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Strong convergence theorems for a finite family of total asymptotically strict pseudocontractive semigroups in Banach spaces

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Abstract

The purpose of this paper is to introduce the concepts of total asymptotically strictly pseudocontractive semigroup, asymptotically strictly pseudocontractive semigroup *etc.*, and to prove some strong convergence theorems of the explicit iteration process for these kinds of semigroups in arbitrary Banach spaces. The results presented in the paper extend and improve some recent results announced in the current literature.
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1 Introduction and preliminaries

Let E be a real Banach space, E^* be the dual space of E. In the sequel we assume that C is a nonempty closed convex subset of E, $\Re^+$ is the set of nonnegative real numbers and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|x\| = \|f\| \right\}, \quad x \in E. \tag{1.1}$$

Let $T : C \to C$ be a mapping, we denote by $F(T)$ the set of fixed points of T. We also use '$\to$' to stand for strong convergence and '$\rightharpoonup$' for weak convergence.

We first recall some definitions.

A one-parameter family $\Im := \{T(t) : t \geq 0\}$ of self-mappings of C is said to be a *nonexpansive semigroup* if the following conditions are satisfied:

(i) $T(t_1 + t_2)x = T(t_1)T(t_2)x$, for any $t_1, t_2 \in \Re^+$ and $x \in C$;

(ii) $T(0)x = x$ for each $x \in C$;

(iii) for each $x \in C$, $t \mapsto T(t)x$ is continuous;

(iv) for any $t \geq 0$, $T(t)$ is a nonexpansive mapping on C, that is, for any $x, y \in C$,

$$\left\| T(t)x - T(t)y \right\| \leq \|x - y\|. \tag{1.2}$$

If the family $\Im := \{T(t) : t \geq 0\}$ satisfies conditions (i)-(iii), then it is:

(a) *Pseudocontractive semigroup* if for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - T(t)y, j(x - y) \rangle \leq \|x - y\|^2. \tag{1.3}$$

(b) *Uniformly Lipschitzian semigroup* if there exists a bounded measurable function $L : [0, \infty) \to (0, \infty)$ such that, for any $x, y \in C$ and $t \geq 0$,

$$\left\| T^n(t)x - T^n(t)y \right\| \leq L(t)\|x - y\|, \quad \forall n \geq 1. \tag{1.4}$$

In this case, we also say that $\Im$ is a uniformly $L(t)$-Lipschitzian semigroup.

(c) *Strictly pseudocontractive semigroup* if there exists a bounded function $\lambda : [0, \infty) \to (0, \infty)$ and for any given $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - T(t)y, j(x - y) \rangle \leq \|x - y\|^2 - \lambda(t)\left\| (I - T(t))x - (I - T(t))y \right\|^2 \tag{1.5}$$

for any $t \geq 0$.

It is easy to see that such a semigroup is $((1 + \lambda(t))/\lambda(t))$-Lipschitzian and pseudocontractive semigroup.

(d) *Demicontractive semigroup* if $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$, and there exists a bounded function $\lambda : [0, \infty) \to (0, \infty)$ for any $t \geq 0$, $x \in C$ and $y \in \bigcap_{t \geq 0} F(T(t))$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - y, j(x - y) \rangle \leq \|x - y\|^2 - \lambda(t)\left\| (I - T(t))x \right\|^2. \tag{1.6}$$

In this paper, we introduce the following semigroups.

Definition 1.1 A one-parameter family $\Im := \{T(t) : t \geq 0\}$ of self-mappings of C satisfies conditions (i)-(iii), then it is:

(e) *Total asymptotically strictly pseudocontractive semigroup* if there exist a bounded function $\lambda : [0, \infty) \to (0, \infty)$ and sequences $\{\mu_n\} \subset [0, \infty)$ and $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. For any given $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n(t)x - T^n(t)y, j(x - y) \rangle \leq \|x - y\|^2 - \lambda(t)\left\| (I - T^n(t))x - (I - T^n(t))y \right\|^2$$
$$+ \mu_n \phi(\|x - y\|) + \xi_n, \quad \forall n \geq 1, \forall t \geq 0, \tag{1.7}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$.

