_L$.*

Lemma 3.5 *For any $x \in \mathcal{Z}_\sigma(s,p)$, $\|x\|_L = 1$ if and only if $\sigma(x) = 1$.*

Lemma 3.6 *If $\liminf p_r > 1$, then for any $x \in \mathcal{Z}_\sigma(s,p)$, there exist $k_0 \in \mathbb{N}$ and $\mu \in (0,1)$ such that*

$$\sigma\left(\frac{x^k}{2}\right) \leq \frac{1-\mu}{2}\sigma\left(x^k\right)$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, where $x^k = (\overbrace{0,0,\ldots,0}^{k-1}, \sum_{2^r \leq i \leq k} |x(i)|, x(k+1), x(k+2), \ldots)$ and $2^r \leq k < 2^{r+1}$.

Proof Let $k \in \mathbb{N}$ be fixed. Then there exists $r_k \in \mathbb{N}$ such that $k \in I_{r_k}$. Let γ be a real number $1 < \gamma \leq \liminf p_r$, and so there exists $k_0 \in \mathbb{N}$ such that $\gamma < p_{r_k}$ for all $k \geq k_0$. Choose $\mu \in (0,1)$ such that $(\frac{1}{2})^\gamma \leq \frac{1-\mu}{2}$. Therefore, we have

$$\begin{aligned}
\sigma\left(\frac{x^k}{2}\right) &= \sum_{r=0}^{\infty}\left(2^{-r}\sum_{r} k^{-s}\left|\frac{\alpha x(k) + (1-\alpha)x(k-1)}{2}\right|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^{p_r}\left(2^{-r}\sum_{r} k^{-s}\left|\alpha x(k) + (1-\alpha)x(k-1)\right|\right)^{p_r} \\
&\leq \left(\frac{1}{2}\right)^{\gamma}\sum_{r=0}^{\infty}\left(2^{-r}\sum_{r} k^{-s}\left|\alpha x(k) + (1-\alpha)x(k-1)\right|\right)^{p_r} \\
&< \frac{1-\mu}{2}\sigma\left(x^k\right)
\end{aligned}$$

for each $x \in \mathcal{Z}_\sigma(s,p)$ and $k \geq k_0$. □

Lemma 3.7 *If $\sigma \in \delta_2^s$, then for any $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\sigma(x) \leq 1-\varepsilon$ implies $\|x\| \leq 1-\delta$.*

Proof Suppose that lemma does not hold. So, there exist $\varepsilon > 0$ and $x_n \in \mathcal{Z}_\sigma(s,p)$ such that $\sigma(x_n) < 1-\varepsilon$ and $\frac{1}{2} \leq \|x_n\| \to 1$. Take $s_n = \frac{1}{\|x_n\|-1}$, and so $s_n \to 0$ as $n \to \infty$. Let $P = \sup\{\sigma(2x_n) : n \in \mathbb{N}\}$. There exists $D \geq 2$ such that

$$\sigma(2u) \leq D\sigma(u) + 1 \tag{3.1}$$

for every $u \in \mathcal{Z}_\sigma(s,p)$ with $\sigma(u) < 1$, since $\sigma \in \delta_2^s$. We have

$$\sigma(2x_n) \leq D\sigma(x_n) + 1 < D + 1$$

for all $n \in \mathbb{N}$ by (3.1). Therefore, $0 < P < \infty$ and from Lemma 3.5 we have

$$\begin{aligned} 1 = \sigma\left(\frac{x_n}{\|x_n\|}\right) &= \sigma\left(2s_n x_n + (1-s_n)x_n\right) \\ &\leq s_n\sigma(2x_n) + (1-s_n)\sigma(x_n) \\ &\leq s_n P + (1-\varepsilon) \to (1-\varepsilon). \end{aligned}$$

