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Common fixed point theorems for generalized contractive mappings with applications

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Abstract

Certain common fixed point results involving four mappings satisfying generalized contractive conditions on a cone metric type space are obtained. Our results substantially improve and extend a number of known results. An example is given in support of the new results developed here. As an application, we establish the existence of a solution for an implicit integral equation.

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1 Introduction

The Banach contraction principle is a remarkable result in the metric fixed point theory. Over the years, it has been generalized in different directions and spaces by several mathematicians (see [1–44]). In 1972, Zamfirescu [44] defined Z-operators on metric spaces: There exist real numbers a, b and c satisfying $0 < a < 1$, $0 < b, c < \frac{1}{2}$ such that for each $x, y \in X$, one of the following holds:

(Z_1) $d(Tx, Ty) \leq ad(x, y)$;
(Z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
(Z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

In 1974, Ćirić [11] defined quasi-contraction (C^q) if for some $0 \leq h < 1$ and all $x, y \in X$,

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{1.1}$$

Clearly, the Zamfirescu operator is quasi-contractive.

In 1983, Naimpally and Singh [33] defined a generalized contractive operator as follows:

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx)\} \tag{1.2}$$

for some $0 \leq h < 1$ and for all $x, y \in X$.

A generalized contractive operator in (1.2) is more general than quasi-contraction.

In [25, 29] the authors investigated quasi-contractions on cone metric spaces and obtained fixed point theorems for such mappings under a different condition. Recently

Cvetković *et al.* [14] and Hussain and Shah [22] introduced the notion of a cone metric type space, which is a generalization of a cone metric space, and proved some common fixed point theorems and KKM-mappings results respectively.

In this paper we consider mappings on a cone metric type space (CMTS) with a solid cone. We prove new common fixed point theorems involving four mappings satisfying a generalized contractive and a generalized nonexpansive conditions. Our results are generalization of theorems proved in [14, 19, 27, 40, 43] and some other authors. Almost all of our results are proved without the assumption of the continuity of mappings.

The paper is organized as follows. In Section 2 we review some definitions and well-known results which will be needed in the sequel. In Section 3 and 4 we prove the existence of common fixed points of four mappings. We also present some corollaries which establish the fact that our results are generalization of several recent results in the literature. In Section 5, we present an application to integral equations to illustrate the usability of the obtained results.

2 Definitions and notation

Let E be a real Banach space and P a subset of E. We use the symbol 0 to denote the zero element of E and the symbol $\operatorname{int} P$ to denote the interior of P. The subset P is called a cone if and only if:

(I) P is closed, nonempty and $P \neq \{0\}$;
(II) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ such that $ax + by \in P$;
(III) $P \cap (-P) = \{0\}$.

For a given cone P, a partial ordering $\leq$ with respect to P is introduced as follows: $x \leq y$ if and only if $y - x \in P$; $x < y$ means $x \leq y$ but $x \neq y$; if $y - x \in \operatorname{int} P$, one writes $x \ll y$; if $\operatorname{int} P \neq \emptyset$, the cone P is called a solid cone.

In this paper we always suppose that E is a real Banach space, $P \subseteq E$ is a solid cone and $\leq$ is partial ordering with respect to P.

Definition 2.1 Let X be a nonempty set and E be a real Banach space with a cone P. A vector-valued function $d : X \times X \to E$ is said to be a cone metric type function on X with the constant $k \geq 1$ if:

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, y) \leq k[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a cone metric type space (CMTS).

Remark 2.2 For $k = 1$, the above definition of CMTS reduces to that of a cone metric space introduced in [19].

Definition 2.3 Let (X, d) be a CMTS and $\{x_n\}$ be a sequence in X:

(I) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We write $\lim_{n\to\infty} x_n = x$, or $x_n \to x$ as $n \to \infty$.
(II) If for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$, then $\{x_n\}$ is called a Cauchy sequence in X.
(III) If every Cauchy sequence is convergent in X, then X is called a complete CMTS.

Let E be an ordered real Banach space with a cone P. The following properties hold ($a, b, c \in E$):

(C_1) If $a \leq b$ and $b \ll c$, then $a \ll c$;
(C_2) If $0 \leq a \ll c$ for all $c \in \operatorname{int} P$, then $a = 0$;
(C_3) If $a \leq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = 0$;
(C_4) Let $x_n \to 0$ in E and $0 \ll c$.

Then there exists a positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

Definition 2.4 Let $F, G : X \to X$ be mappings on a set X:

(i) If $y = Fx = Gx$ for some $x \in X$, then x is called a coincidence point of F and G, and y is called a point of coincidence of F and G;
(ii) The pair $\{F, G\}$ is called weakly compatible if F and G commute at all of their coincidence points, that is, $FGx = GFx$ for all $x \in C(F, G) = \{x \in X : Fx = Gx\}$.
(iii) The pair $\{F, G\}$ is called occasionally weakly compatible (in brief OWC) [28] if $FGx = GFx$ for some $x \in C(F, G)$.

3 Generalized contractive operators

First we show the existence of a unique fixed point.

Theorem 3.1 *Let (X, d) be a cone metric type space with the constant $k \geq 1$ and let P be a solid cone. Let $T : X \to X$ be a mapping satisfying the contractive condition*

$$d(Tx, Ty) \leq \frac{\lambda}{k} m(x, y), \tag{3.1}$$

where

$$m(x, y) \in k\bigl\{d(x, y), d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\bigr\}$$

for all $x, y \in X$ and for some constant $\lambda \in (0, \frac{1}{k+k^2})$. If TX is complete, then T has a unique fixed point in X.

