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Convergence comparison and stability of Jungck-Kirk-type algorithms for common fixed point problems

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Abstract

The aim of this article is to introduce new hybrid iterative schemes, namely Jungck-Kirk-SP and Jungck-Kirk-CR iterative schemes, and prove convergence and stability results for these iterative schemes using certain quasi-contractive operators. Numerical examples showing the comparison of convergence rate and applications of newly introduced iterative schemes are also provided. The obtained results improve, generalize and extend the works of Olatinwo (Acta Math. Univ. Comen. LXXVII(2):299-304, 2008; Fasc. Math. 40:37-43, 2008; Mat. Vesn. 61(4):247-256, 2009; Acta Math. Acad. Paedagog. Nyházi. 25(1):105-118, 2009; Acta Univ. Apulensis 26:225-236, 2011), Chugh and Kumar (Int. J. Contemp. Math. Sci. 7(24):1165-1184, 2012; Int. J. Comput. Appl. 36(12):40-46, 2011), Bosede (Bull. Math. Anal. Appl. 2(3):65-73, 2010), Oleleru and Akewe (Fasc. Math. 47:47-61, 2011) and many others in the literature.

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1 Introduction

In the recent years, fixed and common points of operators have been approximated by using different iterative schemes (see [1–20]). Let X be a Banach space, Y an arbitrary set and $S, T : Y \to X$ such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, consider the following iterative scheme:

$$Sx_{n+1} = Tx_n, \quad n = 0, 1, \ldots. \tag{1.1}$$

This scheme is called Jungck iterative scheme and was essentially introduced by Jungck [21] in 1976. It reduces to the Picard iterative scheme when $S = I_d$ (identity mapping) and $Y = X$.

For $\alpha_n \in [0, 1]$, Singh *et al.* [18] defined the Jungck-Mann iterative scheme as follows:

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n. \tag{1.2}$$

For $\alpha_n, \beta_n, \gamma_n \in [0,1]$, Olatinwo defined the Jungck-Ishikawa [11] and Jungck-Noor [12] iterative schemes as follows:

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Ty_n, \\ Sy_n &= (1-\beta_n)Sx_n + \beta_n Tx_n \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Ty_n, \\ Sy_n &= (1-\beta_n)Sx_n + \beta_n Tz_n, \\ Sz_n &= (1-\gamma_n)Sx_n + \gamma_n Tx_n, \end{aligned} \tag{1.4}$$

respectively.

Chugh and Kumar [6] defined the Jungck-SP iterative scheme as

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sy_n + \alpha_n Ty_n, \\ Sy_n &= (1-\beta_n)Sz_n + \beta_n Tz_n, \\ Sz_n &= (1-\gamma_n)Sx_n + \gamma_n Tx_n, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

Remark 1.1 If $X = Y$ and $S = I_d$ (identity mapping), then Jungck-SP (1.5), Jungck-Noor (1.4), Jungck-Ishikawa (1.3) and the Jungck-Mann (1.2) iterative schemes, respectively, become the SP [16], Noor [22], Ishikawa [23] and Mann [24] iterative schemes.

In 2009, Olatinwo [13] introduced the Kirk-Mann and Kirk-Ishikawa iterative schemes as follows.

(a) Kirk-Mann iterative scheme:

$$x_{n+1} = \sum_{i=1}^{k} \alpha_{n,i} T^i x_n, \quad \sum_{i=1}^{k} \alpha_{n,i} = 1, n = 0,1,2,\ldots, \tag{1.6}$$

where $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0,1]$ and k is a fixed integer.

(b) Kirk-Ishikawa iterative scheme:

$$\begin{aligned} x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{k} \alpha_{n,i} T^i y_n, \quad \sum_{i=1}^{k} \alpha_{n,i} = 1, \\ y_n &= \sum_{j=0}^{s} \beta_{n,j} T^j x_n, \quad \sum_{j=0}^{s} \beta_{n,j} = 1, n = 0,1,2,\ldots, \end{aligned} \tag{1.7}$$

where $k \geq s$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\alpha_{n,i}, \beta_{n,j} \in [0,1]$ and k, s are fixed integers.

Chugh and Kumar [5] introduced the following Jungck-Kirk-Noor iterative scheme:

$$Sx_{n+1} = \alpha_{n,0} Sx_n + \sum_{i=1}^{r} \alpha_{n,i} T^i y_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1,$$

$$Sy_n = \beta_{n,0}Sx_n + \sum_{j=1}^{s}\beta_{n,j}T^j z_n, \quad \sum_{j=0}^{s}\beta_{n,j} = 1, \tag{1.8}$$
$$Sz_n = \gamma_{n,k}Sx_n + \sum_{k=1}^{t}\gamma_{n,k}T^k x_n, \quad \sum_{k=0}^{t}\gamma_{n,k} = 1, n = 0,1,2,\ldots,$$

$r \geq s \geq t$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,l} \geq 0$, $\gamma_{n,0} \neq 0$, $\alpha_{n,i}, \beta_{n,j}, \gamma_{n,l} \in [0,1]$, where r, s and t are fixed integers.

Very recently, Hussain *et al.* [25] defined the Jungck-CR iterative scheme as follows:

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sy_n + \alpha_n Ty_n, \\ Sy_n &= (1-\beta_n)Tx_n + \beta_n Tz_n, \\ Sz_n &= (1-\gamma_n)Sx_n + \gamma_n Tx_n, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

Putting $S = I_d$ (identity mapping) and $\alpha_n = 0$ in the Junck-CR iterative scheme, we get the Agarwal *et al.* iterative scheme [1].

Jungck [21] used iterative scheme (1.1) to approximate the common fixed points of the mappings S and T satisfying the following Jungck-contraction:

$$d(Tx,Ty) \leq \alpha d(Sx,Sy), \quad 0 \leq \alpha < 1. \tag{1.10}$$

Singh *et al.* [17, 18] established some stability results for Jungck and Jungck-Mann iterative schemes for both contractive conditions (1.10) and (1.11): For some $\alpha_n = 0$ and $0 \leq L$,

$$d(Tx,Ty) \leq ad(Sx,Sy) + Ld(Sx,Tx). \tag{1.11}$$

Olatinwo and Imoru [11] studied the generalized Zamfirescu operators for the pair (S,T), satisfying the following condition: For each pair of points x, y in Y, at least one of the following is true:

$$\begin{aligned} &\text{(i)} \quad d(Tx,Ty) \leq ad(Sx,Sy), \\ &\text{(ii)} \quad d(Tx,Ty) \leq b\big[d(Sx,Tx) + d(Sy,Ty)\big], \\ &\text{(iii)} \quad d(Tx,Ty) \leq c\big[d(Sx,Ty) + d(Sy,Tx)\big], \end{aligned} \tag{1.12}$$

where a, b, c are nonnegative constants satisfying $0 \leq a \leq 1$, $0 \leq b, c \leq \frac{1}{2}$.

Any mapping satisfying (1.12)(ii) is called a Kannan mapping, while the mapping satisfying (1.12)(iii) is called a Chatterjea operator.

The contractive condition (1.12) implies

$$d(Tx,Ty) \leq 2\delta d(Sx,Tx) + \delta d(Sx,Sy), \quad \forall x,y \in X, \tag{1.13}$$

where $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ (see Berinde [3]).

Ranganathan [26] used the following more general contractive condition than (1.10) and (1.12) to prove some fixed point theorems:

$$(Tx, Ty) \leq c \max\bigl\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\bigr\}, \tag{1.14}$$

$\forall x, y \in X$ and some $c \in [0, 1)$.

Note that (1.13) and (1.14) are independent of each other but more general than (1.10).

Hussain *et al.* [25] and Olatinwo [14], respectively, used the following more general contractive conditions than (1.13) to prove the stability and strong convergence results for various iterative schemes: There exists $a \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq \phi\bigl(\|Sx - Tx\|\bigr) + a\|Sx - Sy\|, \tag{1.15}$$

$$\|Tx - Ty\| \leq \frac{\phi(\|Sx - Tx\|) + \psi(\|Sx - Sy\|)}{1 + L\|Sx - Tx\|}. \tag{1.16}$$

Recently, Bosede [4] used the following more general contractive condition than (1.13) to prove the convergence results for the Jungck-Ishikawa iteration process:

$$\|Tx - Ty\| \leq e^{L\|Sx - Tx\|}\bigl\{2\delta\bigl(\|Sx - Tx\|\bigr) + \delta\|Sx - Sy\|\bigr\}. \tag{1.17}$$

Definition 1.2 [27, 28] Let f and g be two selfmaps on X. A point x in X is called (1) a fixed point of f if $f(x) = x$; (2) a coincidence point of a pair (f, g) if $fx = gx$; (3) a common fixed point of a pair (f, g) if $x = fx = gx$. If $w = fx = gx$ for some x in X, then w is called a point of coincidence of f and g. A pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points.

The stability theory has extensively been studied by various authors [7, 14, 15, 17, 18, 29, 30] due to its increasing importance in computational mathematics, especially due to revolution in computer programming.

We use the following definition and lemmas to prove our results.

Definition 1.3 [18] Let $S, T : X \to X$ be operators such that $T(X) \subseteq S(X)$ and $p = Sz = Tz$, a point of coincidence of S and T. Let $\{Sx_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative procedure

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1, \ldots, \tag{1.18}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set $\varepsilon_n = d(Sy_n, f(T, y_n))$, $n = 0, 1, \ldots$. Then the iterative procedure (1.18) is said to be (S, T)-stable or stable if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} Sy_n = p$.

Definition 1.4 [2, 3] Let $\{u_n\}$ and $\{v_n\}$ be two iteration procedures that converge to the same fixed point p on a normed space X such that the error estimates

$$\|u_n - p\| \leq a_n$$

and

$$\|v_n - p\| \leq b_n,$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}$ converges faster than $\{b_n\}$, then we say that $\{u_n\}$ converges faster to p than $\{v_n\}$.

Definition 1.5 [31, 32] Suppose that $\{a_n\}$ and $\{b_n\}$ are two real convergent sequences with limits a and b, respectively. Then $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n\to\infty}\left|\frac{a_n - a}{b_n - b}\right|.$$

Definition 1.6 [3] Any function $\psi : R^+ \to R^+$ is called a comparison function if it satisfies the following properties:

(1) ψ is monotonic increasing;

(2) $\lim_{n\to\infty} \psi^n(t) = 0$, $t \geq 0$.

Note that a comparison function always satisfies (i) $\psi(t) < t$, $t \in R^+$, (ii) $\psi(0) = 0$.

Lemma 1.7 [15] *Let $\psi : R^+ \to R^+$ be a subadditive, comparison function and let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \sum_{k=0}^{m} \delta_k \psi^k(u_n) + \varepsilon_n, \quad n = 1, 2, 3, \dots,$$

where $\delta_0, \delta_1, \dots, \delta_m \in [0,1]$ with $0 \leq \sum_{k=0}^{m} \delta_k \leq 1$, we have $\lim_{n\to\infty} u_n = 0$.