In this case, we also say that $\Im$ is a $(\lambda(t), \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive semigroup.

(f) *Asymptotically strictly pseudocontractive semigroup* if there exist a bounded function $\lambda : [0, \infty) \to (0, \infty)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$; for any given $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n(t)x - T^n(t)y, j(x - y) \rangle$$
$$\leq k_n\|x - y\|^2 - \lambda(t)\left\| (I - T^n(t))x - (I - T^n(t))y \right\|^2, \quad \forall n \geq 1, \tag{1.8}$$

for any $t \geq 0$.

In this case, we also say that $\Im$ is a $(\{k_n\}, \lambda(t))$-asymptotically strict pseudocontractive semigroup.

(g) *Asymptotically demicontractive semigroup* if $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ and there exist a bounded function $\lambda : [0, \infty) \to (0, \infty)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$; for any $t \geq 0$, $x \in C$ and $y \in \bigcap_{t \geq 0} F(T(t))$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n(t)x - y, j(x - y) \rangle \leq k_n \|x - y\|^2 - \lambda(t) \| (I - T^n(t))x \|^2, \quad \forall n \geq 1, \tag{1.9}$$

for any $t \geq 0$.

In this case, we also say that $\Im$ is a $(\{k_n\}, \lambda(t))$-asymptotically demicontractive semigroup.

Remark 1.2 If $\phi(\lambda) = \lambda^2$ and $\xi_n = 0$, then a total asymptotically strict pseudocontractive semigroup is an asymptotically strict pseudocontractive semigroup. Every asymptotically strict pseudocontractive semigroup with $\bigcap_{t > 0} F(T(t)) \neq \phi$ is an asymptotically demicontractive semigroup. If $k_n = 1$, $\forall n \geq 1$, an asymptotically strict pseudocontractive semigroup is a strict pseudocontractive semigroup, an asymptotically demicontractive semigroup is a demicontractive semigroup.

It is easy to see that condition (1.7) is equivalent to the following condition: for any $t \geq 0$, $x \in C$ and $y \in F(T(t))$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle x - T^n(t)x, j(x - y) \rangle \geq \lambda(t) \| x - T^n(t)x \|^2 - \mu_n \phi(\|x - y\|) - \xi_n. \tag{1.10}$$

The convergence problem of implicit and explicit iterative sequences for nonexpansive semigroups to a common fixed point has been considered by some authors in various spaces; see, for example, [1–11].

In 1998, Shioji-Takahashi [1] introduced and studied a Halpern-type scheme for a common fixed point of a family of asymptotically nonexpansive semigroups in the framework of a real Hilbert space.

In 2003, Suzuki [2] proved that the implicit scheme defined by $u, x_1 \in C$

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n)(x_n), \quad n \geq 1, \tag{1.11}$$

converges strongly to a common fixed point of the family of nonexpansive semigroups in a real Hilbert space. Xu [3] extended the result of Suzuki to a more general real uniformly convex Banach space having a weakly sequentially continuous duality mapping.

In 2005, Aleyner and Reich [4] proved the strong convergence of an explicit Halpern-type scheme defined by $u, x_1 \in C$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T(t_n)x_n, \quad n \geq 0, \tag{1.12}$$

to a common fixed point of the family $\{T(t) : t \geq 0\}$ of nonexpansive semigroups in a reflexive Banach space with a uniformly Gâteaux differentiable norm.

Recently, Chang *et al.* [11] introduced the following explicit iteration process:

$$x_1 \in C,$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(t_n)x_n, \quad n \geq 1, \tag{1.13}$$

for the Lipschitzian and demicontractive semigroup $\Im := \{T(t) : t \geq 0\}$ in general Banach spaces. Under appropriate assumptions imposed upon the parameter sequences $\{\alpha_n\}$ and $\{t_n\}$, they proved that the sequence $\{x_n\}$ defined by (1.13) converges strongly to some point in $F =: \bigcap_{t \geq 0} F(T(t))$.