This is a contradiction. So, the proof is complete. □

Theorem 3.8 *The space $\mathcal{Z}_\sigma(s,p)$ has property (β).*

Proof Let $\varepsilon > 0$ and $(x_n) \subset B(\mathcal{Z}_\sigma(s,p))$ with $sep(x_n) \geq \varepsilon$ and $x \in B(\mathcal{Z}_\sigma(s,p))$. For each $l \in \mathbb{N}$, we can find $r_k \in \mathbb{N}$ such that $2^{r_k} \leq l < 2^{r_k+1}$. Let

$$x_n^l = \Bigl(\overbrace{0,0,\dots,0}^{l-1}, \sum_{2^{r_k} \leq i \leq l} |x(i)|, x_n(l+1), x_n(l+2), \dots\Bigr).$$

Since for each $i \in \mathbb{N}$, $(x_n(i))_{i=1}^\infty$ is bounded, by using the diagonal method, we can find a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathbb{N}$ with $1 \leq i \leq l$. Therefore, there exists an increasing sequence of positive integers t_l such that $sep((x_{n_j}^l)_{j \geq t_l}) \geq \varepsilon$. Thus, there exists a sequence of positive integers $(r_l)_{l=1}^\infty$ with $r_1 < r_2 < \cdots$ such that $\|x_{r_l}^l\| \geq \frac{\varepsilon}{2}$ for all $l \in \mathbb{N}$. Since $\sigma \in \delta_2^s$, there is $\eta > 0$ such that

$$\sigma\left(x_{r_l}^l\right) \geq \eta \quad \text{for all } l \in \mathbb{N} \tag{3.2}$$

from Lemma 3.3. However, there exist $k_0 \in \mathbb{N}$ and $\mu \in (0,1)$ such that

$$\sigma\left(\frac{\nu^k}{2}\right) \leq \frac{1-\mu}{2}\sigma\left(\nu^k\right) \tag{3.3}$$

for all $\nu \in \mathcal{Z}_\sigma(s,p)$ and $k \geq k_0$ by Lemma 3.6. There exists $\delta > 0$ such that

$$\sigma(y) \leq 1 - \frac{\mu\eta}{4} \quad \Rightarrow \quad \|y\| \leq 1-\delta \tag{3.4}$$

for any $y \in \mathcal{Z}_\sigma(s,p)$ by Lemma 3.7.

By Lemma 3.1, there exists δ_0 such that

$$\left|\sigma(u+v)-\sigma(u)\right| < \frac{\mu\eta}{4}, \tag{3.5}$$

where $\sigma(u) \leq 1$ and $\sigma(v) \leq \delta_0$. Hence, we get that $\sigma(x) \leq 1$ since $x \in B(\mathcal{Z}_\sigma(s,p))$. Then there exists $k \geq k_0$ such that $\sigma(x^k) \leq \delta_0$. Let $u = x^l_{r_l}$ and $v = x^l$. Then

$$\sigma\left(\frac{u}{2}\right) < 1 \quad \text{and} \quad \sigma\left(\frac{v}{2}\right) < \delta_0.$$

We obtain from (3.3) and (3.5) that

$$\sigma\left(\frac{u+v}{2}\right) \leq \sigma\left(\frac{u}{2}\right) + \frac{\mu\eta}{4} \leq \frac{1-\mu}{2}\sigma(u) + \frac{\mu\eta}{4}. \tag{3.6}$$

Choose $s_i = r_{l_i}$. By the inequalities (3.2), (3.3), (3.6) and the convexity of the function $f(u) = |u|^{p_r}$, we have