Now, we define the Jungck-Kirk-SP and Jungck-Kirk-CR iterative schemes as follows:

Let $r \geq s \geq t$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,k} \geq 0$, $\gamma_{n,0} \neq 0$, $\alpha_{n,i}, \beta_{n,j}, \gamma_{n,k} \in [0,1]$, with $\sum_{i=0}^{r} \alpha_{n,i} = \sum_{j=0}^{s} \beta_{n,j} = \sum_{k=0}^{t} \gamma_{n,k} = 1$ where r, s and t are fixed integers. Then we define the Jungck-Kirk-SP iterative scheme as follows:

$$\begin{aligned}
Sx_{n+1} &= \alpha_{n,0} Sy_n + \sum_{i=1}^{r} \alpha_{n,i} T^i y_n, \\
Sy_n &= \beta_{n,0} Sz_n + \sum_{j=1}^{s} \beta_{n,j} T^j z_n, \\
Sz_n &= \gamma_{n,0} Sx_n + \sum_{k=1}^{t} \gamma_{n,k} T^k x_n, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{1.19}$$

and the Jungck-Kirk-CR iterative scheme as follows:

$$Sx_{n+1} = \alpha_{n,0} Sy_n + \sum_{i=1}^{r} \alpha_{n,i} T^i y_n,$$

$$Sy_n = \beta_{n,0} Tx_n + \sum_{j=1}^{s} \beta_{n,j} T^j z_n, \tag{1.20}$$

$$Sz_n = \gamma_{n,0} Sx_n + \sum_{k=1}^{t} \gamma_{n,k} T^k x_n, \quad n = 0,1,2,\ldots.$$

Remark 1.8 Putting $r = s = t = 1$ in Jungck-Kirk-type iterative schemes (*i.e.*, JKCR, JKSP, JKN, JKI, JKM), we get the corresponding Jungck-type iterative schemes (*i.e.*, JCR, JKP, JN, JI, JM).

Also, we shall use the following contractive condition: Let $\psi : R^+ \to R^+$ be a comparison function such that

$$\|T^i x - p\| \le \psi^i(\|Sx - p\|), \quad \forall x \in X \text{ and } \forall i \in N, \tag{1.21}$$

where p is a point of coincidence of S, T, *i.e.*, $p = Sz = Tz$ and T^i, ψ^i denote the ith iterate of T and ψ, respectively.

The following example shows that (1.21) is more general than Jungck contraction (1.10).

Example 1.9 Let $X = Y = [0,1]$. Define T and S by

$$T(x) = \left\{ \begin{matrix} 0, & x \in [0,1) \\ \frac{1}{2}, & x = 1 \end{matrix} \right\}, \qquad Sx = x^2, \qquad \psi(t) = \frac{1}{2}t.$$

It is clear that T and S satisfy (1.21) but not Jungck-contraction (1.10).

If (S, T) are Kannan operators, then from (1.12)(ii) with $i = 1$, $y = z$ (a coincidence point of S, T), we get

$$\|Tx - Tz\| \le a\|Sx - Tx\| \le a\{\|Sx - Tz\| + \|Tz - Tx\|\},$$

which further implies

$$\|Tx - Tz\| \le \frac{a}{1-a}(\|Sx - Tz\|) \quad i.e. \quad \|Tx - p\| \le \frac{a}{1-a}(\|Sx - p\|).$$

Hence every Kannan operator satisfies (1.21) with $\psi(t) = \frac{a}{1-a}t$, $\forall t \in R^+$.

In a similar manner, it can be shown that Chatterjea operators satisfy (1.21) with $\psi(t) = \frac{b}{1-b}t$, $\forall t \in R^+$.

Therefore, we conclude that generalized Zamfirescu operators satisfy (1.21).

Also, if $\psi(t) = \delta t$, $\forall t \in R^+$, then (1.21) reduces to (1.13) as well as (1.17), with $i = 1$, $x = z$, (a coincidence point of S, T).

The condition (1.21) is more general than (1.11) as well as (1.15) and (1.16), with $i = 1$, $x = z$, $\psi(t) = at$, $\forall t \in R^+$.

Moreover, (1.14) reduces to (1.21) as follows: Let $y = z$, then from (1.14), we have

$$\begin{aligned} \|Tx - p\| &\le c\max(\|Sx - p\|, \|Sx - Tx\|, \|p - p\|, \|Sx - p\|, \|p - Tx\|) \\ &\le c\max(\|Sx - p\|, \|Sx - p\| + \|Tx - p\|, \|p - Tx\|) \end{aligned}$$

$$\begin{aligned} &\leq c\bigl(\|Sx-p\|+\|Tx-p\|\bigr) \\ &\leq \frac{c}{1-c}\|Sx-p\|. \end{aligned}$$

Hence every mapping satisfying (1.14) becomes a mapping satisfying (1.21) with $i = 1$,

$$\psi(t) = \frac{c}{1-c}, \quad \forall t \in R^+.$$

2 Main results

Lemma 2.1 *Let $\psi : R^+ \to R^+$ be a subadditive, comparison function, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \sum_{k=0}^{m} \delta_k \psi^k(u_n), \quad n = 1,2,3,\ldots, \tag{2.1}$$

where $\delta_0, \delta_1, \ldots, \delta_m \in [0,1]$ with $0 \leq \sum_{k=0}^{m} \delta_k \leq 1$, we have $\lim_{n\to\infty} u_n = 0$.

Proof Let $\bar{\psi}(u_n) = \sum_{k=0}^{m} \delta_k \psi^k(u_n)$. Now, we know that a linear combination of comparison functions is also a comparison function, hence $\bar{\psi}$ is a comparison function and it satisfies $\bar{\psi}(t) < t$ for all $t \in R^+$. Hence, $\bar{\psi}(u_n) < u_n$. Therefore, from (2.1), we have $u_{n+1} < u_n$. So, $\{u_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive numbers bounded below by 0. Hence $\{u_n\}_{n=0}^{\infty}$ will converge to 0, *i.e.*, $\lim_{n\to\infty} u_n = 0$. □

Theorem 2.2 *Let $(X, \|\cdot\|)$ be a normed linear space, let $S, T : X \to X$ be operators satisfying* (1.21) *such that $T(X) \subseteq S(X)$. Assume that $S(X)$ or $T(X)$ is a complete subspace of X, $\psi : R^+ \to R^+$ is a continuous sublinear comparison function and p is a point of coincidence of S and T, i.e., $p = Sz = Tz$. Then, for $x_0 \in X$, the Jungck-Kirk-SP iteration process $\{Sx_n\}_{n=0}^{\infty}$ defined by* (1.19) *converges to p and is (S,T)-stable. Also, p will be the unique common fixed point of S, T provided S and T are weakly compatible.*

Proof If ψ is sublinear, then ψ^i (iterate of ψ) is also sublinear (see [15]). Now, first we prove the convergence of the Jungck-Kirk-SP iterative scheme.

Using Jungck-Kirk-SP iterative scheme (1.19) and contractive condition (1.21), we have

$$\begin{aligned} \|Sx_{n+1}-p\| &\leq \alpha_{n,0}\|Sy_n-p\| + \sum_{i=1}^{r} \alpha_{n,i}\bigl\|T^i y_n - p\bigr\| \\ &\leq \alpha_{n,0}\|Sy_n-p\| + \sum_{i=1}^{r} \alpha_{n,i}\psi^i\bigl(\|Sy_n-p\|\bigr) = \sum_{i=0}^{r} \alpha_{n,i}\psi^i\bigl(\|Sy_n-p\|\bigr). \end{aligned} \tag{2.2}$$

Similarly, we have the following estimates:

$$\|Sy_n-p\| \leq \sum_{j=0}^{s} \beta_{n,j}\psi^j\bigl(\|Sz_n-p\|\bigr) \tag{2.3}$$

and

$$\|Sz_n-p\| \leq \sum_{k=0}^{t} \gamma_{n,k}\psi^k\bigl(\|Sx_n-p\|\bigr). \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that

$$\|Sx_{n+1} - p\| \leq \sum_{i=0}^{r} \alpha_{n,i} \sum_{j=0}^{s} \beta_{n,j} \sum_{k=0}^{t} \gamma_{n,k} \psi^{i+j+k}\bigl(\|Sx_n - p\|\bigr). \tag{2.5}$$

Using Lemma 2.1, (2.5) yields $\lim_{n\to\infty} Sx_n = p$.

Thus, the Jungck-Kirk-SP iterative scheme converges strongly to p.

Next we prove that the Jungck-Kirk-SP iterative scheme is (S,T)-stable.

Suppose that $\{Sy_n\}_{n=0}^{\infty} \subset X$ is an arbitrary sequence, $\varepsilon_n = \|Sy_{n+1} - \alpha_{n,0}Sb_n - \sum_{i=1}^{k} \alpha_{n,i} \times T^i b_n\|$, $n = 0, 1, 2, \ldots$, where $Sb_n = \beta_{n,0}Sc_n + \sum_{j=1}^{s} \beta_{n,j} T^j Sc_n$, $c_n = \gamma_{n,0}Sy_n + \sum_{k=1}^{t} \gamma_{n,k} T^k y_n$.

First, let $\lim_{n\to\infty} Sy_n = p$. Then, we show that $\lim_{n\to\infty} \varepsilon_n = 0$ as follows:

$$\begin{aligned}
\varepsilon_n &= \left\| Sy_{n+1} - \alpha_{n,0}Sb_n - \sum_{i=1}^{r} \alpha_{n,i} T^i b_n \right\| \leq \|Sy_{n+1} - p\| + \left\| p - \alpha_{n,0}Sb_n - \sum_{i=1}^{r} \alpha_{n,i} T^i b_n \right\| \\
&= \|Sy_{n+1} - p\| + \left\| \sum_{i=1}^{r} \alpha_{n,i} T^i z + \alpha_{n,0}Sz - \alpha_{n,0}Sb_n - \sum_{i=1}^{r} \alpha_{n,i} T^i b_n \right\| \\
&= \|Sy_{n+1} - p\| + \left\| \sum_{i=1}^{k} \alpha_{n,i}\bigl(T^i z - T^i b_n\bigr) + \alpha_{n,0}(Sz - Sb_n) \right\| \\
&\leq \|Sy_{n+1} - p\| + \sum_{i=1}^{k} \alpha_{n,i} \bigl\| T^i z - T^i b_n \bigr\| + \alpha_{n,0}\|Sb_n - p\| \\
&= \|Sy_{n+1} - p\| + \sum_{i=1}^{r} \alpha_{n,i}\psi^i\bigl(\|p - Sb_n\|\bigr) + \alpha_{n,0}\|Sb_n - p\| \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\bigl(\|p - Sb_n\|\bigr) \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\left\| \sum_{j=1}^{s} \beta_{n,j} T^j z + \beta_{n,0}Sz - \beta_{n,0}Sc_n - \sum_{j=1}^{s} \beta_{n,j} a T^j c_n \right\| \right) \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\left\| \sum_{j=1}^{s} \beta_{n,j}\bigl(T^j z - T^j c_n\bigr) + \beta_{n,0}(Sz - Sc_n) \right\| \right) \\
&\leq \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j} \bigl\| T^j z - T^j c_n \bigr\| + \beta_{n,0}\|p - Sc_n\| \right) \\
&\leq \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j}\psi^j\bigl(\|p - Sc_n\|\bigr) + \beta_{n,0}\bigl(\|p - Sc_n\|\bigr) \right) \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=0}^{s} \beta_{n,j}\psi^j\bigl(\|p - Sc_n\|\bigr) \right) \\
&= \|Sy_{n+1} - p\| \\
&\quad + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=0}^{s} \beta_{n,j}\psi^j\left(\left\| \sum_{k=1}^{t} \gamma_{n,k} T^k z + \gamma_{n,0}Sz - \gamma_{n,0}Sy_n - \sum_{k=1}^{t} \gamma_{n,k} T^k y_n \right\| \right) \right)
\end{aligned}$$

$$= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\sum_{j=0}^{s} \beta_{n,j}\psi^{j}\left(\sum_{k=1}^{t} \gamma_{n,k}\left\|T^{k}z - T^{k}y_{n}\right\| + \gamma_{n,0}(Sz - Sy_{n})\right)\right)$$

$$= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\sum_{j=0}^{s} \beta_{n,j}\psi^{j}\left(\sum_{k=1}^{t} \gamma_{n,k}\psi^{k}\left(\|p - Sy_{n}\|\right) + \gamma_{n,0}(Sz - Sy_{n})\right)\right)$$

$$= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\sum_{j=0}^{s} \beta_{n,j}\psi^{j}\left(\sum_{k=0}^{t} \gamma_{n,k}\psi^{k}\left(\|p - Sy_{n}\|\right)\right)\right)$$

$$= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\sum_{j=0}^{s} \beta_{n,j}\sum_{k=0}^{t} \gamma_{n,k}\psi^{i+j+k}\left(\|Sy_{n} - p\|\right). \tag{2.6}$$

Using Lemma 1.7, (2.6) yields $\varepsilon_n = 0$ as $n \to \infty$.