Inspired and motivated by the above works of Shioji-Takahashi [1], Suzuki [2], Xu [3], Aleyner-Reich [4] and Chang [11], in this paper we aim to study the strong convergence to a common fixed point for a finite family of uniformly Lipschitzian and total asymptotically strict pseudocontractive semigroups $\Im_i := \{T_i(t) : t \geq 0\}$, $\forall i \in \{1, 2, \ldots, N\}$ in general Banach spaces. The results presented in the paper extend and improve some recent results given in [4, 5, 7, 9].

The following lemmas will be needed in proving our main results.

Lemma 1.3 *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq n_0, \tag{1.14}$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n \to \infty} a_n$ exists.

Lemma 1.4 [12] *Let E be any real Banach space, let E^* be the dual space of E and let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{1.15}$$

2 Main results

Let E be a real Banach space, and let C be a nonempty, closed convex subset of E. For some fixed $i \in N$, let $\Im_i := \{T_i(t) : t \geq 0\}$ be a uniformly $L_i(t)$-Lipschitzian with a bounded measurable function $L_i : [0, \infty) \to (0, \infty)$ and $(\lambda_i(t), \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$-total asymptotically strictly pseudocontractive semigroup with a bounded function $\lambda_i : [0, \infty) \to (0, \infty)$, sequences $\{\mu_{in}\}, \{\xi_{in}\} \subset [0, \infty)$ with $\mu_{in} \to 0$ and $\xi_{in} \to 0$ as $n \to \infty$, such that

$$L^i := \sup_{t \geq 0} L_i(t) < \infty, \qquad \lambda^i := \inf_{t \geq 0} \lambda_i(t) > 0, \qquad F^i := \bigcap_{t \geq 0} F(T_i(t)) \neq \emptyset. \tag{2.1}$$

Then, for $x, y \in C$, $p \in F^i$ and $t \geq 0$,

$$\langle T_i^n(t)x - p, j(x - p)\rangle \leq \|x - p\|^2 - \lambda^i \|x - T_i^n(t)x\|^2$$
$$+ \mu_{in}\phi_i(\|x - p\|) + \xi_{in}, \quad \forall n \geq 1, \tag{2.2}$$

where $\phi_i : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi_i(0) = 0$, and

$$\|T_i^n(t)x - T_i^n(t)y\| \leq L^i\|x - y\|, \quad \forall n \geq 1. \tag{2.3}$$

Consider a family $\{\Im_i\}_{i=1}^N$ of uniformly $L_i(t)$-Lipschitzian and $(\lambda_i(t), \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$-total asymptotically strict pseudocontractive semigroups of C and let

$$L := \max_{1 \leq i \leq N} L^i < \infty, \qquad \lambda := \min_{1 \leq i \leq N} \lambda^i > 0, \qquad F := \bigcap_{1 \leq i \leq N} \bigcap_{t \geq 0} F(T_i(t)) \neq \emptyset,$$

$$\mu_n = \max_{1 \leq i \leq N} \mu_{in}, \qquad \xi_n = \max_{1 \leq i \leq N} \xi_{in}, \qquad \phi = \max_{1 \leq i \leq N} \phi_i. \tag{2.4}$$

For $x, y \in C$, $p \in F$, $t \geq 0$ and any $i \in \{1, 2, \dots, N\}$,

$$\langle T_i^n(t)x - p, j(x - p) \rangle \leq \|x - p\|^2 - \lambda \|x - T_i^n(t)x\|^2$$
$$+ \mu_n \phi(\|x - p\|) + \xi_n, \quad \forall n \geq 1 \tag{2.5}$$

and

$$\|T_i^n(t)x - T_i^n(t)y\| \leq L\|x - y\|, \quad \forall n \geq 1. \tag{2.6}$$

Now, we are ready to give our main results.