Proof For $x_0 \in X$, we construct a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n \quad \text{for } n \geq 0.$$

We show that $\{x_n\}$ is a Cauchy sequence in X. For $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \frac{\lambda}{k} m(x_n, x_{n-1}),$$

where

$$\begin{aligned} m(x_n, x_{n-1}) &\in k\bigl\{d(x_n, x_{n-1}), d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)\bigr\} \\ &= k\bigl\{d(x_n, x_{n-1}), d(x_n, x_{n+1}) + d(x_{n-1}, x_n), d(x_n, x_n) + d(x_{n-1}, x_{n+1})\bigr\} \\ &= k\bigl\{d(x_n, x_{n-1}), d(x_n, x_{n+1}) + d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})\bigr\}. \end{aligned}$$

Thus we have the following three cases:

(i) $d(x_{n+1},x_n) \leq \lambda d(x_n,x_{n-1})$;
(ii) $d(x_{n+1},x_n) \leq \lambda[d(x_n,x_{n+1}) + d(x_{n-1},x_n)] \leq (\frac{\lambda}{1-\lambda})d(x_{n-1},x_n)$;
(iii) $d(x_{n+1},x_n) \leq \lambda d(x_{n-1},x_{n+1}) \leq \lambda k[d(x_{n-1},x_n) + d(x_n,x_{n+1})] \leq (\frac{\lambda k}{1-\lambda k})d(x_{n-1},x_n)$.

Let $\alpha = \max\{\lambda, \frac{\lambda}{1-\lambda}, \frac{\lambda k}{1-\lambda k}\}$. Then $\alpha = \max\{\lambda, \frac{\lambda}{1-\lambda}, \frac{k\lambda}{1-k\lambda}\} = \frac{k\lambda}{1-k\lambda}$ and so $k\alpha < 1$ if and only if $\frac{k^2\lambda}{1-k\lambda} < 1$ if and only if $\lambda \in (0, \frac{1}{k+k^2})$.

Thus

$$d(x_{n+1},x_n) \leq \alpha d(x_{n-1},x_n) \leq \cdots \leq \alpha^n d(x_1,x_0).$$

Since $k \geq 1$, for $m > n$, we have

$$\begin{aligned} d(x_n,x_m) &\leq kd(x_n,x_{n+1}) + k^2 d(x_{n+1},x_{n+2}) + \cdots + k^{m-n-1} d(x_{m-1},x_m) \\ &\leq \left(k\alpha^n + k^2\alpha^{n+1} + \cdots + k^{m-n-1}\alpha^{m-1}\right)d(x_1,x_0) \\ &\leq \left(\frac{k}{1-k\alpha}\right)\alpha^n d(x_1,x_0) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Now, by (C_4) and (C_1), it follows that for every $c \in \operatorname{int} P$, there exists a positive integer n_0 such that $d(x_n,x_m) \ll c$ for every $m > n > n_0$, so $\{x_n\}$is a Cauchy sequence. Since TX is a complete subspace of X, there exists $p \in TX$ such that $\lim_{n\to\infty} x_n = p = \lim_{n\to\infty} Tx_n$. We show that p is a fixed point of T. Consider

$$\begin{aligned} d(p,Tp) &\leq k\bigl(d(p,x_{n+1}) + d(x_{n+1},Tp)\bigr) = kd(p,x_{n+1}) + kd(Tx_n,Tp) \\ &\leq kd(p,x_{n+1}) + k\frac{\lambda}{k} m(x_n,p), \end{aligned}$$

where

$$m(x_n,p) \in k\bigl\{d(x_n,p), d(x_n,x_{n+1}) + d(p,Tp), d(x_n,Tp) + d(p,x_{n+1})\bigr\}.$$

Let $0 \ll c$. Then at least one of the following three cases holds:

1.

$$d(p,Tp) \leq kd(p,x_{n+1}) + \lambda kd(x_n,p) \ll k\frac{c}{2k} + \lambda k\frac{c}{2\lambda k} = c,$$

whenever $n \geq K_1$.

2.

$$d(p,Tp) \leq kd(p,x_{n+1}) + \lambda kd(x_n,x_{n+1}) + \lambda kd(p,Tp)$$

implies

$$\begin{aligned} d(p,Tp) &\leq \frac{k}{1-\lambda k}d(p,x_{n+1}) + \frac{\lambda k}{1-\lambda k}d(x_n,x_{n+1}) \\ &\ll \frac{k}{1-\lambda k}\frac{1-\lambda k}{2k}c + \frac{\lambda k}{1-\lambda k}\frac{1-\lambda k}{2\lambda k}c \\ &= c, \end{aligned}$$

whenever $n \geq K_2$.

3.

$$\begin{aligned} d(p,Tp) &\le kd(p,x_{n+1}) + \lambda kd(x_n,Tp) + \lambda kd(p,x_{n+1}) \\ &= kd(p,x_{n+1}) + \lambda kd(p,x_{n+1}) + \lambda kd(x_n,Tp) \\ &\le kd(p,x_{n+1}) + \lambda kd(p,x_{n+1}) + \lambda k^2 d(x_n,p) + \lambda k^2 d(p,Tp) \end{aligned}$$

or

$$\begin{aligned} d(p,Tp) &\le \frac{k}{1-\lambda k^2} d(p,x_{n+1}) + \frac{\lambda k}{1-\lambda k^2} d(p,x_{n+1}) + \frac{\lambda k^2}{1-\lambda k^2} d(x_n,p) \\ &\ll \frac{k+\lambda k}{1-\lambda k^2} \frac{1-\lambda k^2}{2(k+\lambda k)} c + \frac{\lambda k^2}{1-\lambda k^2} \frac{1-\lambda k^2}{2\lambda k^2} c \\ &= c, \end{aligned}$$

whenever $n \ge K_3$.

Hence, for $n \ge \max\{K_1, K_2, K_3\}$, $d(p,Tp) \le c$.

Thus $Tp = p$. Suppose that there exists $q \in X$ such that $Tq = q$; then from (3.1), we have

$$d(p,q) = d(Tp,Tq) \le \frac{\lambda}{k} m(p,q),$$

where

$$m(p,q) \in k\bigl\{d(p,q), d(p,Tp) + d(q,Tq), d(p,Tq) + d(q,Tp)\bigr\} = k\bigl\{d(p,q), 2d(p,q)\bigr\}.$$

Thus we have the following two cases:

(i) $d(p,q) = d(Tp,Tq) \le \lambda d(p,q)$, and

(ii) $d(Tp,Tq) \le 2\lambda d(p,q)$.

Since $\lambda \in (0, \frac{1}{k+k^2})$, from both the cases we get $p = q$. □

Remark 3.2 In Theorem 3.1 we have generalized and unified the fixed point theorems of Huang and Zhang [19] and corresponding results in [40].

