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The methods for variational inequality problems and fixed point of κ -strictly pseudononspreading mapping

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Abstract

In this paper, we introduce the methods for finding a common element of the set of fixed points of a κ -strictly pseudononspreading mapping and a finite family of the set of solutions of variational inequality problems. The strong convergence theorem of the proposed method is established under some suitable control conditions. Moreover, by using our main result, we prove interesting theorem involving an iterative scheme for finding a common element of the set of fixed points of a κ -strictly pseudononspreading mapping and a finite family of the set of fixed points of a κ_j -strictly pseudocontractive mappings.

Keywords: variational inequality problems; κ -strictly pseudononspreading mapping; strictly pseudocontractive mapping

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H . Recall that the mapping $T : C \rightarrow C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 2008, Kohsaka and Takahashi [1] introduced the *nonspreading mapping* in Hilbert spaces H which is defined as follows: $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C$. Following the terminology of Browder and Petryshyn [2], in 2011, Osilike and Isiogugu [3] introduced that the mapping $T : C \rightarrow C$ is called a κ -strictly pseudononspreading mapping if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$. Clearly every nonspreading mapping is κ -strictly pseudononspreading; see, for example, [3]. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Let $A : C \rightarrow H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, A)$.

The variational inequality has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences; see, e.g., [4–7].

A mapping A of C into H is called α -inverse strongly monotone (see [8]) if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

In 2003, Takahashi and Toyoda [9] proved a convergence theorem for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequalities for α -inverse strongly monotone mappings as follows.

Theorem 1.1 *Let K be a closed convex subset of a real Hilbert space H . Let $\alpha > 0$. Let A be an α -inverse strongly monotone mapping of K into H , and let S be a nonexpansive mapping of K into itself such that $F(S) \cap VI(K, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in K$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(I - \lambda_n A)x_n$$

for every $n = 0, 1, 2, \dots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then $\{x_n\}$ converges weakly to $z = F(S) \cap VI(K, A)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap VI(K, A)} x_n$.

Recently, Osilike and Isiogugu [3] proved strong convergence theorems for strictly pseudononspreading mappings as follows.

Theorem 1.2 *Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \rightarrow C$ be a κ -strictly pseudononspreading mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k, 1)$ and let $\{\alpha_n\}_{n=1}^\infty$ be a real sequence in $[0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Let $u \in C$ and let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences in C generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, & \forall n \geq 1, \\ z_n = \frac{1}{n} \sum_{k=0}^{n-1} T_\beta^k x_n, & \forall n \geq 1, \end{cases}$$

where $T_\beta = \beta I + (1 - \beta)T$. Then $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ converge strongly to $P_{F(T)} u$, where $P_{F(T)} : H \rightarrow F(T)$ is the metric projection of H onto $F(T)$.

Theorem 1.3 *Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \rightarrow C$ be a κ -strictly pseudononspreading mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k, 1)$ and let $T_\beta = \beta I + (1 - \beta)T$. Let $\{\alpha_n\}_{n=1}^\infty$ be a real sequence in $[0, 1)$ satisfying the conditions*

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and

(C2) $\sum_{n=1}^\infty \alpha_n = \infty$.

Let $u \in C$ be a fixed anchor in C and let $\{x_n\}_{n=1}^\infty$ be a sequence in C generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_\beta x_n, \quad n \geq 1. \tag{1.2}$$

Then $\{x_n\}_{n=1}^\infty$ converges strongly to a fixed point p of T .

Inspired and motivated by [3] and the research in the same direction, we prove a strong convergence theorem of κ -strictly pseudononspreading mappings and introduce the methods for finding a common element of the set of fixed points of a κ -strictly pseudononspreading mapping and a finite family of the set of solutions of variational inequality problems. Moreover, by using our main result, we prove an interesting theorem involving an iterative scheme for finding a common element of the set of fixed points of κ -strictly pseudononspreading mappings and a finite family of the set of fixed points of κ_i -strictly pseudocontractive mappings.

2 Preliminaries

We need the following lemmas to prove our main result. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H , let P_C be the metric projection of H onto C , i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C .

Lemma 2.1 (See [10]) *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if the following inequality holds:*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2 (See [10]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H . Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \iff u \in VI(C, A),$$

where P_C is the metric projection of H onto C .