Theorem 2.1 *Let E be a real Banach space, and let C be a nonempty, closed convex subset of E. Let $\{\Im_i\}_{i=1}^N$ be a finite family of uniformly $L_i(t)$-Lipschitzian and $(\lambda_i(t), \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$-total asymptotically strictly pseudocontractive semigroups of C, $L_i(t)$, $\lambda_i(t)$, $\{\mu_{in}\}$, $\{\xi_{in}\}$, ϕ_i, L^i, λ^i, F^i, L, λ, F, ϕ, $\{\mu_n\}$ and $\{\xi_n\}$ be the same as above. In addition, there exist positive constants M and M^* such that $\phi(\lambda) \leq M^* \lambda^2$ for all $\lambda \geq M$. Let $\{x_n\}$ be the sequence generated by*

$$x_1 \in C,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n^n(t_n)x_n, \quad n \geq 1, \tag{2.7}$$

where $T_n^n(t_n) = T_{n(\mathrm{mod}\,N)}^n(t_n)$, $\forall n \geq 1$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{t_n\}$ is an increasing sequence in $[0, \infty)$. If the following conditions are satisfied:

(1) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n \mu_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \xi_n < \infty$.

(2) *Assume for any $i \in \{1, 2, \dots, N\}$ and for any bounded subset $D \subset C$,*

$$\lim_{n \to \infty} \sup_{x \in D, s \in R^+} \|T_i^n(s + t_n)x - T_i^n(t_n)x\| = 0. \tag{2.8}$$

(3) *There exists a compact subset G of E such that $\bigcap_{t \geq 0} T_l(t)(C) \subset G$ for some $l \in \{1, 2, \dots, N\}$.*

Then the sequence $\{x_n\}$ converges strongly to some element in F.

Proof The proof of Theorem 2.1 is divided into four steps.

Step 1. First we prove that $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in F$.

For any $p \in F$, by (1.4) we have

$$\|T_n^n(t_n)x_n - p\| = \|T_n^n(t_n)x_n - T_n^n(t_n)p\| \leq L\|x_n - p\|. \tag{2.9}$$

It follows from (2.7) and (2.9) that

$$\|x_{n+1} - p\| = \|(1 - \alpha_n)x_n + \alpha_n T_n^n(t_n)x_n - (1 - \alpha_n)p - \alpha_n p\|$$
$$\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n \|T_n^n(t_n)x_n - p\|$$
$$\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n L\|x_n - p\|$$
$$\leq (1 + L)\|x_n - p\| \tag{2.10}$$

and

$$\|x_{n+1} - x_n\| = \alpha_n \|T_n^n(t_n)x_n - x_n\|$$
$$\leq \alpha_n \big(\|T_n^n(t_n)x_n - p\| + \|x_n - p\| \big)$$
$$\leq \alpha_n (1 + L)\|x_n - p\|. \tag{2.11}$$

By (1.10), for the point x_{n+1} and p, there exists $j(x_{n+1} - p) \in J(x_{n+1} - p)$ such that

$$\langle T_n^n(t_n)x_{n+1} - x_{n+1}, j(x_{n+1} - p) \rangle \leq -\lambda \|T_n^n(t_n)x_{n+1} - x_{n+1}\|^2$$
$$+ \mu_n \phi\big(\|x_{n+1} - p\| \big) + \xi_n, \quad \forall n \geq 1. \tag{2.12}$$

Since ϕ is an increasing function, it results that $\phi(\lambda) \leq \phi(M)$ if $\lambda \leq M$ and $\phi(\lambda) \leq M^* \lambda^2$ if $\lambda \geq M$. In either case, we can obtain that