Now we present some new common fixed point theorems involving four mappings defined on a cone metric type space.

Theorem 3.3 *Let (X,d) be a cone metric type space with the constant $k \ge 1$ and let P be a solid cone. Suppose that the self-mappings $F, G, S, T : X \to X$ are such that $SX \subset GX$ and $TX \subset FX$. Let $x_0 \in X$, let the sequences $\{x_n\}$ and $\{z_n\}$ be defined by $z_{2n} = Gx_{2n+1} = Sx_{2n}$ and $z_{2n+1} = Fx_{2n+2} = Tx_{2n+1}$ for $n \ge 0$ and let $\mathcal{O}(x_0) = \{z_n : n \ge 0\}$. Assume that there exists $x_0 \in X$ such that for some constant $\lambda \in (0, \frac{1}{k+k^2})$,*

$$d(Sx,Ty) \le \frac{\lambda}{k} m(x,y), \tag{3.2}$$

where

$$m(x,y) \in k\bigl\{d(Fx,Gy), d(Fx,Sx) + d(Gy,Ty), d(Fx,Ty) + d(Sx,Gy)\bigr\}$$

for all $x, y \in F^{-1}\{\overline{\mathcal{O}(x_0)}\} \cup G^{-1}\{\overline{\mathcal{O}(x_0)}\}$. *If one of* SX, TX, FX, *or* GX *is a complete subspace of* X, *then* $\{S, F\}$ *and* $\{T, G\}$ *have a point of coincidence in* X.

Proof We consider two cases of an odd integer n and of an even n. For $n = 2l + 1$, $l \in \mathbb{N}$, we have $d(z_{2l+1}, z_{2l+2}) = d(Sx_{2l+2}, Tx_{2l+1})$, and from (3.2) there exists some

$$\begin{aligned} & m(x_{2l+2}, x_{2l+1}) \\ & \quad \in k\bigl\{d(Fx_{2l+2}, Gx_{2l+1}), d(Fx_{2l+2}, Sx_{2l+2}) + d(Gx_{2l+1}, Tx_{2l+1}), \\ & \qquad d(Fx_{2l+2}, Tx_{2l+1}) + d(Sx_{2l+2}, Gx_{2l+1})\bigr\} \\ & \quad = k\bigl\{d(z_{2l+1}, z_{2l}), d(z_{2l+1}, z_{2l+2}) + d(z_{2l}, z_{2l+1}), d(z_{2l+2}, z_{2l})\bigr\} \end{aligned}$$

such that

$$d(z_{2l+1}, z_{2l+2}) = d(Sx_{2l+2}, Tx_{2l+1}) \le \left(\frac{\lambda}{k}\right) m(x_{2l+2}, x_{2l+1}).$$

Thus we have the following three cases:

(i) $d(z_{2l+1}, z_{2l+2}) \le \lambda d(z_{2l+1}, z_{2l})$;

(ii) $d(z_{2l+1}, z_{2l+2}) \le \lambda d(z_{2l+1}, z_{2l+2}) + d(z_{2l}, z_{2l+1}) \le (\frac{\lambda}{1-\lambda}) d(z_{2l+1}, z_{2l})$;

(iii) $d(z_{2l+1}, z_{2l+2}) \le \lambda d(z_{2l+2}, z_{2l}) \le \lambda k[d(z_{2l+2}, z_{2l+1}) + d(z_{2l+1}, z_{2l})] \le (\frac{\lambda k}{1-\lambda k}) d(z_{2l+1}, z_{2l})$.

Let $\alpha = \max\{\lambda, \frac{\lambda}{1-\lambda}, \frac{\lambda k}{1-\lambda k}\}$. It is easy to see that $\alpha, k\alpha \in (0,1)$, such that

$$d(z_{n+1}, z_n) \le \alpha d(z_n, z_{n-1}), \quad n \ge 1. \tag{3.3}$$

For $n = 2l$, $l \in \mathbb{N}$, we have

$$d(z_{2l}, z_{2l+1}) = d(Sx_{2l}, Tx_{2l+1}) \le \frac{\lambda}{k} m(x_{2l}, x_{2l+1}), \tag{3.4}$$

where

$$\begin{aligned} & m(x_{2l}, x_{2l+1}) \\ & \quad \in k\bigl\{d(Fx_{2l}, Gx_{2l+1}), d(Fx_{2l}, Sx_{2l}) + d(Gx_{2l+1}, Tx_{2l+1}), \\ & \qquad d(Fx_{2l}, Tx_{2l+1}) + d(Sx_{2l}, Gx_{2l+1})\bigr\} \\ & \quad = k\bigl\{d(z_{2l-1}, z_{2l}), d(z_{2l-1}, z_{2l}) + d(z_{2l}, z_{2l+1}), d(z_{2l-1}, z_{2l+1})\bigr\}. \end{aligned}$$

Thus we have the following three cases:

(i) $d(z_{2l}, z_{2l+1}) \le \lambda d(z_{2l-1}, z_{2l})$;

(ii) $d(z_{2l}, z_{2l+1}) \le \lambda[d(z_{2l-1}, z_{2l}) + d(z_{2l}, z_{2l+1})] \le (\frac{\lambda}{1-\lambda}) d(z_{2l-1}, z_{2l})$;

(iii) $d(z_{2l}, z_{2l+1}) \le \lambda d(z_{2l-1}, z_{2l+1}) \le \lambda k[d(z_{2l-1}, z_{2l}) + d(z_{2l}, z_{2l+1})] \le (\frac{\lambda k}{1-\lambda k}) d(z_{2l-1}, z_{2l})$.

So, inequality (3.3) is satisfied in this case, too. Therefore, (3.3) is satisfied for all $n \in \mathbb{N}$, and by iterating we get

$$d(z_{n+1}, z_n) \le \alpha^n d(z_1, z_0). \tag{3.5}$$

Since $k \geq 1$, for $m > n$ we have

$$\begin{aligned} d(z_n, z_m) &\leq kd(z_n, z_{n+1}) + k^2 d(z_{n+1}, z_{n+2}) + \cdots + K^{m-n-1} d(z_{m-1}, z_m) \\ &\leq \left(k\alpha^n + k^2\alpha^{n+1} + \cdots + K^{m-n-1}\alpha^{m-1}\right) d(z_1, z_0) \\ &\leq \left(\frac{k}{1-k\alpha}\right)\alpha^n d(z_1, z_0) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Now, by (C_4) and (C_1), it follows that for every $c \in \operatorname{int} P$ there exists a positive integer n_0 such that $d(z_n, z_m) \ll c$ for every $m > n > n_0$, so $\{z_n\}$is a Cauchy sequence.