Lemma 2.3 (See [11]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (2) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.4 (See [11]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

- (1) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (2) $\limsup_{n \rightarrow \infty} \beta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.5 (See [12]) *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E and let $S : C \rightarrow C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

In 2009, Kangtanyakarn and Suantai [13] defined an S -mapping and proved their lemmas as follows.

Definition 2.1 Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \rightarrow C$ as follows:

$$\begin{aligned}
 U_0 &= I, \\
 U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
 U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
 U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
 &\vdots \\
 U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
 S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
 \end{aligned} \tag{2.1}$$

This mapping is called an S -mapping generated by T_1, T_2, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.6 *Let C be a nonempty closed convex subset of strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, 3, \dots, N$, where $I = [0, 1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N - 1, \alpha_1^N \in (0, 1], \alpha_2^j, \alpha_3^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let S be a mapping generated by T_1, T_2, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Remark 2.7 It is easy to see that the mapping S is a nonexpansive mapping.

Lemma 2.8 *Let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a κ -strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Then $F(T) = VI(C, (I - T))$.*

Proof It is easy to see that $F(T) \subseteq VI(C, (I - T))$. Put $A = I - T$. Let $z \in VI(C, A)$ and $z^* \in F(T)$. Since $z \in VI(C, A)$, we have

$$\langle y - z, (I - T)z \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Since T is a κ -strictly pseudononspreading mapping, we have

$$\begin{aligned}
 \|Tz - Tz^*\|^2 &= \|(I - A)z - (I - A)z^*\|^2 \\
 &= \|(z - z^*) - (Az - Az^*)\|^2 \\
 &= \|z - z^*\|^2 + \|Az - Az^*\|^2 - 2\langle z - z^*, Az - Az^* \rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \|z - z^*\|^2 + \|Az\|^2 - 2\langle z - z^*, Az \rangle \\
 &\leq \|z - z^*\|^2 + \kappa \|(I - T)z - (I - T)z^*\|^2 + 2\langle z - Tz, z^* - Tz^* \rangle \\
 &= \|z - z^*\|^2 + \kappa \|(I - T)z\|^2,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \frac{1 - \kappa}{2} \|z - Tz\|^2 &\leq \langle z - z^*, (I - T)z \rangle \\
 &= -\langle z^* - z, (I - T)z \rangle \\
 &\leq 0.
 \end{aligned}$$

Then we have $z \in F(T)$. Therefore $VI(C, (I - T)) \subseteq F(T)$. Hence $VI(C, (I - T)) = F(T)$. □

Remark 2.9 From Lemmas 2.2 and 2.8, we have $F(T) = F(P_C(I - \lambda(I - T)))$, $\forall \lambda > 0$.

Example 2.1 Let $T : [-1, 1] \rightarrow [-1, 1]$ be defined by

$$Tx = \begin{cases} \frac{x+4}{5} & \text{if } x \in [0, 1], \\ \frac{4-x}{5} & \text{if } x \in [-1, 0). \end{cases}$$

To see that T is κ -strictly pseudononspreading, if for all $x, y \in [0, 1]$, then we have $Tx = \frac{x+4}{5}$ and $Ty = \frac{y+4}{5}$. From the definition of T , we have

$$\begin{aligned}
 |Tx - Ty|^2 &= \left| \frac{x+4}{5} - \frac{y+4}{5} \right|^2 \\
 &= \frac{1}{25} |x - y|^2, \\
 |(I - T)x - (I - T)y|^2 &= |x - Tx - (y - Ty)|^2 \\
 &= \left| x - \frac{x+4}{5} - \left(y - \frac{y+4}{5} \right) \right|^2 \\
 &= \left| \frac{4x-4}{5} - \frac{4y-4}{5} \right|^2 \\
 &= \left| \frac{4x-4y}{5} \right|^2 \\
 &= \frac{16}{25} |x - y|^2 \geq 0
 \end{aligned}$$

and

$$\begin{aligned}
 2\langle x - Tx, y - Ty \rangle &= 2\left\langle x - \frac{x+4}{5}, y - \frac{y+4}{5} \right\rangle = 2\left\langle \frac{4x-4}{5}, \frac{4y-4}{5} \right\rangle \\
 &= \frac{32}{25} (x-1)(y-1) \geq 0. \quad (\text{Since } 0 \leq x, y \leq 1, (x-1)(y-1) \geq 0.)
 \end{aligned}$$