Conversely, we establish that $\lim_{n\to\infty} Sy_n = p$ as follows:

$$\begin{aligned}
\|Sy_{n+1} - p\| &\leq \left\|Sy_{n+1} - \alpha_{n,0}Sb_{n} - \sum_{i=1}^{r} \alpha_{n,i}T^{i}b_{n}\right\| \\
&\quad + \left\|\alpha_{n,0}Sb_{n} + \sum_{i=1}^{r} \alpha_{n,i}T^{i}b_{n} - p\right\| \\
&= \varepsilon_{n} + \left\|\alpha_{n,0}Sb_{n} + \sum_{i=1}^{r} \alpha_{n,i}T^{i}b_{n} - \sum_{i=1}^{r} \alpha_{n,i}T^{i}z - \alpha_{n,0}Sz\right\| \\
&= \varepsilon_{n} + \left\|\sum_{i=1}^{r} \alpha_{n,i}\left(T^{i}b_{n} - T^{i}z\right) + \alpha_{n,0}(Sb_{n} - Sz)\right\| \\
&\leq \varepsilon_{n} + \sum_{i=1}^{r} \alpha_{n,i}\left\|T^{i}b_{n} - T^{i}z\right\| + \alpha_{n,0}\|Sb_{n} - p\| \\
&\leq \varepsilon_{n} + \sum_{i=1}^{r} \alpha_{n,i}\psi^{i}\left(\|p - Sb_{n}\|\right) + \alpha_{n,0}\|Sb_{n} - p\| \\
&= \varepsilon_{n} + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\left\|\sum_{j=1}^{s} \beta_{n,j}T^{j}z + \beta_{n,0}Sz - \sum_{j=1}^{s} \beta_{n,j}T^{j}c_{n} - \beta_{n,0}Sc_{n}\right\|\right) \\
&= \varepsilon_{n} + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\left\|\sum_{j=1}^{s} \beta_{n,j}\left(T^{j}z - T^{j}c_{n}\right) + \beta_{n,0}(Sz - Sc_{n})\right\|\right) \\
&= \varepsilon_{n} + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\sum_{j=1}^{s} \beta_{n,j}\psi^{j}\left(\|p - Sc_{n}\|\right) + \beta_{n,0}\|Sc_{n} - p\|\right) \\
&= \varepsilon_{n} + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\sum_{j=0}^{s} \beta_{n,j}\psi^{j}\left(\|p - Sc_{n}\|\right)\right) \\
&= \varepsilon_{n} + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\sum_{j=0}^{s} \beta_{n,j}\psi^{j}\left(\left\|\sum_{k=1}^{t} \gamma_{n,k}\left(T^{k}z - T^{k}y_{n}\right) + \gamma_{n,0}\left(Sz - Sy_{n}\right)\right\|\right) \\
&= \varepsilon_{n} + \sum_{i=0}^{r} \alpha_{n,i}\psi^{i}\left(\sum_{j=0}^{s} \beta_{n,j}\psi^{j}\left(\sum_{k=0}^{t} \gamma_{n,k}\psi^{k}\left(\|p - Sy_{n}\|\right)\right)\right).
\end{aligned} \tag{2.7}$$

Using again Lemma 1.7, (2.7) yields $\lim_{n\to\infty} Sy_n = p$.

Now, we prove p is the unique common fixed point of S and T provided S, T are weakly compatible. Let there exist another point of coincidence say p^*. Then there exists $q^* \in X$ such that $Sq^* = Tq^* = p^*$. But from (1.21), we have

$$0 < \|p^* - p\| \leq \|T^i q^* - p\| \leq \psi(\|Sq^* - p\|) < \|p^* - p\|,$$

which implies $p = p^*$.

Now, as S and T are weakly compatible and $p = Tq = Sq$, so $Tp = TTq = TSq = STq$ and hence $Tp = Sp$. Therefore Tp is a point of coincidence of S, T and since the point of coincidence is unique, then $p = Tp$. Thus $Tp = Sp = p$ and therefore p is the unique common fixed point of S and T. □

Remark 2.3 Since the Jungck-Kirk-Mann iteration scheme is a special case of the Jungck-Kirk-SP iteration scheme, the convergence and stability result similar to Theorem 2.2 also holds for the Jungck-Kirk-Mann scheme.

Theorem 2.4 *Let* $(X, \|\cdot\|)$ *be a normed linear space, let* $S, T : X \to X$ *be operators satisfying* (1.21) *such that* $T(X) \subseteq S(X)$. *Assume that* $S(X)$ *or* $T(X)$ *is a complete subspace of* X, $\psi : R^+ \to R^+$ *is a continuous sublinear comparison function and* p *is a point of coincidence of* S *and* T. *Then, for* $x_0 \to X$, *the Jungck-Kirk-CR iteration process* $\{Sx_n\}_{n=0}^{\infty}$ *defined by* (1.20) *converges to* p *and is* (S, T)*-stable. Also,* p *will be the unique common fixed point of* S, T *provided* S *and* T *are weakly compatible.*

Proof Using Jungck-Kirk-CR iterative scheme (1.20), we have

$$\begin{aligned}\|Sx_{n+1} - p\| &\leq \alpha_{n,0}\|Sy_n - p\| + \sum_{i=1}^{r} \alpha_{n,i}\|T^i y_n - p\| \\ &= \alpha_{n,0}\|Sy_n - p\| + \sum_{i=1}^{r} \alpha_{n,i}\psi^i(\|Sy_n - p\|) \\ &= \sum_{i=0}^{r} \alpha_{n,i}\psi^i(\|Sy_n - p\|). \end{aligned} \tag{2.8}$$

In a similar manner, we have the following estimates:

$$\begin{aligned}\|Sy_n - p\| &\leq \beta_{n,0}\|Tx_n - p\| + \sum_{j=1}^{s} \beta_{n,j}\|T^j z_n - p\| \\ &= \beta_{n,0}\psi(\|Sx_n - p\|) + \sum_{j=1}^{s} \beta_{n,j}\psi^j(\|Sz_n - p\|) \end{aligned} \tag{2.9}$$

and

$$\|Sz_n - p\| \leq \sum_{k=0}^{t} \gamma_{n,k}\psi^k(\|Sx_n - p\|). \tag{2.10}$$

It follows from (2.8), (2.9) and (2.10) that

$$\begin{aligned}&\|Sx_{n+1}-p\|\\&\quad\le\sum_{i=0}^{r}\alpha_{n,i}\psi^i\left(\beta_{n,0}\psi\left(\|Sx_n-p\|\right)+\sum_{j=1}^{s}\beta_{n,j}\psi^j\left(\sum_{k=0}^{t}\gamma_{n,k}\psi^k\left(\|Sx_n-p\|\right)\right)\right).\end{aligned}\tag{2.11}$$

Let

$$\bar{\psi}=\beta_{n,0}\psi+\sum_{j=1}^{s}\beta_{n,j}\sum_{k=0}^{t}\gamma_{n,k}\psi^{k+j}.\tag{2.12}$$

Then, obviously, being the linear combination of comparison functions, $\tilde{\psi}$ is also a comparison function and hence (2.11) yields

$$\|Sx_{n+1}-p\|\le\sum_{i=1}^{r}\alpha_{n,i}\psi^i\left(\bar{\psi}\left(\|Sx_n-p\|\right)\right).\tag{2.13}$$

Since $\sum_{i=0}^{k}\alpha_{n,i}=1$, hence using Lemma 2.1, (2.13) yields $\lim_{n\to\infty}Sx_n=p$.

Thus, the Jungck-Kirk-CR iterative scheme converges strongly to p.

Next we prove that the Jungck-Kirk-CR iterative scheme is (S,T)-stable.

Suppose that $\{Sy_n\}_{n=0}^{\infty}\subset X$ is an arbitrary sequence, $\varepsilon_n=\|Sy_{n+1}-\alpha_{n,0}Sb_n-\sum_{i=1}^{r}\alpha_{n,i}\times T^ib_n\|$, $n=0,1,2,\dots$, where $Sb_n=\beta_{n,0}Ty_n+\sum_{j=1}^{s}\beta_{n,j}T^jc_n$, $Sc_n=\gamma_{n,0}Sy_n+\sum_{k=1}^{t}\gamma_{n,k}T^ky_n$ and $\lim_{n\to\infty}\varepsilon_n=0$. We shall establish that $\lim_{n\to\infty}Sy_n=p$ as follows:

$$\begin{aligned}\|Sy_{n+1}-p\|&\le\left\|Sy_{n+1}-\alpha_{n,0}Sb_n-\sum_{i=1}^{r}\alpha_{n,i}T^ib_n\right\|\\&\quad+\left\|\alpha_{n,0}Sb_n+\sum_{i=1}^{r}\alpha_{n,i}T^ib_n-p\right\|\\&=\varepsilon_n+\left\|\alpha_{n,0}Sb_n+\sum_{i=1}^{r}\alpha_{n,i}T^ib_n-\sum_{i=1}^{r}\alpha_{n,i}T^iz-\alpha_{n,0}Sz\right\|\\&=\varepsilon_n+\left\|\sum_{i=1}^{r}\alpha_{n,i}\left(T^ib_n-T^iz\right)+\alpha_{n,0}(Sb_n-Sz)\right\|\\&\le\varepsilon_n+\sum_{i=1}^{r}\alpha_{n,i}\left\|T^ib_n-T^iz\right\|+\alpha_{n,0}\|Sb_n-p\|\\&=\varepsilon_n+\sum_{i=1}^{r}\alpha_{n,i}\psi^i\left(\|p-Sb_n\|\right)+\alpha_{n,0}\|Sb_n-p\|\\&=\varepsilon_n+\sum_{i=0}^{r}\alpha_{n,i}\psi^i\left(\left\|\sum_{j=1}^{s}\beta_{n,j}T^jz+\beta_{n,0}Sz-\beta_{n,0}Ty_n-\sum_{j=1}^{s}\beta_{n,j}T^jc_n\right\|\right)\\&=\varepsilon_n+\sum_{i=0}^{r}\alpha_{n,i}\psi^i\left(\left\|\sum_{j=1}^{s}\beta_{n,j}\left(T^jz-T^jc_n\right)+\beta_{n,0}(Sz-Ty_n)\right\|\right)\end{aligned}$$

$$\begin{aligned}
&= \varepsilon_n + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\left\|\sum_{j=1}^{s} \beta_{n,j}\psi^j(p - Sc_n) + \beta_{n,0}\psi(Sz - Sy_n)\right\|\right) \\
&= \varepsilon_n + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j}\psi^j\left(\left\|\sum_{k=1}^{t} \gamma_{n,k}\left(T^k z - T^k y_n\right)\right.\right.\right. \\
&\quad \left.\left.\left. + \gamma_{n,0}(Sz - Sy_n)\right\|\right) + \beta_{n,0}\psi\left(\|Sy_n - Sz\|\right)\right) \\
&= \varepsilon_n + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j}\psi^j\left(\left\|\sum_{k=1}^{t} \gamma_{n,k}\psi^k(Sz - Sy_n)\right.\right.\right. \\
&\quad \left.\left.\left. + \gamma_{n,0}(Sz - Sy_n)\right\|\right) + \beta_{n,0}\psi\left(\|Sy_n - p\|\right)\right) \\
&\le \varepsilon_n + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j}\psi^j\left(\sum_{k=0}^{t} \gamma_{n,k}\psi^k\left(\|Sz - Sy_n\|\right)\right)\right. \\
&\quad \left. + \beta_{n,0}\psi\left(\|Sy_n - p\|\right)\right).
\end{aligned} \tag{2.14}$$

Then using (2.12) and (2.14), we get

$$\|Sy_{n+1} - p\| \le \varepsilon_n + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\bar{\psi}\left(\|Sy_n - p\|\right)\right). \tag{2.15}$$

Using Lemma 1.7, from (2.15) we obtain $\lim_{n\to\infty} Sy_n = p$.