Let us suppose that SX is a complete subspace of X, then there exists $z \in SX$ such that $\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} Sx_{2n} = z$. Then we have

$$\lim_{n\to\infty} Gx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z,$$

that is, for any $0 \ll c$, for sufficient large n, we have $d(z_n, z) \ll c$. Since $z \in SX \subset GX$, then there exists $y \in X$ such that $z = Gy$. Let us prove that $z = Ty$. From (d_3) and (3.2), we get

$$\begin{aligned} d(Ty, z) &\leq k\left[d(Ty, Sx_{2n}) + d(Sx_{2n}, z)\right] \\ &\leq \lambda m(x_{2n}, y) + kd(z_{2n}, z), \end{aligned}$$

where

$$\begin{aligned} m(x_{2n}, y) &\in k\left[d(Fx_{2n}, Gy), d(Fx_{2n}, Sx_{2n}) + d(Gy, Ty), d(Fx_{2n}, Ty) + d(Sx_{2n}, Gy)\right] \\ &= k\left[d(z_{2n-1}, z), d(z_{2n-1}, z_{2n}) + d(z, Ty), d(z_{2n-1}, Ty) + d(z_{2n}, z)\right]. \end{aligned}$$

Therefore we have three cases:

(i) $d(Ty, z) \leq k\lambda d(z_{2n-1}, z) + kd(z_{2n}, z) \ll k\lambda\frac{c}{2k\lambda} + k\frac{c}{2k} = c$ as $n \to \infty$;

(ii) $d(Ty, z) \leq k\lambda[d(z_{2n-1}, z_{2n}) + d(z, Ty)] + kd(z_{2n}, z) \ll \frac{k}{1-\lambda k}[\lambda\frac{1-\lambda k}{2k\lambda}c + \frac{1-\lambda k}{2k}c] = c$ as $n \to \infty$ (since $1 \leq k \leq 2$ and $\lambda \in (0, \frac{1}{k+k^2})$, we have $\lambda < \frac{1}{k}$, and therefore $1 - \lambda k > 0$);

(iii)

$$\begin{aligned} d(Ty, z) &\leq \lambda k\left[d(z_{2n-1}, Ty) + d(z_{2n}, z)\right] + kd(z_{2n}, z) \\ &\leq \lambda k\left[k\left\{d(z_{2n-1}, z) + d(z, Ty)\right\} + d(z_{2n}, z)\right] + kd(z_{2n}, z) \\ &\leq \frac{k}{1-\lambda k^2}\left[\lambda kd(z_{2n-1}, z) + \lambda d(z_{2n}, z) + d(z_{2n}, z)\right] \\ &\ll \frac{k}{1-\lambda k^2}\left[\lambda k\frac{1-\lambda k^2}{4\lambda k^2}c + \lambda\frac{1-\lambda k^2}{4\lambda k}c + \frac{1-\lambda k^2}{2k}\right] = c \quad \text{as } n \to \infty \end{aligned}$$

(since $k \geq 1$ and $\lambda \in (0, \frac{1}{k+k^2})$, we have $\lambda < \frac{1}{k+k^2} < \frac{1}{k^2}$, and therefore $1 - \lambda k^2 > 0$).

Therefore, $d(Ty, z) \ll c$ for each $c \in \operatorname{int} P$. So, by (C_2) we have $d(Ty, z) = 0$, that is,

$$Ty = Gy = z,$$

so, z is a point of coincidence of T and G.

Since $TX \subset FX$, there exists $v \in X$ such that $z = Fv$. From (d_3) and (3.2), we have

$$d(Sv, z) \leq k\bigl[d(Sv, Tx_{2n+1}) + d(Tx_{2n+1}, z)\bigr] \leq \lambda m(v, x_{2n+1}) + kd(z_{2n+1}, z),$$

where

$$\begin{aligned} m(v, x_{2n+1}) \in k\bigl\{&d(Fv, Gx_{2n+1}), d(Fv, Sv) + d(Gx_{2n+1}, Tx_{2n+1}), \\ &d(Fv, Tx_{2n+1}) + d(Sv, Gx_{2n+1})\bigr\} \\ = k\bigl\{&d(z, z_{2n}), d(z, Sv) + d(z_{2n}, z_{2n+1}), d(z, z_{2n+1}) + d(Sv, z_{2n})\bigr\}. \end{aligned}$$

Therefore we have the following three cases:

(i) $d(Sv, z) \leq \lambda kd(z, z_{2n}) + kd(z_{2n+1}, z)$;

(ii) $d(Sv, z) \leq \lambda k[d(z, Sv) + d(z_{2n}, z_{2n+1})] + kd(z_{2n+1}, z)$;

(iii) $d(Sv, z) \leq \lambda k[d(z, z_{2n+1}) + d(Sv, z_{2n})] + kd(z_{2n+1}, z)$.

By the same argument as above, we get $d(Sv, z) = 0$, that is, $Sv = Fv = z$. So, z is also a point of coincidence of S and F. □

Theorem 3.4 *In addition to the hypothesis of Theorem* 3.3, *suppose that* $\{S, F\}$ *and* $\{T, G\}$ *are weakly compatible pairs, then* F, G, S, *and* T *have a common fixed point.*

Proof From Theorem 3.3, we get that the mappings $\{S, F\}$ and $\{T, G\}$ have a point of coincidence in $z \in X$. Now we prove that z is a unique point of coincidence of pairs $\{S, F\}$ and $\{T, G\}$. Suppose that there exists $z^* \in X$ such that $Fv^* = Gy^* = Sv^* = Ty^* = z^*$, from (3.2),

$$d(z, z^*) = d(Sv, Ty^*) \leq \frac{\lambda}{k} m(v, y^*), \tag{3.6}$$

where

$$\begin{aligned} m(v, y^*) &\in k\bigl\{d(Fv, Gy^*), d(Fv, Sv) + d(Gy^*, Ty^*), d(Fv, Ty^*) + d(Sv, Gy^*)\bigr\} \\ &= k\bigl\{d(z, z^*), d(z, z^*) + d(z, z^*)\bigr\} = k\bigl\{d(z, z^*), 2d(z, z^*)\bigr\}. \end{aligned}$$

Using (C_3), we get $d(z, z^*) = 0$, that is, $z = z^*$. Therefore, z is a unique point of coincidence of the pairs $\{S, F\}$ and $\{T, G\}$. That is, $Sv = Fv = Ty = Gy = z$.