From the above, then there exists $\kappa \in [0, 1)$ such that

$$\begin{aligned} |x - y|^2 + \kappa |(I - T)x - (I - T)y|^2 + 2\langle x - Tx, y - Ty \rangle &\geq |x - y|^2 \\ &\geq \frac{1}{25} |x - y|^2 \\ &= |Tx - Ty|^2. \end{aligned}$$

For every $x, y \in [-1, 0)$, we have $Tx = \frac{4-x}{5}$, $Ty = \frac{4-y}{5}$. From the definition of T , we have

$$\begin{aligned} |Tx - Ty|^2 &= \left| \frac{4-x}{5} - \frac{4-y}{5} \right|^2 \\ &= \left| \frac{y-x}{5} \right|^2 \\ &= \frac{1}{25} |x - y|^2, \\ |(I - T)x - (I - T)y|^2 &= \left| x - Tx - (y - Ty) \right|^2 \\ &= \left| x - \frac{4-x}{5} - \left(y - \frac{4-y}{5} \right) \right|^2 \\ &= \left| \frac{6x-4-(6y-4)}{5} \right|^2 \\ &= \frac{36}{25} |x - y|^2 \geq 0 \end{aligned}$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty \rangle &= 2 \left\langle x - \frac{4-x}{5}, y - \frac{4-y}{5} \right\rangle \\ &= 2 \left\langle \frac{6x-4}{5}, \frac{6y-4}{5} \right\rangle \\ &= \frac{8}{25} (3x-2)(3y-2) \\ &= \frac{8}{25} (3x(3y-2) - 2(3y-2)) \\ &= \frac{8}{25} (9xy - 6x - 6y + 4) \\ &> 0. \quad (\text{Since } -1 \leq x, y < 0, 9xy, -6x, -6y > 0.) \end{aligned}$$

From the above, then there exists $\kappa \in [0, 1)$ such that

$$\begin{aligned} |x - y|^2 + \kappa |(I - T)x - (I - T)y|^2 + 2\langle x - Tx, y - Ty \rangle &> |x - y|^2 \\ &\geq \frac{1}{25} |x - y|^2 \\ &= |Tx - Ty|^2. \end{aligned}$$

Finally, for every $x \in [0, 1]$ and $y \in [-1, 0)$, we have $Tx = \frac{x+4}{5}$ and $Ty = \frac{4-y}{5}$. From the definition of T , we have

$$\begin{aligned} |Tx - Ty|^2 &= \left| \frac{x+4}{5} - \frac{4-y}{5} \right|^2 \\ &= \frac{1}{25} |x+y|^2, \\ |(I-T)x - (I-T)y|^2 &= \left| x - Tx - (y - Ty) \right|^2 \\ &= \left| x - \frac{x+4}{5} - \left(y - \frac{4-y}{5} \right) \right|^2 \\ &= \left| \frac{4x-4-(6y-4)}{5} \right|^2 \\ &= \frac{1}{25} |4x-6y|^2 \geq 0 \end{aligned}$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty \rangle &= 2 \left\langle x - \frac{x+4}{5}, y - \frac{4-y}{5} \right\rangle \\ &= 2 \left\langle \frac{4x-4}{5}, \frac{6y-4}{5} \right\rangle \\ &= \frac{16}{25} (x-1)(3y-2) \\ &= \frac{16}{25} (3y(x-1) - 2(x-1)) \\ &= \frac{16}{25} (3y(x-1) + 2(1-x)) \geq 0. \end{aligned}$$

(Since $0 \leq x \leq 1$ and $-1 \leq y < 0$, then $3y(x-1), 2(1-x) \geq 0$.)

From the above, then there exists $\kappa \in [0, 1)$ such that

$$\begin{aligned} |x-y|^2 + \kappa |(I-T)x - (I-T)y|^2 + 2\langle x - Tx, y - Ty \rangle &\geq |x-y|^2 \\ &= x^2 - 2xy + y^2 \\ &= x^2 + 2xy + y^2 - 4xy \\ &\geq (x+y)^2 \quad (\text{Since } -4xy \geq 0.) \\ &\geq \frac{1}{25} |x+y|^2 \\ &= |Tx - Ty|^2. \end{aligned}$$

Then, for all $x, y \in [-1, 1]$, we have

$$|Tx - Ty|^2 \leq |x-y|^2 + \kappa |(I-T)x - (I-T)y|^2 + 2\langle x - Tx, y - Ty \rangle$$

for some $\kappa \in [0, 1)$. Hence T is a κ -strictly pseudononspreading mapping. Observe that $1 \in F(T)$. From Lemma 2.8, we have $1 \in VI([-1, 1], I - T)$.