$$\phi(\lambda) \leq \phi(M) + M^* \lambda^2. \tag{2.13}$$

Thus, by Lemma 1.4, (2.10)-(2.13), we have

$$\|x_{n+1} - p\|^2 \leq \|x_n - p + \alpha_n \big(T_n^n(t_n)x_n - x_n \big)\|^2$$
$$\leq \|x_n - p\|^2 + 2\alpha_n \langle T_n^n(t_n)x_n - x_n, j(x_{n+1} - p) \rangle$$
$$= \|x_n - p\|^2 + 2\alpha_n \langle T_n^n(t_n)x_n - T_n^n(t_n)x_{n+1}, j(x_{n+1} - p) \rangle$$
$$\quad + 2\alpha_n \langle T_n^n(t_n)x_{n+1} - x_{n+1}, j(x_{n+1} - p) \rangle$$
$$\quad + 2\alpha_n \langle x_{n+1} - x_n, j(x_{n+1} - p) \rangle$$
$$\leq \|x_n - p\|^2 + 2\alpha_n L \|x_{n+1} - x_n\| \|x_{n+1} - p\|$$
$$\quad - 2\alpha_n \lambda \|T_n^n(t_n)x_{n+1} - x_{n+1}\|^2 + 2\alpha_n \mu_n \phi\big(\|x_{n+1} - p\| \big)$$
$$\quad + 2\alpha_n \xi_n + 2\alpha_n \|x_{n+1} - x_n\| \|x_{n+1} - p\|$$
$$\leq \big(1 + 2\alpha_n^2 (1 + L)^3 + 2\alpha_n \mu_n M^* (1 + L)^2 \big) \|x_n - p\|^2$$
$$\quad - 2\alpha_n \lambda \|T_n^n(t_n)x_{n+1} - x_{n+1}\|^2$$
$$\quad + 2\alpha_n \mu_n \phi(M) + 2\alpha_n \xi_n$$
$$\leq \big(1 + 2\alpha_n^2 (1 + L)^3 + 2\alpha_n \mu_n M^* (1 + L)^2 \big) \|x_n - p\|^2$$
$$\quad + 2\alpha_n \mu_n \phi(M) + 2\alpha_n \xi_n. \tag{2.14}$$

By condition (1), it follows from Lemma 1.3 that the limit $\lim_{n \to \infty} \|x_n - p\|$ exists and so the sequence $\{x_n\}$ is bounded in C.

Step 2. Now we prove that

$$\liminf_{n \to \infty} \|x_n - T_n^n(t)x_n\| = 0. \tag{2.15}$$

In fact, it follows from (2.14) that

$$
2\alpha_n\lambda \left\| T_n^n(t_n)x_{n+1} - x_{n+1} \right\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n^2(1+L)^3K^2
$$
$$
+ \left(2M^*(1+L)^2K^2 + 2\phi(M)\right)\alpha_n\mu_n + 2\alpha_n\xi_n, \tag{2.16}
$$

where $K = \sup_{n\geq 1}\|x_n - p\|$. Hence, for some $m \geq 1$,

$$
2\lambda \sum_{n=1}^{m} \alpha_n \left\| T_n^n(t_n)x_{n+1} - x_{n+1} \right\|^2
$$
$$
\leq \sum_{n=1}^{m} \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right) + 2(1+L)^3K^2 \sum_{n=1}^{m} \alpha_n^2
$$
$$
+ \left(2M^*(1+L)^2K^2 + 2\phi(M)\right) \sum_{n=1}^{m} \alpha_n\mu_n + 2\sum_{n=1}^{m} \alpha_n\xi_n
$$
$$
\leq \|x_1 - p\|^2 + 2(1+L)^3K^2 \sum_{n=1}^{m} \alpha_n^2
$$
$$
+ \left(2M^*(1+L)^2K^2 + 2\phi(M)\right) \sum_{n=1}^{m} \alpha_n\mu_n + 2\sum_{n=1}^{m} \alpha_n\xi_n. \tag{2.17}
$$

Letting $m \to \infty$, we have

$$
2\lambda \sum_{n=1}^{\infty} \alpha_n \left\| T_n^n(t_n)x_{n+1} - x_{n+1} \right\|^2
$$
$$
\leq \|x_1 - p\|^2 + 2(1+L)^3K^2 \sum_{n=1}^{\infty} \alpha_n^2
$$
$$
+ \left(2M^*(1+L)^2K^2 + 2\phi(M)\right) \sum_{n=1}^{\infty} \alpha_n\mu_n + 2\sum_{n=1}^{\infty} \alpha_n\xi_n. \tag{2.18}
$$