The weak compatibility of the pair $\{S, F\}$ implies $SFv = FSv = SSv = FFv = Sz = Fz$.

Now we show that z is a common fixed point of S and F.

Consider

$$d(Sz, z) = d(Sz, Ty) \leq \frac{\lambda}{k} m(z, y),$$

where

$$\begin{aligned} m(z, y) &\in k\bigl\{d(Fz, Gy), d(Fz, Sz) + d(Gy, Ty), d(Fz, Ty) + d(Sz, Gy)\bigr\} \\ &= k\bigl\{d(Sz, z), d(Sz, z) + d(Sz, z)\bigr\} \\ &= k\bigl\{d(Sz, z), 2d(Sz, z)\bigr\}. \end{aligned}$$

Using (C_3), we get $Sz = z = Fz$. Similarly, the weak compatibility of the pair $\{T, G\}$ implies $Tz = z = Gz$. Hence $Sz = Fz = Tz = Gz = z$. □

Remark 3.5 Theorem 3.4 is a generalization of Theorem 3.1 of [14].

If we choose $T = S$ and $G = F$ in Theorems 3.3 and 3.4, we get the following results for two mappings on a cone metric type space.

Corollary 3.6 *Let (X, d) be a cone metric type space with the constant $k \geq 1$ and let P be a solid cone. Suppose that the self-mappings $F, S : X \to X$ are such that $SX \subset FX$. Let $x_0 \in X$, let the sequences $\{x_n\}$ and $\{z_n\}$ be defined by $z_{2n} = Fx_{2n+1} = Sx_{2n}$ and $z_{2n+1} = Fx_{2n+2} = Sx_{2n+1}$ for $n \geq 0$, and let $\mathcal{O}(x_0) = \{z_n : n \geq 0\}$. Assume that there exists $x_0 \in X$ such that for some constant $\lambda \in (0, \frac{1}{k+k^2})$,*

$$d(Sx, Sy) \leq \frac{\lambda}{k} m(x, y),$$

where

$$m(x, y) \in k\big\{d(Fx, Fy), d(Fx, Sx) + d(Fy, Sy), d(Fx, Sy) + d(Sx, Fy)\big\}$$

for all $x, y \in F^{-1}\{\overline{\mathcal{O}(x_0)}\} \cup S^{-1}\{\overline{\mathcal{O}(x_0)}\}$. If one of SX or FX is a complete subspace of X, then S and F have a unique point of coincidence in X. Moreover, if $\{S, F\}$ is a weakly compatible pair, then S and F have a common fixed point.

The following example shows that Theorem 3.4 is different from Theorem 3.1 of Cvetković *et al.* [14].

Example 3.7 Let $X = \{0, 1, 2\}$, and let $d(x, y) = |x - y|$ for each $x, y \in X$. Let $T : X \to X$ be given by $T0 = 2$, $T2 = 2$ and $T1 = 1$. Let $S = T$ and let $F = G = I$. We first show that we cannot invoke Theorem 3.1 of [14] to show the existence of a fixed point for T. On the contrary, assume that there exists $\lambda \in (0, 1)$ such that

$$|Tx - Ty| \leq \lambda \max\left\{|x - y|, |x - Tx|, |y - Ty|, \frac{|x - Ty| + |y - Tx|}{2}\right\}$$

for each $x, y \in X$. Let $x = 1$ and $y = 2$. Then, from the above, we get $1 \leq \lambda$, a contradiction.

Now let $x_0 = 0$. Then $\mathcal{O}(x_0) = \{0, 2\}$ and so $F^{-1}\{\overline{\mathcal{O}(x_0)}\} \cup G^{-1}\{\overline{\mathcal{O}(x_0)}\} = \{0, 2\}$. Since for each $x, y \in \{0, 2\}$, $Tx = Ty$, then the assumptions of Theorem 3.4 are satisfied by any $\lambda \in (0, \frac{1}{2})$. Thus T has a fixed point.

4 Generalized nonexpansive maps

The aim of this section is to present coincidence points results for four mappings without satisfying the condition of continuity and commutation on cone metric type spaces with a non-normal cone. Common fixed point theorems are obtained under the weak compatible condition. Our results generalize and unify main results in [40, 43] and many others.

First, we give some common fixed point theorems for generalized nonexpansive mappings defined on a cone metric type space.

Theorem 4.1 *Let (X,d) be a cone metric type space with the constant $k \geq 1$ and let P be a solid cone. Suppose that the mappings $F, G, S, T : X \to X$ are such that $SX \subset GX$, $TX \subset FX$ satisfying*

$$d(Sx, Ty) \leq ad(Fx, Gy) + b\{d(Fx, Sx) + d(Gy, Ty)\} + c\{d(Fx, Ty) + d(Gy, Sx)\} \tag{4.1}$$

for all $x, y \in X$, where a, b and c ($\neq 0$) are nonnegative constants such that $b + (a + b + c)k + ck^2 < 1$. If one of SX, TX, FX, or GX is a complete subspace of X, then $\{S, F\}$ and $\{T, G\}$ have a unique point of coincidence in X. Moreover, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then F, G, S, and T have a unique common fixed point.

Proof We define sequences $\{x_n\}$ and $\{z_n\}$ as in the proof of Theorem 3.3. We prove that $\{z_n\}$ is a Cauchy sequence by considering the cases when n is odd and even, respectively.