Conversely, let $\lim_{n\to\infty} Sy_n = p$. Then we shall show that $\lim_{n\to\infty} \varepsilon_n = 0$ as follows:

$$\begin{aligned}
\varepsilon_n &= \left\|Sy_{n+1} - \alpha_{n,0}Sb_n - \sum_{i=1}^{r} \alpha_{n,i}T^i b_n\right\| \\
&\le \|Sy_{n+1} - p\| + \left\|p - \alpha_{n,0}Sb_n - \sum_{i=1}^{r} \alpha_{n,i}T^i b_n\right\| \\
&= \|Sy_{n+1} - p\| + \left\|\sum_{i=1}^{r} \alpha_{n,i}T^i z + \alpha_{n,0}Sz - \alpha_{n,0}Sb_n - \sum_{i=1}^{r} \alpha_{n,i}T^i b_n\right\| \\
&= \|Sy_{n+1} - p\| + \left\|\sum_{i=1}^{r} \alpha_{n,i}\left(T^i z - T^i b_n\right) + \alpha_{n,0}(Sz - Sb_n)\right\| \\
&\le \|Sy_{n+1} - p\| + \sum_{i=1}^{r} \alpha_{n,i}\left\|T^i z - T^i b_n\right\| + \alpha_{n,0}\|Sb_n - p\| \\
&\le \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\|p - Sb_n\|\right) + \alpha_{n,0}\|Sb_n - p\| \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\left\|\sum_{j=1}^{s} \beta_{n,j}T^j z + \beta_{n,0}Sz - \beta_{n,0}Ty_n - \sum_{j=1}^{s} \beta_{n,j}T^j c_n\right\|\right) \\
&= \|Sy_{n+1} - p\| + \left(\sum_{i=0}^{k} \alpha_{n,i}\psi^i\left(\left\|\sum_{j=1}^{s} \beta_{n,j}\left(T^j z - T^j c_n\right) + \beta_{n,0}(Sz - Ty_n)\right\|\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \|Sy_{n+1} - p\| + \left(\sum_{i=0}^{k} \alpha_{n,i}\psi^i\left(\left\|\sum_{j=1}^{s} \beta_{n,j}\psi^j(Sz - Sc_n) + \beta_{n,0}\psi(Sz - Sy_n)\right\|\right)\right) \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{k} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j}\psi^j\left(\left\|\sum_{k=1}^{t} \gamma_{n,k}T^k z + \gamma_{n,0}Sz\right.\right.\right. \\
&\quad \left.\left. - \sum_{k=1}^{t} \gamma_{n,k}T^k y_n - \gamma_{n,0}Sy_n\right\|\right) + \beta_{n,0}\psi\big(\|Sy_n - p\|\big)\Bigg) \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j}\psi^j\left(\left\|\sum_{k=1}^{t} \gamma_{n,k}\big(T^k z - T^k y_n\big)\right.\right.\right. \\
&\quad \left.\left. + \gamma_{n,0}(Sz - Sy_n)\right\|\right) + \beta_{n,0}\psi\big(\|Sy_n - p\|\big)\Bigg) \\
&= \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\left(\sum_{j=1}^{s} \beta_{n,j}\psi^j\left(\sum_{k=1}^{t} \gamma_{n,k}\psi^k\big(\|Sz - Sy_n\|\big)\right.\right. \\
&\quad \left. + \gamma_{n,0}\|Sz - Sy_n\|\right) + \beta_{n,0}\psi\big(\|Sy_n - p\|\big)\Bigg).
\end{aligned}
\tag{2.16}
$$

Using (2.12) and (2.16), we get

$$
\varepsilon_n \leq \|Sy_{n+1} - p\| + \sum_{i=0}^{r} \alpha_{n,i}\psi^i\big(\bar{\psi}\big(\|Sx_n - p\|\big)\big). \tag{2.17}
$$

Lemma 1.7 implies that $\lim_{n\to\infty} \varepsilon_n = 0$.

Thus, the Jungck-Kirk-CR iterative scheme is (S,T)-stable.

The uniqueness of a common fixed point can be proved in the same lines as in Theorem 2.2. □

The following example shows the validity of our Theorems 2.2 and 2.4.

Example 2.5 Let $X = [0,1]$, $T(x) = \frac{1}{2}(\frac{1}{2} + x)$, $S(x) = 1 - x$, $\alpha_{n,0} = 1 - \alpha_{n,1} - \alpha_{n,2}$, $\beta_{n,0} = 1 - \beta_{n,1} - \beta_{n,2}$, $\gamma_{n,0} = 1 - \gamma_{n,1} - \gamma_{n,2}$, $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \alpha_{n,2} = \beta_{n,2} = \gamma_{n,2} = \frac{1}{\sqrt{2n+4}}$ and $\psi(t) = \frac{t}{2}$. It is clear that T and S are weakly compatible operators satisfying (1.21) with a unique common fixed point 0.5. Convergence of the Junck-Kirk-CR iterative scheme as well as the Junck-Kirk-SP iterative scheme to 0.5 is shown in Example 3.2.

3 Results on direct comparison

Various authors [2, 7, 8, 16, 19, 20, 25, 31–34] have worked on convergence speed of iterative schemes. In [2] Berinde showed that Picard iteration is faster than Mann iteration for quasi-contractive operators. In [31], Qing and Rhoades by taking an example showed that Ishikawa iteration is faster than Mann iteration for a certain class of quasi-contractive operators. Chugh and Kumar [7] showed that the SP iterative scheme with error terms converges faster than Ishikawa and Noor iterative schemes for accretive type mappings. Very recently, Hussain *et al.* [25] showed that Jungck-CR and Jungck-SP iterative schemes have a better convergence rate as compared to other Jungck-type iterative schemes existing in the literature.

Theorem 3.1 *Let $(X, \|\cdot\|)$ be a normed linear space, let $S, T : X \to X$ be operators satisfying* (1.21) *such that $T(X) \subseteq S(X)$. Assume that $S(X)$ or $T(X)$ is a complete subspace of X, $\psi : R^+ \to R^+$ is a continuous sublinear comparison function and p is a point of coincidence of S, T (i.e., $Sz = Tz = p$). If $\lim_{n\to\infty} \alpha_n = 0$ then for $x_0 \in X$,*

1. *Jungck-Kirk-Mann* (*JKM*) *iterative scheme is faster than Jungck-Mann* (*JM*) *iterative scheme*;
2. *Jungck-Kirk-Ishikawa* (*JKI*) *iterative scheme is faster than Jungck-Ishikawa* (*JI*) *iterative scheme*;
3. *Jungck-Kirk-Noor* (*JKN*) *iterative scheme is faster than Jungck-Noor* (*JN*) *iterative scheme*;
4. *Jungck-Kirk-SP* (*JKSP*) *iterative scheme is faster than Jungck-SP* (*JSP*) *iterative scheme*;
5. *Jungck-Kirk-CR* (*JKCR*) *iterative scheme is faster than Jungck-CR* (*JCR*) *iterative scheme.*

Proof For a Jungck-type iterative scheme, we have the following estimates:

$$\begin{aligned}
\|JM_{n+1} - p\| &\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\big(\|Tx_n - p\|\big) \\
&\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\psi\big(\|Sx_n - p\|\big) \\
&\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\big(\|Sx_n - p\|\big) \quad \big(\text{using } \psi(t) < t, \forall t \in R^+\big) \\
&\geq (1-2\alpha_n)\|Sx_n - p\| \\
&\dots \\
&\geq \prod_{l=0}^{n}(1-2\alpha_l)\|Sx_0 - p\|,
\end{aligned}$$

$$\begin{aligned}
\|JI_{n+1} - p\| &\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\psi\big(\|Sy_n - p\|\big) \\
&\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\big(\|Sy_n - p\|\big) \\
&\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\big(\|Sx_n - p\|\big) \\
&\geq (1-2\alpha_n)\|Sx_n - p\| \\
&\dots \\
&\geq \prod_{l=0}^{n}(1-2\alpha_l)\|Sx_0 - p\|,
\end{aligned}$$

$$\begin{aligned}
\|JN_{n+1} - p\| &\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\psi\big(\|Sy_n - p\|\big) \\
&\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\big(\|Sy_n - p\|\big) \\
&\geq (1-\alpha_n)\|Sx_n - p\| - \alpha_n\big(\|Sz_n - p\|\big) \\
&\geq (1-2\alpha_n)\|Sx_n - p\| \\
&\dots \\
&\geq \prod_{l=0}^{n}(1-2\alpha_l)\|Sx_0 - p\|,
\end{aligned}$$

$$\begin{aligned}
\|JSP_{n+1}-p\| &\geq [1-2\alpha_n]\|Sy_n-p\| \\
&\geq [1-2\alpha_n][1-2\beta_n][1-2\gamma_n]\|Sx_n-p\| \\
&\geq [1-2\alpha_n]\|Sx_n-p\| \\
&\cdots \\
&\geq \prod_{l=0}^{n}(1-2\alpha_l)\|Sx_0-p\|
\end{aligned}$$

and

$$\begin{aligned}
\|JCR_{n+1}-p\| &\leq (1-\alpha_n)\|Sy_n-p\| + \alpha_n\psi\big(\|Sy_n-p\|\big) \\
&\leq (1-\alpha_n)\big[(1-\beta_n)\|Tx_n-p\| + \beta_n\big(\|Tz_n-p\|\big)\big] \\
&\quad + \alpha_n\psi\big((1-\beta_n)\|Tx_n-p\| + \beta_n\big(\|Tz_n-p\|\big)\big) \\
&\leq (1-\alpha_n)(1-\beta_n)\psi\big(\|Sx_n-p\|\big) + (1-\alpha_n)\beta_n\psi\big(\|Sz_n-p\|\big) \\
&\quad + \alpha_n(1-\beta_n)\psi^2\big(\|Sx_n-p\|\big) + \alpha_n\beta_n\psi^2\big(\|Sz_n-p\|\big) \\
&\leq (1-\alpha_n)(1-\beta_n)\psi\big(\|Sx_n-p\|\big) + (1-\alpha_n)\beta_n\psi\big((1-\gamma_n)\|Sx_n-p\| \\
&\quad + \gamma_n\big(\|Tx_n-p\|\big)\big) + \alpha_n(1-\beta_n)\psi^2\big(\|Sx_n-p\|\big) \\
&\quad + \alpha_n\beta_n\psi^2\big((1-\gamma_n)\|Sx_n-p\| + \gamma_n\big(\|Tx_n-p\|\big)\big) \\
&\leq (1-\alpha_n)(1-\beta_n)\psi\big(\|Sx_n-p\|\big) + (1-\alpha_n)\beta_n(1-\gamma_n)\psi\big(\|Sx_n-p\|\big) \\
&\quad + (1-\alpha_n)\beta_n\gamma_n\psi^2\big(\|Sx_n-p\|\big) + \alpha_n(1-\beta_n)\psi^2\big(\|Sx_n-p\|\big) \\
&\quad + \alpha_n\beta_n(1-\gamma_n)\psi^2\big(\|Sx_n-p\|\big) \\
&\quad + \alpha_n\beta_n\gamma_n\psi^3\big(\|Sx_n-p\|\big)
\end{aligned}$$