By condition (1), we obtain

$$
\sum_{n=1}^{\infty} \alpha_n \left\| T_n^n(t_n)x_{n+1} - x_{n+1} \right\|^2 < \infty, \tag{2.19}
$$

which implies

$$
\liminf_{n\to\infty} \left\| x_{n+1} - T_n^n(t_n)x_{n+1} \right\| = 0. \tag{2.20}
$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$ and $\lim_{n\to\infty}\alpha_n = 0$, using (2.11), we have

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.21}
$$

This implies that

$$
\lim_{n\to\infty} \|x_{n+i} - x_n\| = 0, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{2.22}
$$

It follows from (2.20) and (2.21) that

$$
\liminf_{n\to\infty}\left\|x_n - T_n^n(t_n)x_n\right\|
$$

$$
\leq \liminf_{n\to\infty}\left\{\|x_n - x_{n+1}\| + \left\|x_{n+1} - T_n^n(t_n)x_{n+1}\right\| + \left\|T_n^n(t_n)x_{n+1} - T_n^n(t_n)x_n\right\|\right\}
$$

$$
\leq \liminf_{n\to\infty}\left\{(1+L)\|x_n - x_{n+1}\| + \left\|x_{n+1} - T_n^n(t_n)x_{n+1}\right\|\right\} = 0. \tag{2.23}
$$

For any $t \geq 0$, we have

$$
\left\|x_n - T_n^n(t)x_n\right\|
$$

$$
\leq \left\|x_n - T_n^n(t_n)x_n\right\| + \left\|T_n^n(t_n)x_n - T_n^n(t_n + t)x_n\right\| + \left\|T_n^n(t_n + t)x_n - T_n^n(t)x_n\right\|
$$

$$
\leq (1+L)\left\|x_n - T_n^n(t_n)x_n\right\| + \sup_{z\in\{x_n\},t\in R^+}\left\|T_n^n(t + t_n)z - T_n^n(t_n)z\right\|, \tag{2.24}
$$

from (2.8) and (2.23), conclusion (2.15) is proved.

Step 3. Now we prove that

$$
\liminf_{n\to\infty}\left\|x_n - T_m(t)x_n\right\| = 0, \quad \forall m \in \{1, 2, \dots, N\}. \tag{2.25}
$$

For each $j = 1, 2, \dots, N$, let

$$
\eta_{iN+j} := \left\|x_{iN+j} - T_j^{iN+j}(t)x_{iN+j}\right\|.
$$

Since T_j is uniformly $L_j(t)$-Lipschitzian continuous, it follows from (2.1) and (2.4) that

$$
\left\|x_{iN+j} - T_j(t)x_{iN+j}\right\|
$$

$$
\leq \left\|x_{iN+j} - T_j^{iN+j}(t)x_{iN+j}\right\| + \left\|T_j^{iN+j}(t)x_{iN+j} - T_j(t)x_{iN+j}\right\|
$$

$$
\leq \eta_{iN+j} + L\left\|T_j^{iN+j-1}(t)x_{iN+j} - x_{iN+j}\right\|
$$

$$
\leq \eta_{iN+j} + L\left\{\left\|T_j^{iN+j-1}(t)x_{iN+j} - T_j^{iN+j-1}(t)x_{iN+j-1}\right\|\right.
$$

$$
\left. + \left\|T_j^{iN+j-1}(t)x_{iN+j-1} - x_{iN+j-1}\right\| + \|x_{iN+j-1} - x_{iN+j}\|\right\}
$$

$$
\leq \eta_{iN+j} + L(1+L)\|x_{iN+j} - x_{iN+j-1}\| + L\eta_{iN+j-1},
$$

from (2.15) and (2.21), we have

$$
\liminf_{n\to\infty}\left\|x_{iN+j} - T_j(t)x_{iN+j}\right\| = 0. \tag{2.26}
$$

This implies that

$$
\liminf_{n\to\infty}\left\|x_n - T_n(t)x_n\right\| = 0, \tag{2.27}
$$

where $T_n(t) = T_{n(\mathrm{mod}\,N)}(t), \forall n \geq 1$.