For $n = 2l + 1$, $l \in \mathbb{N}$, we have

$$\begin{aligned}
d(z_{2l+1}, z_{2l+2}) &= d(Sx_{2l+2}, Tx_{2l+1}) \\
&\leq ad(Fx_{2l+2}, Gx_{2l+1}) + b\{d(Fx_{2l+2}, Sx_{2l+2}) + d(Gx_{2l+1}, Tx_{2l+1})\} \\
&\quad + c\{d(Fx_{2l+2}, Tx_{2l+1}) + d(Gx_{2l+1}, Sx_{2l+2})\} \\
&= ad(z_{2l+1}, z_{2n}) + b\{d(z_{2l+1}, z_{2l+2}) + d(z_{2l}, z_{2l+1})\} \\
&\quad + c\{d(z_{2l+1}, z_{2l+1}) + d(z_{2l}, z_{2l+2})\} \\
&= (a + b)d(z_{2l+1}, z_{2n}) + cd(z_{2l}, z_{2l+2}) + bd(z_{2l+1}, z_{2l+2}) \\
&\leq (a + b)d(z_{2l+1}, z_{2n}) + ck[d(z_{2l}, z_{2l+1}) + d(z_{2l+1}, z_{2l+2})] + bd(z_{2l+1}, z_{2l+2}) \\
&= (a + b + kc)d(z_{2l+1}, z_{2n}) + (b + ck)d(z_{2l+1}, z_{2l+2}).
\end{aligned}$$

Therefore

$$d(z_{2l+1}, z_{2l+2}) \leq \left(\frac{a + b + ck}{1 - b - ck}\right) d(z_{2l+1}, z_{2n}). \tag{4.2}$$

Similarly, for $n = 2l$, $l \in \mathbb{N}$, we have

$$\begin{aligned}
d(z_{2l}, z_{2l+1}) &= d(Sx_{2l}, Tx_{2l+1}) \\
&\leq ad(Fx_{2l}, Gx_{2l+1}) + b\{d(Fx_{2l}, Sx_{2l}) + d(Gx_{2l+1}, Tx_{2l+1})\} \\
&\quad + c\{d(Fx_{2l}, Tx_{2l+1}) + d(Gx_{2l+1}, Sx_{2l})\} \\
&= ad(z_{2l-1}, z_{2n}) + b\{d(z_{2l-1}, z_{2l}) + d(z_{2l}, z_{2l+1})\} \\
&\quad + c\{d(z_{2l-1}, z_{2l+1}) + d(z_{2l}, z_{2l})\} \\
&= (a + b)d(z_{2l-1}, z_{2n}) + bd(z_{2l}, z_{2l+1}) + cd(z_{2l-1}, z_{2l+1}) \\
&\leq (a + b)d(z_{2l-1}, z_{2n}) + bd(z_{2l}, z_{2l+1}) \\
&\quad + ck[d(z_{2l-1}, z_{2n}) + d(z_{2l}, z_{2l+1})].
\end{aligned}$$

Thus,

$$d(z_{2l}, z_{2l+1}) \le \left(\frac{a+b+ck}{1-b-ck}\right) d(z_{2l-1}, z_{2n}).$$

Let $\alpha = \frac{a+b+ck}{1-b-ck}$. Clearly $\alpha < 1$. Hence, inequality (4.2) holds for both the cases. By the same arguments as in Theorem 3.3, we conclude that $\{z_n\}$ is a Cauchy sequence. Let us suppose that SX is a complete subspace of X, then there exists $z \in SX$ such that $\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} Sx_{2n} = z$. Then we have

$$\lim_{n\to\infty} Gx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z, \tag{4.3}$$

that is, for any $0 \ll c$, for sufficient large n, we have $d(z_n, z) \ll c$. Since $z \in SX \subset GX$, then there exists $y \in X$ such that $z = Gy$. We prove that $z = Ty$. From (d_3) and (4.1), we have

$$\begin{aligned} d(Ty, z) &\le k\big[d(Sx_{2n}, Ty) + d(Sx_{2n}, z)\big] \\ &\le k\big[ad(Fx_{2n}, Gy) + b\big\{d(Fx_{2n}, Sx_{2n}) + d(Gy, Ty)\big\} \\ &\quad + c\big\{d(Fx_{2n}, Ty) + d(Gy, Sx_{2n})\big\}\big] + kd(Sx_{2n}, z) \\ &= k\big[ad(z_{2n-1}, z) + b\big\{d(z_{2n-1}, z_{2n}) + d(z, Ty)\big\} \\ &\quad + c\big\{d(z_{2n-1}, Ty) + d(z, z_{2n})\big\}\big] + kd(z_{2n}, z) \\ &\le k\big[ad(z_{2n-1}, z) + b\big\{d(z_{2n-1}, z_{2n}) + d(z, Ty)\big\} \\ &\quad + c\big\{k\big[d(z_{2n-1}, z) + d(z, Ty)\big] + d(z, z_{2n})\big\}\big] + kd(z_{2n}, z) \\ &= k\big[(a+kc)d(z_{2n-1}, z) + (c+1)d(z, z_{2n}) + (b+ck)d(z, Ty) + bd(z_{2n-1}, z_{2n})\big]. \end{aligned}$$

Since $\{z_n\}$ converges to z, so for each $t \in \operatorname{int} P$, there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$,

$$\begin{aligned} d(Ty, z) &\le \left(\frac{k}{1-bk-ck^2}\right)\big[(a+kc)d(z_{2n-1}, z) + (c+1)d(z, z_{2n}) + bd(z_{2n-1}, z_{2n})\big] \\ &\ll \left(\frac{k}{1-bk-ck^2}\right)\left[\frac{(a+kc)t}{3}\frac{(1-bk-ck^2)}{k(a+kc)}\right. \\ &\quad \left. + \frac{(c+1)t}{3}\frac{(1-bk-ck^2)}{k(c+1)} + \frac{bt}{3}\frac{(1-bk-ck^2)}{bk}\right] \\ &= t. \end{aligned}$$

Now, by (C_2) it follows that $d(z, Ty) = 0$, that is, $Ty = z$. So, we have $Ty = Gy = z$, that is, z is a point of coincidence of T and G.