3 Main results

Theorem 3.1 *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . For every $i = 1, 2, \dots, N$, let $B_i : C \rightarrow H$ be δ_i -inverse strongly monotone mappings and let $T : C \rightarrow C$ be a κ -strictly pseudononspreading mapping for some $\kappa \in [0, 1)$. Let $G_i : C \rightarrow C$ be defined by $G_i x = P_C(I - \eta B_i)x$ for every $x \in C$ and $\eta \in (0, 2\delta_i)$ for every $i = 1, 2, \dots, N$, and let $\delta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N - 1$, $\alpha_1^N \in (0, 1]$, $\alpha_2^j, \alpha_3^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let $S : C \rightarrow C$ be the S -mapping generated by G_1, G_2, \dots, G_N and $\delta_1, \delta_2, \dots, \delta_N$. Assume that $\mathfrak{F} = F(T) \cap \bigcap_{i=1}^N VI(C, B_i) \neq \emptyset$. For every $n \in \mathbb{N}$, $i = 1, 2, \dots, N$, let $x_1, u \in C$ and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n Sx_n, \quad \forall n \in \mathbb{N}, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n \in [c, d] \subset (0, 1)$, $\{\lambda_n\} \subset (0, 1 - \kappa)$ and suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- (ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$,
- (iii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_{\mathfrak{F}}u$.

Proof Let $x^* \in \mathfrak{F}$. First, we show that $\|P_C(I - \lambda_n A)x_n - x^*\| \leq \|x_n - x^*\|$, where $A = I - T$. From Remark 2.9, we have $x^* \in F(P_C(I - \lambda_n A))$. From the nonexpansiveness of P_C , we have

$$\begin{aligned} \|P_C(I - \lambda_n A)x_n - x^*\|^2 &= \|P_C(I - \lambda_n A)x_n - P_C(I - \lambda_n A)x^*\|^2 \\ &\leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|^2. \end{aligned} \tag{3.2}$$

Since T is a κ -strictly pseudononspreading mapping and $A = I - T$, we have

$$\begin{aligned} \|Tx_n - Tx^*\|^2 &= \|(I - A)x_n - (I - A)x^*\|^2 \\ &= \|(x_n - x^*) - (Ax_n - Ax^*)\|^2 \\ &= \|x_n - x^*\|^2 - 2\langle x_n - x^*, Ax_n \rangle + \|Ax_n\|^2 \\ &\leq \|x_n - x^*\|^2 + \kappa \|Ax_n - Ax^*\|^2 + 2\langle Ax_n, Ax^* \rangle \\ &= \|x_n - x^*\|^2 + \kappa \|Ax_n\|^2, \end{aligned}$$

which implies that

$$(1 - \kappa) \|Ax_n\|^2 \leq 2\langle x_n - x^*, Ax_n \rangle. \tag{3.3}$$

From (3.3), we have

$$\begin{aligned} \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|^2 &= \|(x_n - x^*) - \lambda_n(Ax_n - Ax^*)\|^2 \\ &= \|x_n - x^*\|^2 - 2\lambda_n \langle x_n - x^*, Ax_n \rangle + \lambda_n^2 \|Ax_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \lambda_n(1 - \kappa) \|Ax_n\|^2 + \lambda_n^2 \|Ax_n\|^2 \end{aligned}$$

$$\begin{aligned}
 &= \|x_n - x^*\|^2 - \lambda_n((1 - \kappa) - \lambda_n) \|Ax_n\|^2 \\
 &\leq \|x_n - x^*\|^2.
 \end{aligned}
 \tag{3.4}$$

From (3.4) and (3.2), we can imply that

$$\|P_C(I - \lambda_n A)x_n - x^*\| \leq \|x_n - x^*\|.
 \tag{3.5}$$

Next, we will show that the mapping G_i is a nonexpansive mapping for every $i = 1, 2, \dots, N$. Let $x, y \in H$. Since B_i is δ_i -inverse strongly monotone and $0 < \eta < 2\delta_i$, for every $i = 1, 2, \dots, N$, we have

$$\begin{aligned}
 \|(I - \eta B_i)x - (I - \eta B_i)y\|^2 &= \|x - y - \eta(B_i x - B_i y)\|^2 \\
 &= \|x - y\|^2 - 2\eta \langle x - y, B_i x - B_i y \rangle + \eta^2 \|B_i x - B_i y\|^2 \\
 &\leq \|x - y\|^2 - 2\delta_i \eta \|B_i x - B_i y\|^2 + \eta^2 \|B_i x - B_i y\|^2 \\
 &= \|x - y\|^2 + \eta(\eta - 2\delta_i) \|B_i x - B_i y\|^2 \\
 &\leq \|x - y\|^2.
 \end{aligned}
 \tag{3.6}$$