$$\begin{aligned}
\sigma\left(\frac{x+x_{s_k}}{2}\right) &= \sum_{r=0}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\frac{\alpha(x(k)+x_{s_i}(k))+(1-\alpha)(x(k-1)+x_{s_i}(k-1))}{2}\right|\right)^{p_r} \\
&= \sum_{r=0}^{r_k-1}\left(2^{-r}\sum_r k^{-s}\left|\frac{\alpha(x(k)+x_{s_i}(k))+(1-\alpha)(x(k-1)+x_{s_i}(k-1))}{2}\right|\right)^{p_r} \\
&\quad + \sum_{r=r_k}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\frac{\alpha(x(k)+x_{s_i}(k))+(1-\alpha)(x(k-1)+x_{s_i}(k-1))}{2}\right|\right)^{p_r} \\
&\leq \frac{1}{2}\sum_{r=0}^{r_k-1}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r} \\
&\quad + \frac{1}{2}\sum_{r=0}^{r_k-1}\left(2^{-r}\sum_r k^{-s}\left|\alpha x_{s_i}(k)+(1-\alpha)x_{s_i}(k-1)\right|\right)^{p_r} \\
&\quad + \sum_{r=r_k}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\frac{\alpha x_{s_i}(k)+(1-\alpha)x_{s_i}(k-1)}{2}\right|\right)^{p_r} + \frac{\mu\eta}{4} \\
&\leq \frac{1}{2}\sum_{r=0}^{r_k-1}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r} \\
&\quad + \frac{1}{2}\sum_{r=0}^{r_k-1}\left(2^{-r}\sum_r k^{-s}\left|\alpha x_{s_i}(k)+(1-\alpha)x_{s_i}(k-1)\right|\right)^{p_r} \\
&\quad + \frac{1-\mu}{2}\sum_{r=r_k}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\frac{\alpha x_{s_i}(k)+(1-\alpha)x_{s_i}(k-1)}{2}\right|\right)^{p_r} + \frac{\mu\eta}{4} \\
&\leq \frac{1}{2}\sum_{r=0}^{r_k-1}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r} \\
&\quad + \frac{1}{2}\sum_{r=0}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\alpha x_{s_i}(k)+(1-\alpha)x_{s_i}(k-1)\right|\right)^{p_r}
\end{aligned}$$

$$\begin{aligned}
&-\frac{\mu}{2}\sum_{r=r_k}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\frac{\alpha x_{s_i}(k)+(1-\alpha)x_{s_i}(k-1)}{2}\right|\right)^{p_r}+\frac{\mu\eta}{4}\\
&\leq \frac{1}{2}+\frac{1}{2}-\frac{\mu\eta}{2}+\frac{\mu\eta}{4}\\
&=1-\frac{\mu\eta}{4}.
\end{aligned}$$

So, the inequality (3.4) implies that $\|\frac{x+x_{s_k}}{2}\|\leq 1-\delta$. Consequently, the space $\mathcal{Z}_\sigma(s,p)$ possesses property (β). □

Since property (β) implies NUC, NUC implies property (D) and property (D) implies reflexivity, we can give the following result from Theorem 3.8.

Corollary 3.9 *The space $\mathcal{Z}_\sigma(s,p)$ is nearly uniform convex, reflexive and also it has property (D).*

Theorem 3.10 *The space $\mathcal{Z}_\sigma(s,p)$ has the uniform Opial property.*

Proof Let $\varepsilon>0$ and $x\in\mathcal{Z}_\sigma(s,p)$ be such that $\|x\|\geq\varepsilon$ and (x_n) be a weakly null sequence in $S(\mathcal{Z}_\sigma(s,p))$. By $\sigma\in\delta_2^s$, there exists $\zeta\in(0,1)$ independent of x such that $\sigma(x)>\zeta$ by Lemma 3.2. Also since $\sigma\in\delta_2^s$, by Lemma 3.1, there is $\zeta_1\in(0,\zeta)$ such that

$$\left|\sigma(y+z)-\sigma(y)\right|<\frac{\zeta}{4}\tag{3.7}$$

whenever $\sigma(y)\leq 1$ and $\sigma(z)\leq\zeta_1$. Take $r_0\in\mathbb{N}$ such that

$$\sum_{r=r_0+1}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r}<\frac{\zeta_1}{4}.\tag{3.8}$$

Hence, we have

$$\begin{aligned}
\zeta &< \sum_{r=1}^{r_0}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r}\\
&\quad+\sum_{r=r_0+1}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r}\\
&\leq \sum_{r=1}^{r_0}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r}+\frac{\zeta_1}{4}
\end{aligned}\tag{3.9}$$

and this implies that

$$\begin{aligned}
\sum_{r=1}^{r_0}\left(2^{-r}\sum_r k^{-s}\left|\alpha x(k)+(1-\alpha)x(k-1)\right|\right)^{p_r} &> \zeta-\frac{\zeta_1}{4}\\
&> \zeta-\frac{\zeta}{4}\\
&= \frac{3\zeta}{4}.
\end{aligned}\tag{3.10}$$