Since $TX \subset FX$, there exists $u \in X$ such that $z = Fu$. From (d_3) and (4.1), we have

$$\begin{aligned} d(Su, z) &\le k\big[d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, z)\big] \\ &\le k\big[ad(Fu, Gx_{2n+1}) + b\big\{d(Fu, Su) + d(Gx_{2n+1}, Tx_{2n+1})\big\} \\ &\quad + c\big\{d(Fu, Tx_{2n+1}) + d(Gx_{2n+1}, Su)\big\} + d(Tx_{2n+1}, z)\big] \\ &= k\big[ad(z, z_{2n}) + b\big\{d(z, Su) + d(z_{2n}, z_{2n+1})\big\} \end{aligned}$$

$$\begin{aligned}
&\quad + c\bigl\{d(z, z_{2n+1}) + d(z_{2n}, Su)\bigr\} + d(z_{2n+1}, z)\bigr] \\
&\leq k\bigl[ad(z, z_{2n}) + b\bigl\{d(z, Su) + d(z_{2n}, z_{2n+1})\bigr\} \\
&\quad + c\bigl\{d(z, z_{2n+1}) + k\bigl[d(z_{2n}, z) + d(z, Su)\bigr]\bigr\} + d(z_{2n+1}, z)\bigr],
\end{aligned}$$

so by the same arguments as above, we have $d(Su, z) = 0$, that is, $Su = Fu = Ty = Gy = z$.

Suppose that z^* is another point of coincidence of these four mappings, that is, $Fu^* = Su^* = Gy^* = Ty^* = z^*$. From (4.1) we get

$$\begin{aligned}
d(z, z^*) &= d(Su, Ty^*) \\
&\leq ad(Fu, Gy^*) + b\bigl\{d(Fu, Su) + d(Gy^*, Ty^*)\bigr\} + c\bigl\{d(Fu, Ty^*) + d(Gy^*, Su)\bigr\} \\
&\leq (a + 2ck)d(z, z^*),
\end{aligned}$$

because of (C_3), we get $z = z^*$. Therefore, z is the unique point of coincidence of S, F, G and T. The weak compatibility of the pair $\{S, F\}$ implies $SFu = FSu = SSu = FFu = Sz = Fz$, and the weak compatibility of the pair $\{T, G\}$ gives $GTy = TGy = GGy = TTy = Gz = Tz$. Consider

$$\begin{aligned}
d(z, Sz) &= d(Ty, Sz) \\
&\leq ad(Fz, Gy) + b\bigl\{d(Fz, Sz) + d(Gy, Ty)\bigr\} + c\bigl\{d(Fz, Ty) + d(Gy, Sz)\bigr\} \\
&= (a + 2c)d(Sz, z).
\end{aligned}$$

Since $b + (a + b + c)k + ck^2 < 1$ and using (C_3), we get $Sz = Fz = z$; similarly we have $Tz = Gz = z$. This shows that z is a common fixed point of G, T, F and S. The proof for the cases in which FX or GX or TX is complete are similar. Hence the theorem follows. □

Choosing $k = 1$ in Theorem 4.1, we obtain the following generalized form of the results in [40] and [43].

Corollary 4.2 *Let (X, d) be a cone metric space and P be a solid cone. Suppose that the mappings $F, G, S, T : X \to X$ are such that $SX \subset GX$, $TX \subset FX$ and that there exist non-negative constants a, b andc satisfying*

$$a + 2b + 2c < 1$$

such that for all $x, y \in X$,

$$d(Sx, Ty) \leq ad(Fx, Gy) + b\bigl\{d(Fx, Sx) + d(Gy, Ty)\bigr\} + c\bigl\{d(Fx, Ty) + d(Gy, Sx)\bigr\}.$$

If one of SX, TX, FX, or GX is a complete subspace of X, then $\{S, F\}$ and $\{T, G\}$ have a unique point of coincidence in X. Moreover, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then F, G, S, and T have a unique common fixed point.

If we choose $T = S$ and $G = F$ in Theorem 4.1, we get the following result for two mappings on a cone metric type space.

Corollary 4.3 *Let (X, d) be a cone metric type space with the constant $k \geq 1$ and let P be a solid cone. Suppose that the mappings $G, T : X \to X$ are such that $TX \subset GX$ and that there exist nonnegative constants a, b andc, satisfying $b + (a + b + c)k + ck^2 < 1$, such that for all $x, y \in X$,*

$$d(Sx, Ty) \preceq ad(Gx, Gy) + b\{d(Gx, Tx) + d(Gy, Ty)\} + c\{d(Gx, Ty) + d(Gy, Tx)\}.$$

If one of TX or GX is a complete subspace of X, then G and T have a unique point of coincidence in X. Moreover, if $\{T, G\}$ is a weakly compatible pair, then G and T have a unique common fixed point.

Now, we use the following lemma, that is a consequence of the axiom of choice, to obtain coincidence point results for two self-mappings defined on a subset of a partially ordered cone metric type space (see [45, 46]).

Lemma 4.4 *Let X be a nonempty set and let $g : X \to X$ be a mapping. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

The following result contains properly the recent main result (Theorem 2.1) of Shi and Xu [39]. Here we shall establish this result in the set up of a partially ordered cone metric type space relative to a solid cone P.

Theorem 4.5 (Theorem 2.10 of [38]) *Let (X, d) be a cone metric type space with the constant $k \geq 1$ and let P be a cone having a nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X and that there exist nonnegative constants $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ with $2ka_1 + (k+1)(a_2 + a_3) + (k^2 + k)(a_4 + a_5) < 2$ such that, for each $x, y \in X$,*

$$d(fx, fy) \preceq a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 d(gy, fy) + a_4 d(gx, fy) + a_5 d(gy, fx).$$

Then f and g have a unique point of coincidence in X. Moreover, if f and g are OWC, f and g have a unique common fixed point.

Recall that if $(X, \preceq)$ is a partially ordered set and $f, g : X \to X$ is such that for $x, y \in X$, $x \preceq y$ implies $fx \preceq fy$, then the mapping f is said to be nondecreasing. Further, for $x, y \in X$, $gx \preceq gy$ implies $fx \preceq fy$, then the mapping f is said to be g-nondecreasing [34].