Thus $(I - \eta B_i)$ is a nonexpansive mapping for every $i = 1, 2, \dots, N$. The proof of the above result can be also found in Imnang and Suantai [14]. From the definition of G_i , we have $G_i = P_C(I - \eta B_i)$ are nonexpansive mappings for all $i = 1, 2, \dots, N$. Since $x^* \in \mathfrak{F}$, by Lemma 2.2, we have

$$x^* = G_i x^* = P_C(I - \eta B_i)x^*, \quad \forall i = 1, 2, \dots, N.
 \tag{3.7}$$

From Lemma 2.6, we have $x^* \in F(S)$. Next, we will show that $\{x_n\}$ is bounded. From the definition of x_n and (3.5), we have

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + \beta_n(P_C(I - \lambda_n(I - T))x_n - x^*) + \gamma_n(Sx_n - x^*)\| \\
 &\leq \alpha_n \|u - x^*\| + \beta_n \|P_C(I - \lambda_n(I - T))x_n - x^*\| + \gamma_n \|Sx_n - x^*\| \\
 &\leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|x_n - x^*\|.
 \end{aligned}
 \tag{3.8}$$

Put $K = \max\{\|u - x^*\|, \|x_1 - x^*\|\}$. From (3.8) we can show by induction that $\|x_n - x^*\| \leq K$, $\forall n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so are $\{Sx_n\}, \{P_C(I - \lambda_n(I - T))x_n\}$. Next, we will show that

$$\|Tx_n - x^*\| \leq \frac{1 + \kappa}{1 - \kappa} \|x_n - x^*\|.
 \tag{3.9}$$

Since T is κ -strictly pseudononspreading, we have

$$\begin{aligned}
 \|Tx_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + \kappa \|(I - T)x_n - (I - T)x^*\|^2 + 2\langle x_n - Tx_n, x^* - Tx^* \rangle \\
 &= \|x_n - x^*\|^2 + \kappa \|(x_n - x^*) - (Tx_n - x^*)\|^2 \\
 &= \|x_n - x^*\|^2 + \kappa (\|x_n - x^*\|^2 + \|Tx_n - x^*\|^2 - 2\langle x_n - x^*, Tx_n - x^* \rangle),
 \end{aligned}$$

which implies that

$$(1 - \kappa) \|Tx_n - x^*\|^2 \leq (1 + \kappa) \|x_n - x^*\|^2 + 2\kappa \|x_n - x^*\| \|Tx_n - x^*\|. \quad (3.10)$$

Putting $A = \|Tx_n - x^*\|$ and $B = \|x_n - x^*\|$ in (3.10), we have

$$\begin{aligned} 0 &\geq (1 - \kappa)A^2 - (1 + \kappa)B^2 - 2\kappa AB \\ &= (1 - \kappa)A^2 - \kappa AB - ((1 + \kappa)B^2 + \kappa AB) \\ &= (1 - \kappa)A^2 - \kappa AB + AB - ((1 + \kappa)B^2 + \kappa AB + AB) \\ &= (1 - \kappa)(A^2 + AB) - ((1 + \kappa)(B^2 + AB)) \\ &= (1 - \kappa)A(A + B) - ((1 + \kappa)B(B + A)), \end{aligned}$$

which implies that

$$A \leq \frac{1 + \kappa}{1 - \kappa} B. \quad (3.11)$$

From (3.11) we have (3.9). Since $\|x_n - x^*\| \leq K, \forall n \in \mathbb{N}$ and (3.9), we have $\{Tx_n\}$ is bounded.