Since $x_n \to^w 0$, by the inequality (3.10), there exists $r_0 \in \mathbb{N}$ such that

$$\sum_{r=1}^{r_0}\left(2^{-r}\sum_r k^{-s}\left|\alpha\left(x_n(k)+x(k)\right)+(1-\alpha)\left(x_n(k-1)+x(k-1)\right)\right|\right)^{p_r} > \frac{3\zeta}{4}. \tag{3.11}$$

Again, by $x_n \to^w 0$, there is $r_1 > r_0$ such that for all $r > r_1$

$$\|x_{n_{|r_0}}\| < 1-\left(1-\frac{\zeta}{4}\right)^{1/M}, \tag{3.12}$$

where $p_r \leq M \in \mathbb{N}$ for all $r \in \mathbb{N}$. Therefore, we obtain that

$$\|x_{n_{|\mathbb{N}-r_0}}\| > \left(1-\frac{\zeta}{4}\right)^{1/M} \tag{3.13}$$

by the triangle inequality of the norm. It follows from the definition of the Luxemburg norm that

$$\begin{aligned} 1 &\leq \sigma\left(\frac{x_{n_{|\mathbb{N}-r_0}}}{(1-\frac{\zeta}{4})^{1/M}}\right) \\ &= \sum_{r=r_0+1}^{\infty}\left(\frac{2^{-r}\sum_r k^{-s}|\alpha x_n(k)+(1-\alpha)x_n(k-1)|}{(1-\frac{\zeta}{4})^{1/M}}\right)^{p_r} \\ &\leq \left(\frac{1}{(1-\frac{\zeta}{4})^{1/M}}\right)^{M}\sum_{r=r_0+1}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\alpha x_n(k)+(1-\alpha)x_n(k-1)\right|\right)^{p_r} \end{aligned} \tag{3.14}$$

and this implies that

$$\sum_{r=r_0+1}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\alpha x_n(k)+(1-\alpha)x_n(k-1)\right|\right)^{p_r} \geq 1-\frac{\zeta}{4}. \tag{3.15}$$

By (3.7), (3.8), (3.11), (3.15) and since $x_n \to^w 0 \Rightarrow x_n \to 0$ (coordinatewise), we have for any $r > r_1$ that

$$\begin{aligned} \sigma(x_n+x) &= \sum_{r=1}^{r_0}\left(2^{-r}\sum_r k^{-s}\left|\alpha\left(x_n(k)+x(k)\right)+(1-\alpha)\left(x_n(k-1)+x(k-1)\right)\right|\right)^{p_r} \\ &\quad+\sum_{r=r_0+1}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\alpha\left(x_n(k)+x(k)\right)+(1-\alpha)\left(x_n(k-1)+x(k-1)\right)\right|\right)^{p_r} \\ &\geq \sum_{r=r_0+1}^{\infty}\left(2^{-r}\sum_r k^{-s}\left|\alpha\left(x_n(k)+x(k)\right)+(1-\alpha)\left(x_n(k-1)+x(k-1)\right)\right|\right)^{p_r} \\ &\quad-\frac{\zeta}{4}+\frac{3\zeta}{4} \\ &\geq \frac{3\zeta}{4}+\left(1-\frac{\zeta}{4}\right)-\frac{\zeta}{4} \\ &= 1+\frac{\zeta}{4}. \end{aligned}$$

Since $\sigma \in \delta_2^s$, it follows from Lemma 3.3 that there is τ depending on ζ only such that $\|x_n + x\| \geq 1 + \tau$. □

Corollary 3.11 *The space $\mathcal{Z}_\sigma(s,p)$ has property (L) and the fixed point property.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Received: 14 December 2012 Accepted: 5 June 2013 Published: 25 June 2013

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doi:10.1186/1687-1812-2013-165
Cite this article as: Et et al.: Some geometric properties of a new modular space defined by Zweier operator. *Fixed Point Theory and Applications* 2013 **2013**:165.