Also, for Jungck-Kirk-type iterative schemes, we have the following estimates:

$$\begin{aligned}
\|JKM_{n+1}-p\| &\leq \sum_{i=0}^{r}\alpha_{n,i}\psi^i\big(\|Sx_n-p\|\big) \\
&\leq \psi\big(\|Sx_n-p\|\big) \quad \left(\text{using } \psi^i \leq \psi \text{ and } \sum_{i=0}^{r}\alpha_{n,i}=1\right) \\
&\cdots \\
&\leq \psi^n\big(\|Sx_0-p\|\big), \\
\|JKI_{n+1}-p\| &\leq \alpha_{n,0}\|Sx_n-p\| + \sum_{i=1}^{r}\alpha_{n,i}\sum_{j=0}^{s}\beta_{n,j}\psi^{i+j}\big(\|Sx_n-p\|\big) \\
&\leq \sum_{i=0}^{r}\alpha_{n,i}\psi^i\big(\|Sx_n-p\|\big) \quad \left(\text{using } \psi^{i+j} \leq \psi^i \text{ and } \sum_{j=0}^{s}\beta_{n,j}=1\right) \\
&\leq \psi\big(\|Sx_n-p\|\big) \\
&\cdots \\
&\leq \psi^n\big(\|Sx_0-p\|\big),
\end{aligned}$$

$$\begin{aligned}
\|JKN_{n+1}-p\| &\le \alpha_{n,0}\|Sx_n-p\| + \sum_{i=1}^{r}\alpha_{n,i}\psi^i\left(\beta_{n,0}\big(\|Sx_n-p\|\big)+\sum_{j=1}^{s}\beta_{n,j}\psi^j\big(\|Sz_n-p\|\big)\right) \\
&\le \alpha_{n,0}\|Sx_n-p\| \\
&\quad + \sum_{i=1}^{r}\alpha_{n,i}\psi^i\left(\beta_{n,0}\big(\|Sx_n-p\|\big)+\sum_{j=1}^{s}\beta_{n,j}\psi^j\left(\sum_{k=0}^{t}\gamma_{n,k}\psi^k\big(\|Sx_n-p\|\big)\right)\right) \\
&\le \sum_{i=1}^{r}\alpha_{n,i}\psi^i\big(\|Sx_n-p\|\big) \\
&\quad \left(\text{using } \psi^l \le \psi \text{ for } l=k,j \text{ and } \sum_{j=0}^{s}\beta_{n,j}=\sum_{k=0}^{t}\gamma_{n,k}=1\right) \\
&\le \psi\big(\|Sx_n-p\|\big) \\
&\cdots \\
&\le \psi^n\big(\|Sx_0-p\|\big), \\
\|JKSP_{n+1}-p\| &\le \left(\sum_{i=0}^{r}\alpha_{n,i}\right)\left(\sum_{j=0}^{s}\beta_{n,j}\right)\left(\sum_{k=0}^{t}\gamma_{n,k}\right)\psi^{i+j+k}\big(\|Sx_n-p\|\big) \\
&\le \psi\big(\|Sx_n-p\|\big) \\
&\quad \left(\text{using } \psi^{i+j+k} \le \psi \text{ and } \left(\sum_{i=0}^{r}\alpha_{n,i}\right)\left(\sum_{j=0}^{s}\beta_{n,j}\right)\left(\sum_{k=0}^{t}\gamma_{n,k}\right)=1\right) \\
&\cdots \\
&\le \psi^{n+1}\big(\|Sx_0-p\|\big)
\end{aligned}$$

and

$$\|JKCR_{n+1}-p\| \le \sum_{i=1}^{r}\alpha_{n,i}\psi^i\left(\beta_{n,0}\psi\big(\|Sx_n-p\|\big)+\sum_{j=1}^{s}\beta_{n,j}\psi^j\left(\sum_{k=0}^{t}\alpha_{n,k}\psi^k\big(\|Sz_n-p\|\big)\right)\right)$$

(using (2.13)).

Using the above estimates, we have

$$\frac{\|JKM_{n+1}-p\|}{\|JM_{n+1}-p\|} \le \frac{\psi^{n+1}(\|Sx_0-p\|)}{\prod_{l=0}^{n}(1-2\alpha_l)\|Sx_0-p\|}.$$

Let

$$p_n = \frac{\psi^{n+1}(\|Sx_0-p\|)}{\prod_{l=0}^{n}(1-2\alpha_l)\|Sx_0-p\|}.$$

Then

$$\frac{p_{n+1}}{p_n} = \frac{\psi^{n+2}(\|Sx_0-p\|)}{\psi^{n+1}(\|Sx_0-p\|)(1-2\alpha_{n+1})}.$$

But $\psi^{n+2}(t) < \psi^{n+1}(t)$, $t \in R^+$. Hence

$$\frac{\psi^{n+2}(\|Sx_0 - p\|)}{\psi^{n+1}(\|Sx_0 - p\|)} = \delta \text{ (say)} < 1.$$

Therefore, $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = \lim_{n\to\infty} \frac{\delta}{1-2\alpha_{n+1}} = \delta < 1$. So, by ratio test $\sum p_n$ is convergent.

Hence $\lim_{n\to\infty} p_n = 0$, which further implies $\lim_{n\to\infty} \frac{\|JKM_{n+1}-p\|}{\|JM_{n+1}-p\|} = 0$, *i.e.*, the JKM iterative scheme converges faster than the JM iterative scheme in view of Definition 1.5.

Now, since the estimates for JKI, JKN and JKSP are similar to that of JKM, also estimates of JI, JN and JSP are similar to that of JM, therefore using a very similar argument, it can be easily shown that JKI, JKN, JKSP iterative schemes converge faster than JI, JN, JSP iterative schemes.

Now we compare JCR and JKCR iterative schemes.

From estimates of JCR and JKCR iterative schemes, we have

$$\|JCR_{n+1} - p\| \le \delta_1$$

and

$$\|JKCR_{n+1} - p\| \le \delta_2,$$

where

$$\begin{aligned}\delta_1 &= (1-\alpha_n)(1-\beta_n)\psi\big(\|Sx_n - p\|\big) + (1-\alpha_n)\beta_n(1-\gamma_n)\psi\big(\|Sx_n - p\|\big)\\ &\quad + (1-\alpha_n)\beta_n\gamma_n\psi^2\big(\|Sx_n - p\|\big) + \alpha_n(1-\beta_n)\psi^2\big(\|Sx_n - p\|\big)\\ &\quad + \alpha_n\beta_n(1-\gamma_n)\psi^2\big(\|Sx_n - p\|\big) + \alpha_n\beta_n\gamma_n\psi^3\big(\|Sx_n - p\|\big)\end{aligned}$$

and

$$\delta_2 = \sum_{i=1}^{r} \alpha_{n,i}\psi^i\left(\beta_{n,0}\psi\big(\|Sx_n - p\|\big) + \left(\sum_{j=1}^{s}\beta_{n,j}\psi^j\left(\sum_{k=0}^{t}\alpha_{n,k}\psi^k\big(\|Sx_n - p\|\big)\right)\right)\right)$$

(using (2.15)).

Using $\psi^j < \psi^k$, for $j > k$, it can be easily observed that $\delta_2 < \delta_1$.

Therefore, in view of Berinde's Definition 1.4, Jungck-Kirk-CR have better convergence rate as compared to the Jungck-CR iterative scheme. □

Example 3.2 Let S, T and X be the same as in Example 2.5. Then convergence speed comparison of Jungck-Kirk-type iterative schemes with corresponding Jungck-type iterative schemes is shown in Table 1 with initial approximation $x_0 = 0.9$ and $r = s = t = 2$.

Remark 3.3 Although direct comparison among Jungck-Kirk-type iterative schemes is not possible in view of Rhoades Definition 1.5, yet the following example shows that newly introduced iterative schemes have better convergence rate.

Table 1 Comparison of Jungck-Kirk-type iterative schemes with corresponding Jungck-type iterative schemes

n	JKCR	JCR	JKSP	JSP	JKN	JN	JKI	KI	JKM	KM
0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
1	0.521094	0.53125	0.521094	0.49375	0.521094	0.34375	0.44375	0.325	0.65	0.4
2	0.50014	0.495457	0.499961	0.499636	0.502443	0.423889	0.487565	0.41029	0.481598	0.461237
3	0.499999	0.500867	**0.5**	0.499962	0.500531	0.458379	0.496449	0.449456	0.49949	0.481794
4	**0.5**	0.499806	**0.5**	0.499995	0.500153	0.475493	0.498805	0.46964	0.499933	0.49043
5	**0.5**	0.500048	**0.5**	0.499999	0.500052	0.484795	0.499548	0.480889	0.499986	0.494574
6	**0.5**	0.499987	**0.5**	**0.5**	0.50002	0.490185	0.499814	0.487527	0.499996	0.496749
7	**0.5**	0.500004	**0.5**	**0.5**	0.500008	0.493463	0.499918	0.491619	0.499999	0.497968
8	**0.5**	0.499999	**0.5**	**0.5**	0.500004	0.495534	0.499962	0.494233	**0.5**	0.498687
9	**0.5**	**0.5**	**0.5**	**0.5**	0.500002	0.496883	0.499981	0.495951	**0.5**	0.499127
10	**0.5**	**0.5**	**0.5**	**0.5**	0.500001	0.497785	0.49999	0.497108	**0.5**	0.499406
11	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.498401	0.499995	0.497903	**0.5**	0.499588
12	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.49883	0.499997	0.498459	**0.5**	0.499709
13	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499133	0.499999	0.498855	**0.5**	0.499792
14	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499351	0.499999	0.49914	**0.5**	0.499849
15	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499509	**0.5**	0.499348	**0.5**	0.499889
16	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	-	**0.5**	-	**0.5**	-
-	-	-	-	-	-	-	-	-	-	-
36	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499995	**0.5**	0.499993	**0.5**	0.499999
37	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499996	**0.5**	0.499994	**0.5**	0.499999
38	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499996	**0.5**	0.499995	**0.5**	0.499999
39	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499997	**0.5**	0.499996	**0.5**	**0.5**
40	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499997	**0.5**	0.499996	**0.5**	**0.5**
-	-	-	-	-	-	-	-	-	-	-
48	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499999	**0.5**	0.499999	**0.5**	**0.5**
49	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499999	**0.5**	0.499999	**0.5**	**0.5**
50	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499999	**0.5**	0.499999	**0.5**	**0.5**
51	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499999	**0.5**	**0.5**
52	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	0.499999	**0.5**	**0.5**
53	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**
54	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**

Example 3.4 Let $X = [0,1]$, $S : X \to X = \frac{x}{2}$, $T : X \to X = \frac{x}{4}$, $\alpha_{n,j} = \beta_{n,j} = \gamma_{n,j} = 0$, $j = 0,1,2$, $n = 1,2,\dots,n_0$ for some $n_0 \in N$ and $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \alpha_{n,0} = \beta_{n,0} = \gamma_{n,0} = \frac{4}{\sqrt{n}}$, $\alpha_{n,2} = \beta_{n,2} = \gamma_{n,2} = 1 - \frac{8}{\sqrt{n}}$, $n > n_0$. It is clear that T and S are operators satisfying (1.21) with a unique common fixed point 0. Also, it is easy to see that Example 3.4 satisfies all the conditions of Theorems 2.2 and 2.4.