Next, we will show that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.12)$$

From the definition of x_n , we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|\alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n Sx_n \\ &\quad - \alpha_{n-1} u - \beta_{n-1} P_C(I - \lambda_{n-1}(I - T))x_{n-1} - \gamma_{n-1} Sx_{n-1}\| \\ &= \|\alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n - \beta_n P_C(I - \lambda_{n-1}(I - T))x_{n-1} \\ &\quad + \beta_n P_C(I - \lambda_{n-1}(I - T))x_{n-1} + \gamma_n Sx_n - \gamma_n Sx_{n-1} + \gamma_n Sx_{n-1} \\ &\quad - \alpha_{n-1} u - \beta_{n-1} P_C(I - \lambda_{n-1}(I - T))x_{n-1} - \gamma_{n-1} Sx_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|P_C(I - \lambda_n(I - T))x_n - P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| + \gamma_n \|Sx_n - Sx_{n-1}\| \\ &\quad + |\gamma_n - \gamma_{n-1}| \|Sx_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|(x_n - x_{n-1}) - \lambda_n(I - T)x_n + \lambda_n(I - T)x_{n-1} \\ &\quad - \lambda_n(I - T)x_{n-1} + \lambda_{n-1}(I - T)x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| \\ &\quad + \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|Sx_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n (\|x_n - x_{n-1}\| + \lambda_n \|(I - T)x_n - (I - T)x_{n-1}\| \\ &\quad + |\lambda_n - \lambda_{n-1}| \|(I - T)x_{n-1}\|) + |\beta_n - \beta_{n-1}| \|P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| \\ &\quad + \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|Sx_{n-1}\| \end{aligned}$$

$$\begin{aligned} &\leq |\alpha_n - \alpha_{n-1}|L + (1 - \alpha_n)\|x_n - x_{n-1}\| + \lambda_n L \\ &\quad + |\lambda_n - \lambda_{n-1}|L + |\beta_n - \beta_{n-1}|L + |\gamma_n - \gamma_{n-1}|L, \end{aligned} \tag{3.13}$$

where $L = \max_{n \in \mathbb{N}} \{\|u\|, \|(I - T)x_n - (I - T)x_{n-1}\|, \|(I - T)x_n\|, \|P_C(I - \lambda_n(I - T))x_n\|, \|Sx_n\|\}$. From Lemma 2.3 and conditions (i)-(iii), we have (3.12). Next, we will show that

$$\lim_{n \rightarrow \infty} \|x_n - P_C(I - \lambda_n(I - T))x_n\| = 0. \tag{3.14}$$

From the definition of x_n and (3.5), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) + \beta_n(P_C(I - \lambda_n(I - T))x_n - x^*) + \gamma_n(Sx_n - x^*)\|^2 \\ &\leq \alpha_n\|u - x^*\|^2 + \beta_n\|P_C(I - \lambda_n(I - T))x_n - x^*\|^2 + \gamma_n\|Sx_n - x^*\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))x_n - Sx_n\|^2 \\ &\leq \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))x_n - Sx_n\|^2 \\ &\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))x_n - Sx_n\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))x_n - Sx_n\|^2 &\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq \alpha_n\|u - x^*\|^2 + (\|x_n - x^*\| \\ &\quad + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|. \end{aligned} \tag{3.15}$$

From (3.15) and (3.12), we have

$$\lim_{n \rightarrow \infty} \|P_C(I - \lambda_n(I - T))x_n - Sx_n\| = 0. \tag{3.16}$$

Since

$$x_{n+1} - P_C(I - \lambda_n(I - T))x_n = \alpha_n(u - P_C(I - \lambda_n(I - T))x_n) + \gamma_n(Sx_n - P_C(I - \lambda_n(I - T))x_n),$$

from condition (i) and (3.16), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - P_C(I - \lambda_n(I - T))x_n\| = 0. \tag{3.17}$$

Since

$$\|x_n - P_C(I - \lambda_n(I - T))x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - P_C(I - \lambda_n(I - T))x_n\|,$$

from (3.12) and (3.17), we have (3.14). Since

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \beta_n(P_C(I - \lambda_n(I - T))x_n - x_n) + \gamma_n(Sx_n - x_n),$$

from condition (i) (3.12) and (3.14), we have

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0. \tag{3.18}$$

Next, we will show that

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle \leq 0, \tag{3.19}$$

where $z_0 = P_{\mathfrak{F}}u$. To show this equality, take a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{m \rightarrow \infty} \langle u - z_0, x_{n_m} - z_0 \rangle. \tag{3.20}$$