The following result extends Theorem 4.5 and corresponding results in Altun *et al.* [5, 6] and many others (see also Lemma 2.3 [34]).

Theorem 4.6 *Let $(X, d, \preceq)$ be a partially ordered cone metric type space relative to a solid cone P with the constant $k \geq 1$. Suppose that the mappings $f, g : X \to X$ are such that f is g-nondecreasing mapping w.r.t. $\preceq$, $fX \subseteq gX$ and gX is a complete subspace of X. Suppose that the following assertions hold:*

(a) *f is continuous or X has the following property: if a nondecreasing sequence $\{gx_n\} \to gx$, then $gx_n \preceq gx$ for all $n \geq 0$;*
(b) *there exist nonnegative constants $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ with $2ka_1 + (k + 1)(a_2 + a_3) + (k^2 + k)(a_4 + a_5) < 2$;*

(c) *for each $x, y \in X$ with $gy \preceq gx$, we have*

$$d(fx,fy) \leq a_1 d(gx,gy) + a_2 d(gx,fx) + a_3 d(gy,fy) + a_4 d(gx,fy) + a_5 d(gy,fx). \tag{4.4}$$

If there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$, then f and g have a coincidence point in X.

Proof Using Lemma 4.4, there exists $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one. We define a mapping $G : g(E) \to g(E)$ by

$$G(gx) = f(x) \tag{4.5}$$

for all $gx \in g(E)$. As g is one-to-one on $g(E) = g(X)$ and $f(X) \subseteq g(X)$, so G is well defined. Thus, it follows from (4.4) and (4.5) that

$$d(Ggx, Ggy) = d(fx,fy) \leq a_1 d(gx,gy) + a_2 d(gx,fx) + a_3 d(gy,fy) + a_4 d(gx,fy) + a_5 d(gy,fx)$$

for all $gx, gy \in g(X)$ for which $g(y) \preceq g(x)$. Since f is a g-nondecreasing mapping, for all $gy \preceq gx \in g(X)$, it implies $fy \preceq fx$, which gives $Ggy \preceq Ggx$. Thus G is a nondecreasing mapping on $g(X)$. Also, there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0) = G(gx_0)$. Suppose that the assumption (a) holds. Since f is continuous, G is also continuous. Using Theorem 2.2 [4] to the mapping G, it follows that G has a fixed point $u \in g(X)$.

Suppose that the assumption (b) holds. We conclude similarly from Theorem 2.3 [4] that the mapping G has a fixed point $u \in g(X)$. Finally, we prove that f and g have a coincidence point. Since u is a fixed point of G, we get $u = Gu$. Since $u \in g(X)$, there exists a point $u_0 \in X$ such that $u = gu_0$. It follows that $gu_0 = u = Gu = Ggu_0 = fu_0$. Thus, u_0 is a required coincidence point of f and g. This completes the proof. □

5 Application to the existence of solutions of integral equations

Fixed point theorems for operators in ordered Banach spaces are widely investigated and have found various applications in differential and integral equations (see [1, 24] and references therein). Let $X = C([0;T];\mathbb{R})$ be the set of real continuous functions defined on $[0;T]$ and let $d : X \times X \to [0;+\infty)$ be defined by $d(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|^2$. Then (X,d) is a complete metric type space with the constant $k = 2$.

Consider the integral equation

$$Fu(t) = p(t) + \int_0^T G(t,s) f\bigl(s,u(s)\bigr)\,ds, \tag{5.1}$$

and let $S : X \to X$ be defined by

$$(Su)(t) = p(t) + \int_0^T G(t,s) f\bigl(s,u(s)\bigr)\,ds \quad \text{for each } t \in [0,T]. \tag{5.2}$$

We assume that

(i) $p : [0,T] \to \mathbb{R}$ is continuous;
(ii) $G(t,s) : [0,T] \times [0,T] \to \mathbb{R}$ is continuous;

(iii) $|f(s,u(s)) - f(s,v(s))| \le \mu \max\{|Fu(s) - Fv(s)|, |Fu(s) - Su(s)| + |Fv(s) - Sv(s)|, |Fu(s) - Sv(s)| + |Su(s) - Fv(s)|\}$ for each $u, v \in X$ and $s \in [0, T]$, where μ is a constant such that $2T\mu^2 \sup_{t\in[0,T]} \int_0^T |G(t,s)|^2\,ds < \frac{1}{4}$.

Theorem 5.1 *Under the assumptions* (i)-(iii), *the integral equation* (5.1) *has a solution in* $X = C([0;T];\mathbb{R})$.

Proof Consider the mapping $S : X \to X$ defined by (5.2). Notice first that the existence of a solution for the integral equation (5.1) is equivalent to the existence of a common fixed point for the mappings F and S. For each $u, v \in X$, we have

$$\begin{aligned}
d(Su,Sv) &= \sup_{t\in[0,T]} \left|Su(t) - Sv(t)\right|^2 \\
&\le \sup_{t\in[0,T]} \left\{\int_0^T \left|G(t,s)\right|\left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right| ds\right\}^2 \\
&\le \sup_{t\in[0,T]} \int_0^T \left|G(t,s)\right|^2 ds \int_0^T \left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right|^2 ds \\
&\le 2\mu^2 \sup_{t\in[0,T]} \int_0^T \left|G(t,s)\right|^2 ds \\
&\quad \times \int_0^T \max\left\{\left|Fu(s) - Fv(s)\right|^2, \left|Fu(s) - Su(s)\right|^2 + \left|Fv(s) - Sv(s)\right|^2, \right. \\
&\quad \left. \left|Fu(s) - Sv(s)\right|^2 + \left|Su(s) - Fv(s)\right|^2\right\} ds \\
&\le 2T\mu^2 \sup_{t\in[0,T]} \int_0^T \left|G(t,s)\right|^2 ds \\
&\quad \times \max\left\{d(Fu,Fv), d(Fu,Su) + d(Fv,Sv), d(Fu,Sv) + d(Su,Fv)\right\}.
\end{aligned}$$

Hence all the hypotheses of Corollary 3.6 are satisfied, and so the mappings F and S have a common fixed point that is a solution in $X = C([0;T];\mathbb{R})$ of the integral equation (5.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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