Proof For JM, JI, JN, JCR, JSP, JKM, JKI, JKN, JKCR and JKSP, iterative schemes with initial approximation $x_0 \neq 0$, we have the following equations:

$$JM_n = \prod_{i=n_0}^{n} \left(\frac{1}{2} - \frac{1}{\sqrt{i}} \right) x_0,$$

$$JI_n = \prod_{i=n_0}^{n} \left(\frac{1}{2} - \frac{1}{\sqrt{i}} - \frac{2}{i} \right) x_0,$$

$$JN_n = \prod_{i=n_0}^{n} \left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i} - \frac{8}{i^{\frac{3}{2}}} \right) x_0,$$

$$JCR_n = \prod_{i=n_0}^{n} \left(\frac{1}{4} - \frac{1}{2\sqrt{i}} - \frac{2}{i} + \frac{4}{i^{\frac{3}{2}}} \right) x_0,$$

$$JSP_n = \prod_{i=n_0}^{n}\left(\frac{1}{2} - \frac{3}{\sqrt{i}} + \frac{6}{i} - \frac{4}{i^{\frac{3}{2}}}\right)x_0,$$

$$JKM_n = \prod_{i=n_0}^{n}\left(\frac{1}{16} + \frac{5}{2\sqrt{i}}\right)x_0,$$

$$JKI_n = \prod_{i=n_0}^{n}\left(\frac{1}{128} + \frac{19}{8\sqrt{i}} + \frac{5}{2i}\right)x_0,$$

$$JKN_n = \prod_{i=n_0}^{n}\left(\frac{1}{1{,}024} + \frac{295}{128\sqrt{i}} + \frac{43}{16i} + \frac{5}{2i^{\frac{3}{2}}}\right)x_0,$$

$$JKCR_n = \prod_{i=n_0}^{n}\left(\frac{1}{1{,}024} + \frac{27}{128\sqrt{i}} + \frac{115}{16i} + \frac{25}{2i^{\frac{3}{2}}}\right)x_0$$

and

$$JKSP_n = \prod_{i=n_0}^{n}\left(\frac{1}{1{,}024} + \frac{15}{128\sqrt{i}} + \frac{75}{16i} + \frac{125}{2i^{\frac{3}{2}}}\right)x_0,$$

respectively.

First we compare Jungck-Kirk-type iterative schemes with their corresponding Jungck-type iterative schemes.

For $n_0 = 80$, consider

$$\left|\frac{JKM_{n+1}}{JM_{n+1}}\right| = \left|\frac{\prod_{i=81}^{n}(\frac{1}{16} + \frac{5}{2\sqrt{i}})x_0}{\prod_{i=81}^{n}(\frac{1}{2} - \frac{1}{\sqrt{i}})x_0}\right| = \left|\prod_{i=81}^{n}\left[1 - \frac{(\frac{7}{16} - \frac{7}{2\sqrt{i}})}{(\frac{1}{2} - \frac{1}{\sqrt{i}})}\right]\right| = \left|\prod_{i=81}^{n}\left[1 - \frac{(7\sqrt{i} - 56)}{(8\sqrt{i} - 16)}\right]\right|.$$

It is easy to see that

$$0 \le \lim_{n\to\infty}\prod_{i=81}^{n}\left[1 - \frac{(7\sqrt{i} - 56)}{(8\sqrt{i} - 16)}\right] \le \lim_{n\to\infty}\prod_{i=81}^{n}\left(1 - \frac{1}{i}\right) = \lim_{n\to\infty}\frac{80}{n} = 0.$$

Hence, $\lim_{n\to\infty}|\frac{JKM_{n+1}}{JM_{n+1}}| = 0$.

Similarly,

$$\begin{aligned}\left|\frac{JKI_n}{JI_n}\right| &= \left|\frac{\prod_{i=81}^{n}(\frac{1}{128} + \frac{19}{8\sqrt{i}} + \frac{5}{2i})x_0}{\prod_{i=81}^{n}(\frac{1}{2} - \frac{1}{\sqrt{i}} - \frac{2}{i})x_0}\right| \\ &= \left|\prod_{i=81}^{n}\left[1 - \frac{(\frac{63}{128} - \frac{27}{8\sqrt{i}} - \frac{9}{2i})}{(\frac{1}{2} - \frac{1}{\sqrt{i}} - \frac{2}{i})}\right]\right| = \left|\prod_{i=81}^{n}\left[1 - \frac{(63i - 432\sqrt{i} - 576)}{(64i - 128\sqrt{i} - 256)}\right]\right|\end{aligned}$$

with

$$0 \le \lim_{n\to\infty}\prod_{i=81}^{n}\left[1 - \frac{(63i - 432\sqrt{i} - 576)}{(64i - 128\sqrt{i} - 256)}\right] \le \lim_{n\to\infty}\prod_{i=81}^{n}\left(1 - \frac{1}{i}\right) = \lim_{n\to\infty}\frac{80}{n} = 0$$

implies $\lim_{n\to\infty}|\frac{JKI_n}{JI_n}| = 0$.

Again, similarly,

$$\begin{aligned}\left|\frac{JKN_n}{JN_n}\right| &= \left|\frac{\prod_{i=81}^{n}(\frac{1}{1{,}024}+\frac{295}{128\sqrt{i}}+\frac{43}{16i}+\frac{5}{2i^{\frac{3}{2}}})x_0}{\prod_{i=81}^{n}(1-\frac{2}{\sqrt{i}}-\frac{4}{i}-\frac{8}{i^{\frac{3}{2}}})x_0}\right| \\ &= \left|\prod_{i=81}^{n}\left[1-\frac{(\frac{1{,}023}{1{,}024}-\frac{551}{128\sqrt{i}}-\frac{107}{16i}-\frac{21}{2i^{\frac{3}{2}}})}{(1-\frac{2}{\sqrt{i}}-\frac{4}{i}-\frac{8}{i^{\frac{3}{2}}})}\right]\right| \\ &= \left|\prod_{i=81}^{n}\left[1-\frac{(1{,}023i^{\frac{3}{2}}-4{,}408i-6{,}848\sqrt{i}-10{,}752)}{(1{,}024i^{\frac{3}{2}}-2{,}048i-4{,}096\sqrt{i}-8{,}192)}\right]\right|\end{aligned}$$

with

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\prod_{i=81}^{n}\left[1-\frac{(1{,}023i^{\frac{3}{2}}-4{,}408i-6{,}848\sqrt{i}-10{,}752)}{(1{,}024i^{\frac{3}{2}}-2{,}048i-4{,}096\sqrt{i}-8{,}192)}\right] \\ &\leq \lim_{n\to\infty}\prod_{i=81}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{80}{n} = 0\end{aligned}$$

implies $|\frac{JKN_n}{JN_n}| = 0$.

Again, similarly,

$$\begin{aligned}\left|\frac{JKCR_n}{JCR_n}\right| &= \left|\frac{\prod_{i=81}^{n}(\frac{1}{1{,}024}+\frac{27}{128\sqrt{i}}+\frac{115}{16i}+\frac{25}{2i^{\frac{3}{2}}})x_0}{\prod_{i=81}^{n}(\frac{1}{4}-\frac{1}{2\sqrt{i}}-\frac{2}{i}+\frac{4}{i^{\frac{3}{2}}})x_0}\right| \\ &= \left|\prod_{i=81}^{n}\left[1-\frac{(\frac{255}{1{,}024}-\frac{91}{128\sqrt{i}}-\frac{147}{16i}-\frac{17}{2i^{\frac{3}{2}}})}{(\frac{1}{4}-\frac{1}{2\sqrt{i}}-\frac{2}{i}+\frac{4}{i^{\frac{3}{2}}})}\right]\right| \\ &= \left|\prod_{i=81}^{n}\left[1-\frac{(255i^{\frac{3}{2}}-728i-9{,}408\sqrt{i}-8{,}704)}{(256i^{\frac{3}{2}}-512i-2{,}048\sqrt{i}+4{,}096)}\right]\right|\end{aligned}$$

with

$$0 \leq \lim_{n\to\infty}\prod_{i=81}^{n}\left[1-\frac{(255i^{\frac{3}{2}}-728i-9{,}408\sqrt{i}-8{,}704)}{(256i^{\frac{3}{2}}-512i-2{,}048\sqrt{i}+4{,}096)}\right] \leq \lim_{n\to\infty}\prod_{i=81}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{80}{n} = 0$$

implies $\lim_{n\to\infty}|\frac{JKCR_n}{JCR_n}| = 0$.

Again, similarly,

$$\begin{aligned}\left|\frac{JKSP_n}{JSP_n}\right| &= \left|\frac{\prod_{i=81}^{n}(\frac{1}{1{,}024}+\frac{15}{128\sqrt{i}}+\frac{75}{16i}+\frac{125}{2i^{\frac{3}{2}}})x_0}{\prod_{i=81}^{n}(\frac{1}{2}-\frac{3}{\sqrt{i}}+\frac{6}{i}-\frac{4}{i^{\frac{3}{2}}})x_0}\right| \\ &= \left|\prod_{i=81}^{n}\left[1-\frac{(\frac{511}{1{,}024}-\frac{399}{128\sqrt{i}}+\frac{21}{16i}-\frac{133}{2i^{\frac{3}{2}}})}{(\frac{1}{2}-\frac{3}{\sqrt{i}}+\frac{6}{i}-\frac{4}{i^{\frac{3}{2}}})}\right]\right| \\ &= \left|\prod_{i=81}^{n}\left[1-\frac{(511i^{\frac{3}{2}}-3{,}192i+1{,}344\sqrt{i}-68{,}096)}{(512i^{\frac{3}{2}}-3{,}072i+6{,}144\sqrt{i}-4{,}096)}\right]\right|\end{aligned}$$

with

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\prod_{i=81}^{n}\left[1-\frac{(511i^{\frac{3}{2}}-3{,}192i+1{,}344\sqrt{i}-68{,}096)}{(512i^{\frac{3}{2}}-3{,}072i+6{,}144\sqrt{i}-4{,}096)}\right] \\ &\le \lim_{n\to\infty}\prod_{i=81}^{n}\left(1-\frac{1}{i}\right)=\lim_{n\to\infty}\frac{80}{n}=0 \end{aligned}$$

implies $\lim_{n\to\infty}|\frac{JKSP_n}{JSP_n}|=0$.

Therefore, by Definition 1.5, Jungck-Kirk-type iterative schemes converge faster than corresponding Jungck-type iterative schemes to the common fixed point 0 of T and S.

Now, we compare Jungck-Kirk-type iterative schemes with each other.

For $n_0=63$, we have

$$\begin{aligned} \left|\frac{JKI_n}{JKM_n}\right| &= \left|\frac{\prod_{i=64}^{n}(\frac{1}{128}+\frac{19}{8\sqrt{i}}+\frac{5}{2i})x_0}{\prod_{i=64}^{n}(\frac{1}{16}+\frac{5}{2\sqrt{i}})x_0}\right| \\ &= \left|\prod_{i=64}^{n}\left[1-\frac{(\frac{7}{128}+\frac{1}{8\sqrt{i}}-\frac{5}{2i})}{(\frac{1}{16}+\frac{5}{2\sqrt{i}})}\right]\right| = \left|\prod_{i=64}^{n}\left[1-\frac{(7i+16\sqrt{i}-320)}{(8i+320\sqrt{i})}\right]\right| \end{aligned}$$

with

$$0\le \lim_{n\to\infty}\prod_{i=64}^{n}\left[1-\frac{(7i+16\sqrt{i}-320)}{(8i+320\sqrt{i})}\right]\le \lim_{n\to\infty}\prod_{i=64}^{n}\left(1-\frac{1}{i}\right)=\lim_{n\to\infty}\frac{63}{n}=0$$

implies $\lim_{n\to\infty}|\frac{JKI_n}{JKM_n}|=0$.

Similarly, for $n_0=63$,

$$\begin{aligned} \left|\frac{JKN_n}{JKI_n}\right| &= \left|\frac{\prod_{i=64}^{n}(\frac{1}{1{,}024}+\frac{295}{128\sqrt{i}}+\frac{43}{16i}+\frac{5}{2i^{\frac{3}{2}}})x_0}{\prod_{i=64}^{n}(\frac{1}{128}+\frac{19}{8\sqrt{i}}+\frac{5}{2i})x_0}\right| \\ &= \left|\prod_{i=64}^{n}\left[1-\frac{(\frac{7}{1{,}024}+\frac{9}{128\sqrt{i}}-\frac{3}{16i}-\frac{5}{2i^{\frac{3}{2}}})}{(\frac{1}{128}+\frac{19}{8\sqrt{i}}+\frac{5}{2i})}\right]\right| \\ &= \left|\prod_{i=64}^{n}\left[1-\frac{(7i^{\frac{3}{2}}+72i-192\sqrt{i}-2{,}560)}{(8i^{\frac{3}{2}}+2{,}432i+2{,}560\sqrt{i})}\right]\right| \end{aligned}$$

with

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\prod_{i=64}^{n}\left[1-\frac{(7i^{\frac{3}{2}}+72i-192\sqrt{i}-2{,}560)}{(8i^{\frac{3}{2}}+2{,}432i+2{,}560\sqrt{i})}\right] \\ &\le \lim_{n\to\infty}\prod_{i=64}^{n}\left(1-\frac{1}{i}\right)=\lim_{n\to\infty}\frac{63}{n}=0 \end{aligned}$$

implies $\lim_{n\to\infty}|\frac{JKN_n}{JKI_n}|=0$.