Without loss of generality, we may assume that $x_{n_m} \rightharpoonup \omega$ as $m \rightarrow \infty$ where $\omega \in C$. We shall show that $\omega \in F(T)$. From Remark 2.9, we have $F(T) = F(P_C(I - \lambda_{n_m}(I - T)))$. Assume that $\omega \neq P_C(I - \lambda_{n_m}(I - T))\omega$. Since $x_{n_m} \rightharpoonup \omega$ as $m \rightarrow \infty$, by Opial's property, (3.14) and condition (ii), we have

$$\begin{aligned} \liminf_{m \rightarrow \infty} \|x_{n_m} - \omega\| &< \liminf_{m \rightarrow \infty} \|x_{n_m} - P_C(I - \lambda_{n_m}(I - T))\omega\| \\ &\leq \liminf_{m \rightarrow \infty} (\|x_{n_m} - P_C(I - \lambda_{n_m}(I - T))x_{n_m}\| \\ &\quad + \|P_C(I - \lambda_{n_m}(I - T))x_{n_m} - P_C(I - \lambda_{n_m}(I - T))\omega\|) \\ &\leq \liminf_{m \rightarrow \infty} (\|x_{n_m} - P_C(I - \lambda_{n_m}(I - T))x_{n_m}\| \\ &\quad + \|x_{n_m} - \omega\| + \lambda_{n_m} \|(I - T)x_{n_m} - (I - T)\omega\|) \\ &\leq \liminf_{m \rightarrow \infty} \|x_{n_m} - \omega\|. \end{aligned}$$

This is a contradiction. Then $\omega \in F(T)$. From (3.18), we have

$$\lim_{m \rightarrow \infty} \|Sx_{n_m} - x_{n_m}\| = 0.$$

From the nonexpansiveness of S , $x_{n_m} \rightharpoonup \omega$ as $m \rightarrow \infty$ and Lemma 2.5, we can imply that

$$\omega \in F(S). \tag{3.21}$$

Since $P_C(I - \eta B_i)x = G_i x$ for every $x \in C$ and $i = 1, 2, \dots, N$, by Lemma 2.2, we have

$$VI(C, B_i) = F(P_C(I - \eta B_i)) = F(G_i), \quad \forall i = 1, 2, \dots, N.$$

By Lemma 2.6, we have

$$F(S) = \bigcap_{i=1}^N F(G_i) = \bigcap_{i=1}^N VI(C, B_i). \tag{3.22}$$

From (3.21) and (3.22), we have $\omega \in \bigcap_{i=1}^N VI(C, B_i)$. Hence $\omega \in \mathfrak{F}$. Since $x_{n_m} \rightharpoonup \omega$ as $m \rightarrow \infty$ and $\omega \in \mathfrak{F}$, we have

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{m \rightarrow \infty} \langle u - z_0, x_{n_m} - z_0 \rangle = \langle u - z_0, \omega - z_0 \rangle \leq 0.$$

Finally, we show that $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u$. From the definition of x_n and (3.5), we have

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &= \|\alpha_n(u - z_0) + \beta_n(P_C(I - \lambda_n(I - T))x_n - z_0) + \gamma_n(Sx_n - z_0)\|^2 \\ &\leq \|\beta_n(P_C(I - \lambda_n(I - T))x_n - z_0) + \gamma_n(Sx_n - z_0)\|^2 \\ &\quad + 2\langle \alpha_n(u - z_0), x_{n+1} - z_0 \rangle \\ &\leq \beta_n \|P_C(I - \lambda_n(I - T))x_n - z_0\|^2 + \gamma_n \|Sx_n - z_0\|^2 \\ &\quad + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle. \end{aligned}$$

From (3.19) and Lemma 2.4, we have $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u$. This completes the proof. \square

The following result can be obtained from Theorem 3.1. We, therefore, omit the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $B : C \rightarrow H$ be a δ -inverse strongly monotone mapping and let $T : C \rightarrow C$ be a κ -strictly pseudononspreading mapping for some $\kappa \in [0, 1)$. Assume that $\mathfrak{F} = F(T) \cap VI(C, B) \neq \emptyset$. For every $n \in \mathbb{N}$, let $x_1, u \in C$ and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n P_C(I - \eta B)x_n, \quad \forall n \in \mathbb{N}, \tag{3.23}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n \in [c, d] \subset (0, 1)$, $\lambda_n \in (0, 1 - \kappa)$, and $\eta \in (0, 2\delta)$, and suppose that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- (ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$,
- (iii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_{\mathfrak{F}}u$.

4 Application

In this section, by using our main result, we prove strong a convergence theorem involving a strictly pseudononspreading mapping and a finite family of strictly pseudocontractive mappings. Before proving the next theorem, we need the following definition.

Definition 4.1 The mapping $T : C \rightarrow C$ is said to be *strictly pseudocontractive* [2] with the coefficient $\kappa \in [0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C. \tag{4.1}$$

Remark 4.1 If C is a nonempty closed convex subset of H and $T : C \rightarrow C$ is a κ -strictly pseudocontractive mapping with $F(T) \neq \emptyset$, then $F(T) = VI(C, (I - T))$. To show this, put $A = I - T$. Let $z \in VI(C, (I - T))$ and $z^* \in F(T)$. Since $z \in VI(C, (I - T))$, $\langle y - z, (I - T)z \rangle \geq 0$,

$\forall y \in C$. Since $T : C \rightarrow C$ is a κ -strictly pseudocontractive mapping, we have

$$\begin{aligned} \|Tz - Tz^*\|^2 &= \|(I - A)z - (I - A)z^*\|^2 = \|z - z^* - (Az - Az^*)\|^2 \\ &= \|z - z^*\|^2 - 2\langle z - z^*, Az - Az^* \rangle + \|Az - Az^*\|^2 \\ &= \|z - z^*\|^2 - 2\langle z - z^*, (I - T)z \rangle + \|(I - T)z\|^2 \\ &\leq \|z - z^*\|^2 + \kappa \|(I - T)z\|^2. \end{aligned}$$

It implies that

$$(1 - \kappa) \|(I - T)z\|^2 \leq 2\langle z - z^*, (I - T)z \rangle \leq 0.$$

Then we have $z = Tz$, therefore $z \in F(T)$. Hence $VI(C, (I - T)) \subseteq F(T)$. It is easy to see that $F(T) \subseteq VI(C, (I - T))$.

Remark 4.2 $A = I - T$ is a $\frac{1-\kappa}{2}$ -inverse strongly monotone mapping. To show this, let $x, y \in C$, we have

$$\begin{aligned} \|Tx - Ty\|^2 &= \|(I - A)x - (I - A)y\|^2 = \|x - y - (Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\langle x - y, Ax - Ay \rangle + \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 \\ &= \|x - y\|^2 + \kappa \|Ax - Ay\|^2. \end{aligned}$$

Then we have

$$\langle x - y, Ax - Ay \rangle \geq \frac{1 - \kappa}{2} \|Ax - Ay\|^2.$$

Theorem 4.3 Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . Let $T_i^k : C \rightarrow C$ be κ_i -strictly pseudocontractive mappings for every $i = 1, 2, \dots, N$, and let $T : C \rightarrow C$ be a κ -strictly pseudononspreading mapping for some $\kappa \in [0, 1)$. Let $F_i : C \rightarrow C$ be defined by $P_C(I - \eta(I - T_i^k))x = F_i x$ for every $x \in C$ and $\eta \in (0, 1 - \kappa_i)$ for every $i = 1, 2, \dots, N$, and let $\delta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N - 1$, $\alpha_1^N \in (0, 1]$, $\alpha_2^j, \alpha_3^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let $S : C \rightarrow C$ be the S -mapping generated by F_1, F_2, \dots, F_N and $\delta_1, \delta_2, \dots, \delta_N$. Assume that $\mathfrak{F} = F(T) \cap \bigcap_{i=1}^N F(T_i^k) \neq \emptyset$. For every $n \in \mathbb{N}$, $i = 1, 2, \dots, N$, let $x_1, u \in C$ and $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n Sx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n \in [c, d] \subset (0, 1)$, $\lambda_n \subset (0, 1 - \kappa)$ and suppose that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- (ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$,
- (iii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_{\mathfrak{F}}u$.

Proof From Remark 4.2, we have $I - T_i^k$ is $\frac{1-\kappa_i}{2}$ -inverse strongly monotone for every $i = 1, 2, \dots, N$. From Remark 4.1 and Lemma 2.2, we have $F(T_i^k) = VI(C, (I - T_i^k)) = F(P_C(I - \eta(I - T_i^k)))$ for every $i = 1, 2, \dots, N$. Put $I - T_i^k \equiv B_i$ and $\delta_i = \frac{1-\kappa_i}{2}$ for every $i = 1, 2, \dots, N$ in Theorem 3.1. The conclusion of Theorem 4.3 can be obtained from Theorem 3.1 \square

Competing interests

The author declares that they have no competing interests.

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