Also, for $n_0 = 63$,

$$\begin{aligned}\left|\frac{JKCR_n}{JKN_n}\right| &= \left|\frac{\prod_{i=64}^{n}(\frac{1}{1{,}024} + \frac{27}{128\sqrt{i}} + \frac{115}{16i} + \frac{25}{2i^{\frac{3}{2}}})x_0}{\prod_{i=64}^{n}(\frac{1}{1{,}024} + \frac{295}{128\sqrt{i}} + \frac{43}{16i} + \frac{5}{2i^{\frac{3}{2}}})x_0}\right| \\ &= \left|\prod_{i=64}^{n}\left[1 - \frac{(\frac{268}{128\sqrt{i}} - \frac{72}{16i} - \frac{20}{2i^{\frac{3}{2}}})}{(\frac{1}{1{,}024} + \frac{295}{128\sqrt{i}} + \frac{43}{16i} + \frac{5}{2i^{\frac{3}{2}}})}\right]\right| \\ &= \left|\prod_{i=64}^{n}\left[1 - \frac{(268i - 576\sqrt{i} - 1{,}280)}{(0.125i^{\frac{3}{2}} + 295i + 344\sqrt{i} + 320)}\right]\right|\end{aligned}$$

with

$$0 \leq \lim_{n\to\infty}\prod_{i=64}^{n}\left[1 - \frac{(268i - 576\sqrt{i} - 1{,}280)}{(0.125i^{\frac{3}{2}} + 295i + 344\sqrt{i} + 320)}\right] \leq \lim_{n\to\infty}\prod_{i=64}^{n}\left(1 - \frac{1}{i}\right) = \lim_{n\to\infty}\frac{63}{n} = 0$$

implies $\lim_{n\to\infty}|\frac{JKCR_n}{JKN_n}| = 0$.

Similarly, for $n_0 = 999$,

$$\begin{aligned}\left|\frac{JKSP_n}{JKCR_n}\right| &= \left|\frac{\prod_{i=1{,}000}^{n}(\frac{1}{1{,}024} + \frac{15}{128\sqrt{i}} + \frac{75}{16i} + \frac{125}{2i^{\frac{3}{2}}})x_0}{\prod_{i=1{,}000}^{n}(\frac{1}{1{,}024} + \frac{27}{128\sqrt{i}} + \frac{115}{16i} + \frac{25}{2i^{\frac{3}{2}}})x_0}\right| \\ &= \left|\prod_{i=1{,}000}^{n}\left[1 - \frac{(\frac{12}{128\sqrt{i}} + \frac{40}{16i} - \frac{100}{2i^{\frac{3}{2}}})}{(\frac{1}{1{,}024} + \frac{27}{128\sqrt{i}} + \frac{115}{16i} + \frac{25}{2i^{\frac{3}{2}}})}\right]\right| \\ &= \left|\prod_{i=1{,}000}^{n}\left[1 - \frac{(96i + 2{,}560\sqrt{i} - 51{,}200)}{(i^{\frac{3}{2}} + 216i + 7{,}360\sqrt{i} + 12{,}800)}\right]\right|\end{aligned}$$

with

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\prod_{i=1{,}000}^{n}\left[1 - \frac{(96i + 2{,}560\sqrt{i} - 51{,}200)}{(i^{\frac{3}{2}} + 216i + 7{,}360\sqrt{i} + 12{,}800)}\right] \\ &\leq \lim_{n\to\infty}\prod_{i=1{,}000}^{n}\left(1 - \frac{1}{i}\right) = \lim_{n\to\infty}\frac{999}{n} = 0\end{aligned}$$

implies $\lim_{n\to\infty}|\frac{JKSP_n}{JKCR_n}| = 0$.

Hence, in view of Definition 1.5, we observe that the decreasing order of Jungck-Kirk-type iterative schemes is as follows:

JKSP, JKCR, JKN, JKI and JKM iterative scheme. □

4 Applications

In this section, with the help of computer programs in *C*++, we explain how and why the newly introduced Jungck-Kirk-type iterative schemes can be applied to solve different types of problems. The outcome is listed in the form of Tables 2, 3 and 4, by taking $r = s = t = 2$, for all iterative schemes.

Table 2 Goat problem

Number of iterations	Jungck-Kirk-Mann			Jungck-Kirk-Ishikawa			Jungck-Kirk-Noor			Jungck-Kirk-CR			Jungck-Kirk-SP		
n	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}
0	1.568685	1.568685	0.223278	1.568685	1.568685	0.456892	1.568685	1.568685	0.492505	1.568685	1.568685	1.885776	1.568685	1.568685	1.879253
1	1.720615	1.720615	1.473891	1.736372	1.736372	1.473055	1.728006	1.728006	1.468275	0.9895	0.9895	1.001579	0.978659	0.978659	1.014704
2	0.953808	0.953808	0.924859	0.954403	0.954403	0.925895	0.957829	0.957829	0.928822	1.388138	1.388138	1.270059	1.375728	1.375728	1.257049
–	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
56	1.235762	1.235762	1.158618	1.235762	1.235762	1.158618	1.235762	1.235762	1.158618	1.235764	1.235764	1.15862	1.235763	1.235763	1.15862
57	1.235764	1.235764	1.15862	1.235764	1.235764	1.15862	1.235764	1.235764	1.15862	1.235763	1.235763	1.158619	1.235763	1.235763	1.158619
58	1.235762	1.235762	1.158619	1.235762	1.235762	1.158619	1.235762	1.235762	1.158619	1.235764	1.235764	1.15862	1.235764	1.235764	1.15862
59	1.235764	1.235764	1.15862	1.235764	1.235764	1.15862	1.235764	1.235764	1.15862	1.235763	1.235763	1.158619	**1.235763**	**1.235763**	**1.158619**
60	1.235762	1.235762	1.158619	1.235762	1.235762	1.158619	1.235762	1.235762	1.158619	1.235763	1.235763	1.15862	**1.235763**	**1.235763**	**1.158619**
61	1.235764	1.235764	1.15862	1.235764	1.235764	1.15862	1.235764	1.235764	1.15862	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**
62	1.235763	1.235763	1.158619	1.235763	1.235763	1.158619	1.235763	1.235763	1.158619	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**
63	1.235763	1.235763	1.15862	1.235763	1.235763	1.15862	1.235763	1.235763	1.15862	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**
64	1.235763	1.235763	1.158619	1.235763	1.235763	1.158619	1.235763	1.235763	1.158619	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**
65	1.235763	1.235763	1.15862	1.235763	1.235763	1.15862	1.235763	1.235763	1.15862	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**
66	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**
67	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**
68	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**	**1.235763**	**1.235763**	**1.158619**

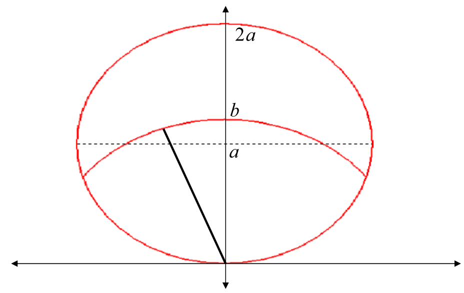
Table 3 Solution of equation

Number of iterations	Jungck-Kirk-Mann			Jungck-Kirk-Ishikawa			Jungck-Kirk-Noor			Jungck-Kirk-CR			Jungck-Kirk-SP		
n	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}
0	0.8	0.8	0.8	0.8	0.8	0.251845	0.8	0.8	0.879634	0.8	0.8	0.750212	0.8	0.8	0.490599
1	0.8	0.8	0.612962	0.251845	0.251845	0.369233	0.879634	0.879634	0.700824	0.750212	0.750212	0.44583	0.490599	0.490599	0.495894
2	0.612962	0.612962	0.511105	0.369233	0.369233	0.410876	0.700824	0.700824	0.517399	0.44583	0.44583	0.412812	0.495894	0.495894	0.460743
3	0.511105	0.511105	0.458584	0.410876	0.410876	0.412387	0.517399	0.517399	0.440228	0.412812	0.412812	0.412391	0.460743	0.460743	0.431079
4	0.458584	0.458584	0.43267	0.412387	0.412387	0.412391	0.440228	0.440228	0.418184	**0.412391**	**0.412391**	**0.412391**	0.431079	0.431079	0.417575
5	0.43267	0.43267	0.420663	**0.412391**	**0.412391**	**0.412391**	0.418184	0.418184	0.413388	**0.412391**	**0.412391**	**0.412391**	0.417575	0.417575	0.413452
6	0.420663	0.420663	0.415511	**0.412391**	**0.412391**	**0.412391**	0.413388	0.413388	0.412536	**0.412391**	**0.412391**	**0.412391**	0.413452	0.413452	0.412555
7	0.415511	0.415511	0.413478	**0.412391**	**0.412391**	**0.412391**	0.412536	0.412536	0.412409	**0.412391**	**0.412391**	**0.412391**	0.412555	0.412555	0.412411
8	0.413478	0.413478	0.412741	**0.412391**	**0.412391**	**0.412391**	0.412409	0.412409	0.412393	**0.412391**	**0.412391**	**0.412391**	0.412411	0.412411	0.412393
9	0.412741	0.412741	0.412496	**0.412391**	**0.412391**	**0.412391**	0.412393	0.412393	0.412391	**0.412391**	**0.412391**	**0.412391**	0.412393	0.412393	0.412391
10	0.412496	0.412496	0.41242	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**
11	0.41242	0.41242	0.412399	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**
12	0.412399	0.412399	0.412393	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**
13	0.412393	0.412393	0.412392	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**
14	0.412392	0.412392	0.412391	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**
15	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**
16	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**	**0.412391**

Table 4 Oscillating function

Number of iterations	Jungck-Kirk-Mann			Jungck-Kirk-Ishikawa			Jungck-Kirk-Noor			Jungck-Kirk-CR			Jungck-Kirk-SP		
n	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}	Tx_n	Sx_n	x_{n+1}
0	0.25	0.25	2	0.25	0.25	2	0.25	0.25	2	0.25	0.25	0.707107	0.25	0.25	1.189207
1	0.25	0.25	1.916621	0.25	0.25	1.837722	0.25	0.25	1.7634	2	2	1.152313	0.707107	0.707107	1.006249
2	0.272225	0.272225	1.792281	0.296101	0.296101	1.611392	0.321587	0.321587	1.458178	0.753111	0.753111	0.994516	0.987617	0.987617	1.000077
3	0.311306	0.311306	1.64864	0.385121	0.385121	1.378543	0.470304	0.470304	1.196768	1.011059	1.011059	1.000139	0.999846	0.999846	1
4	0.367916	0.367916	1.499982	0.52621	0.52621	1.18467	0.698201	0.698201	1.057246	0.999722	0.999722	1	**1**	**1**	**1**
5	0.444455	0.444455	1.357403	0.712533	0.712533	1.065618	0.894639	0.894639	1.013039	**1**	**1**	**1**	**1**	**1**	**1**
6	0.542728	0.542728	1.230991	0.880636	0.880636	1.017436	0.974423	0.974423	1.002468	**1**	**1**	**1**	**1**	**1**	**1**
7	0.659918	0.659918	1.130418	0.966019	0.966019	1.003771	0.995081	0.995081	1.000388	**1**	**1**	**1**	**1**	**1**	**1**
8	0.782567	0.782567	1.062306	0.9925	0.9925	1.000694	0.999224	0.999224	1.000051	**1**	**1**	**1**	**1**	**1**	**1**
9	0.886136	0.886136	1.025036	0.998613	0.998613	1.00011	0.999899	0.999899	1.000005	**1**	**1**	**1**	**1**	**1**	**1**
10	0.951748	0.951748	1.008718	0.999779	0.999779	1.000015	0.999989	0.999989	1	**1**	**1**	**1**	**1**	**1**	**1**
11	0.982789	0.982789	1.002744	0.99997	0.99997	1.000002	0.999999	0.999999	1	**1**	**1**	**1**	**1**	**1**	**1**
12	0.994535	0.994535	1.000803	0.999996	0.999996	1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
13	0.998396	0.998396	1.000222	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
14	0.999556	0.999556	1.000058	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
15	0.999884	0.999884	1.000015	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
16	0.999971	0.999971	1.000003	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
17	0.999993	0.999993	1.000001	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
18	0.999998	0.999998	1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
19	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
20	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Figure 1 Goat problem.



Goat problem

A farmer has a fenced circular pasture of radius a and wants to tie a goat to the fence with a rope of length b (see Figure 1) so as to allow the goat to graze half the pasture. How long should the rope be to accomplish this?

The length of the rope b must be longer than a and shorter than $\sqrt{2}a$, *i.e.*, $a < b < \sqrt{2}a$. Using polar coordinates, we find the grazing area

$$= 2\left[\frac{1}{2}\cdot\int_0^{\sin^{-1}(\frac{b}{2a})} 4a^2\sin^2\theta\,d\theta + \frac{1}{2}\cdot\int_{\sin^{-1}(\frac{b}{2a})}^{\frac{\pi}{2}} b^2\,d\theta\right]$$
$$= \int_0^{\sin^{-1}(\frac{b}{2a})} 4a^2\sin^2\theta\,d\theta + \int_{\sin^{-1}(\frac{b}{2a})}^{\frac{\pi}{2}} b^2\,d\theta.$$

We want this to equal half the pasture area, which is $\frac{\pi a^2}{2}$, so we get the equation

$$\int_0^{\sin^{-1}(\frac{b}{2a})} 4a^2\sin^2\theta\,d\theta + \int_{\sin^{-1}(\frac{b}{2a})}^{\frac{\pi}{2}} b^2\,d\theta = \frac{\pi a^2}{2}.$$

Multiplying both sides by $\frac{2}{a^2}$ and integrating, we get

$$\left(4 - 2\frac{b^2}{a^2}\right)\sin^{-1}\left(\frac{b}{2a}\right) - \frac{b}{a}\sqrt{4 - \frac{b^2}{a^2}} + \frac{\pi b^2}{a^2} = \pi.$$

After putting $x = \frac{b}{a}$, we get the simplified equation $(4-2x^2)\sin^{-1}(\frac{x}{2}) - x\sqrt{4-x^2} + \pi x^2 = \pi$ and we are looking for the solution x, with $1 < x < \sqrt{2}$.

Now, we rearrange the above equation as $Sx = Tx$, with S, T defined on $[0, 2]$ as

$$Sx = 2\sin^{-1}\left(\frac{x}{2}\right) \quad\text{and}\quad Tx = \frac{\pi + x\sqrt{4-x^2} - \pi x^2 + 2x^2\sin^{-1}(\frac{x}{2})}{2}.$$

By taking the initial approximation $x_1 = 2$, $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \alpha_{n,2} = \beta_{n,2} = \gamma_{n,2} = \frac{2}{(1+2n)^3}$, $\alpha_{n,0} = 1 - \alpha_{n,1} - \alpha_{n,2}\beta_{n,0} = 1 - \beta_{n,1} - \beta_{n,2}$ and $\gamma_{n,0} = 1 - \gamma_{n,1} - \gamma_{n,2}$, the comparison of convergence of Jungck-Kirk-type iterative schemes to the point of coincidence **1.158619** of S, T, is listed in Table 2. So the rope length b should be approximately **1.158619**a.

Solution of equation $e^{(1-x)^2} - x - 1 = 0$

To solve this equation, we rearrange it as $Sx = Tx$, with S, T defined on $[0, 2]$ defined by $Sx = x$ and $Tx = e^{(1-x)^2} - 1$.

With the initial approximation $x_0 = 0.8$, $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \alpha_{n,2} = \beta_{n,2} = \gamma_{n,2} = \frac{1}{\sqrt[4]{1+n}}$, $\alpha_{n,0} = 1 - \alpha_{n,1} - \alpha_{n,2}\beta_{n,0} = 1 - \beta_{n,1} - \beta_{n,2}$ and $\gamma_{n,0} = 1 - \gamma_{n,1} - \gamma_{n,2}$, the comparison of convergence of Jungck-Kirk-type iterative schemes to the common fixed point **0.412391** of S, T, is listed in Table 3.

Oscillating function $1/x^2$

In order to solve this function by Jungck-type iterative schemes, we write it in the form $Sx = Tx$, where the functions T, S are defined on R^+ as $Tx = 1/x^2$ and $Sx = x^4$, respectively. By taking the initial approximation $x_0 = 2$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt[4]{n+1}}$, the obtained results are listed in Table 4.

For detailed study, these programs are again executed after changing the parameters and some observations are made as given below.

5 Observations

Goat problem

1. Taking initial guess $x_0 = 1.9$ (near coincidence point), Jungck-Kirk-Mann, Jungck-Kirk-Ishikawa, Jungck-Kirk-Noor and Jungck-Kirk-CR iterative schemes converge in 64 iterations, while the Jungck-Kirk-SP iterative scheme converges in 63 iterations.

2. Taking $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \alpha_{n,2} = \beta_{n,2} = \gamma_{n,2} = \frac{1}{(1+2n)^3}$ and $x_0 = 2$, we observe that Jungck-Kirk-Mann, Kirk-Ishikawa and Jungck-Kirk-Noor iterative schemes converge in 65 iterations, while the Jungck-Kirk-CR iterative scheme converges in 64 iterations and the Jungck-Kirk-SP iterative scheme converges in 63 iterations.

Equation $e^{(1-x)^2} - x - 1 = 0$

1. Taking initial guess $x_0 = 1.6$ (somewhat nearer to the common fixed point), the Jungck-Kirk-Mann iterative scheme converges in 15 iterations, the Jungck-Kirk-Ishikawa iterative scheme converges in 5 iterations, the Jungck-Kirk-Noor iterative scheme converges in 11 iterations and the Jungck-Kirk-CR iterative scheme converges in 5 iterations, while the Jungck-Kirk-SP iterative scheme converges in 11 iterations.

2. Taking $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \alpha_{n,2} = \beta_{n,2} = \gamma_{n,2} = \frac{1}{\sqrt[2]{1+n}}$ and $x_0 = 1.8$, we observe that the Jungck-Kirk-Mann iterative scheme converges in 8 iterations, the Jungck-Kirk-Ishikawa iterative scheme converges in 7 iterations, the Jungck-Kirk-Noor iterative scheme converges in 8 iterations and Jungck-Kirk-CR as well as Jungck-Kirk-SP iterative schemes converge in 6 iterations.

Oscillating function $1/x^2$

1. Taking initial guess $x_0 = 1.2$ (near to the common fixed point), the Jungck-Kirk-Mann iterative scheme converges in 18 iterations, the Jungck-Kirk-Ishikawa iterative scheme converges in 13 iterations, the Jungck-Kirk-Noor iterative scheme converges in 12 iterations and the Jungck-Kirk-CR iterative scheme converges in 6 iterations, while the Jungck-Kirk-SP iterative scheme converges in 5 iterations.

2. Taking $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \alpha_{n,2} = \beta_{n,2} = \gamma_{n,2} = \frac{1}{\sqrt[2]{1+n}}$ and $x_0 = 2$, we observe that Jungck-Kirk-Mann, Jungck-Kirk-Ishikawa and Jungck-Kirk-Noor iterative schemes converge in 10 iterations, while the Jungck-Kirk-CR Noor iterative scheme converges in 6 iterations and the Jungck-Kirk-SP iterative scheme converges in 5 iterations.

6 Conclusions

Goat problem

1. Decreasing order of convergence rate of Jungck-Kirk-type iterative schemes is as follows:

Jungck-Kirk-SP, Jungck-Kirk-CR, and Jungck-Kirk-Noor iterative scheme, where Jungck-Kirk-Noor shows equivalence with Jungck-Kirk-Ishikawa and Jungck-Kirk-Mann iterative schemes.

2. For initial guess somewhat near to the point of coincidence, the number of iterations increases in case of Jungck-Kirk-SP and Jungck-Kirk-CR iterative schemes, while the number of iterations decreases in case of Jungck-Kirk-Noor, Jungck-Kirk-Ishikawa and Jungck-Kirk-Mann iterative schemes.

3. The speed of iterative schemes depends on $\alpha_{n,i}$, $\beta_{n,i}$ and $\gamma_{n,i}$. On decreasing the value of these parameters, Jungck-Kirk-SP and Jungck-Kirk-CR iterative schemes show an increase, while Jungck-Kirk-Noor, Jungck-Kirk-Ishikawa and Jungck-Kirk-Mann iterative schemes show a decrease in the number of iterations to converge.

Equation $e^{(1-x)^2} - x - 1 = 0$

1. Decreasing order of convergence rate of Jungck-Kirk-type iterative schemes is as follows:

Jungck-Kirk-CR, Jungck-Kirk-Ishikawa, Jungck-Kirk-SP and Jungck-Kirk-Mann iterative scheme, while Jungck-Kirk-Noor shows equivalence with the Jungck-Kirk-SP iterative scheme.

2. For initial guess somewhat near to the common fixed point, the number of iterations decreases in case of Jungck-Kirk-Ishikawa and Jungck-Kirk-Mann iterative schemes, while Jungck-Kirk-SP, Jungck-Kirk-CR and Jungck-Kirk-Noor iterative schemes show no change in the number of iterations to converge.

3. The speed of iterative schemes depends on $\alpha_{n,i}$, $\beta_{n,i}$ and $\gamma_{n,i}$. On decreasing the value of these parameters, Jungck-Kirk-Ishikawa and Jungck-Kirk-CR iterative schemes show an increase, while Jungck-Kirk-Noor, Jungck-Kirk-SP and Jungck-Kirk-Mann iterative schemes show a decrease in the number of iterations to converge.

Oscillating function $1/x^2$

1. Decreasing order of convergence rate of Jungck-Kirk-type iterative schemes is as follows:

Jungck-Kirk-SP, Jungck-Kirk-CR, Jungck-Kirk-Noor iterative scheme, Jungck-Kirk-Ishikawa and Jungck-Kirk-Mann iterative scheme.

2. For initial guess nearer to the common fixed point, the number of iterations decreases in case of Jungck-Kirk-Noor, Jungck-Kirk-Ishikawa and Jungck-Kirk-Mann iterative schemes, while Jungck-Kirk-SP as well as Jungck-Kirk-CR iterative schemes show no change in the number of iterations to converge.

3. The speed of iterative schemes depends on $\alpha_{n,i}$, $\beta_{n,i}$ and $\gamma_{n,i}$. On decreasing the value of these parameters, Jungck-Kirk-Noor, Jungck-Kirk-Ishikawa and Jungck-Kirk-Mann iterative schemes show a decrease while Jungck-Kirk-SP and Jungck-Kirk-CR iterative schemes show no change in the number of iterations to converge.

Open problem

It is still an open problem to compare Jungck-Kirk-type iterative schemes with each other in view of Rhoades Definition 1.5 and also to study the same using nonself contractive-type operators.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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