For $l \in \{1, 2, \ldots, N\}$, we have

$$
\begin{aligned}
\big\| x_n &- T_{n+l}(t)x_n \big\| \\
&\leq \| x_n - x_{n+l} \| + \big\| x_{n+l} - T_{n+l}(t)x_{n+l} \big\| + \big\| T_{n+l}(t)x_{n+l} - T_{n+l}(t)x_n \big\| \\
&\leq (1+L)\| x_n - x_{n+l} \| + \big\| x_{n+l} - T_{n+l}(t)x_{n+l} \big\|,
\end{aligned}
$$

from (2.22) and (2.27), we obtain

$$
\liminf_{n \to \infty} \big\| x_n - T_{n+l}(t)x_n \big\| = 0. \tag{2.28}
$$

It follows from (2.28) that $\liminf_{n \to \infty} \| x_n - T_m(t)x_n \| = 0, \forall m \in \{1, 2, \ldots, N\}$. This completes the proof.

Step 4. Finally, we prove the sequence $\{x_n\}$ converges strongly to some element in F.

By (2.25), we have $\liminf_{n \to \infty} \| x_n - T_m(t)x_n \| = 0, \forall m \in \{1, 2, \ldots, N\}$. If $\bigcap_{t \geq 0} T_l(t)(C) \subset G$ for some compact subset G of E and some $l \in \{1, 2, \ldots, N\}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in C$ such that

$$
\lim_{k \to \infty} T_l(t)x_{n_k} = q, \qquad \lim_{k \to \infty} \big\| x_{n_k} - T_l(t)x_{n_k} \big\| = 0. \tag{2.29}
$$

Hence, it follows from (2.29) that

$$
\lim_{k \to \infty} x_{n_k} = q. \tag{2.30}
$$

Now, for any $m \in \{1, 2, \ldots, N\}$, since $\liminf_{k \to \infty} \| x_{n_k} - T_m(t)x_{n_k} \| = 0$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\lim_{j \to \infty} \| x_{n_{k_j}} - T_m(t)x_{n_{k_j}} \| = 0$. Using (2.30) and the fact that T_m is Lipschitzian, we get $q \in \bigcap_{t \geq 0} F(T_m(t))$. Since $m \in \{1, 2, \ldots, N\}$ is arbitrarily chosen, we have $q \in F$.

Since $x_{n_k} \to q$ as $k \to \infty$ and the limit $\lim_{n \to \infty} \| x_n - q \|$ exists, this implies that $x_n \to q \in F$ as $n \to \infty$. This completes the proof. □

The following theorem can be obtained from Theorem 2.1 immediately.

Theorem 2.2 *Let E be a real Banach space, and let C be a nonempty, closed convex subset of E. Let $\{\Im_i\}_{i=1}^N$ be a finite family of uniformly $L_i(t)$-Lipschitzian and $(\{k_{in}\}, \lambda_i(t))$-asymptotically strictly pseudocontractive semigroups of C, $L_i(t)$, $\lambda_i(t)$, L^i, λ^i, F^i, L, λ, F be as in Theorem 2.1. $k_n = max_{1 \leq i \leq N}\{k_{in}\}$. Let $\{x_n\}$ be the sequence defined by (2.7), $\{\alpha_n\}$ is a sequence in $(0,1)$ and let $\{t_n\}$ be an increasing sequence in $[0, \infty)$. If the following conditions are satisfied:*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

(2) *Assume for any $i \in \{1, 2, \ldots, N\}$ and for any bounded subset $D \subset C$,*

$$
\lim_{n \to \infty} \sup_{x \in D, s \in R^+} \big\| T_i^n(s + t_n)x - T_i^n(t_n)x \big\| = 0.
$$

(3) *There exists a compact subset G of E such that $\bigcap_{t \geq 0} T_l(t)(C) \subset G$ for some $l \in \{1, 2, \ldots, N\}$.*

Then the sequence $\{x_n\}$ converges strongly to some element in F.

Proof Taking $\phi(\lambda) = \lambda^2$, $\xi_n = 0$, $\mu_n = k_n - 1$ in Theorem 2.1, since all conditions in Theorem 2.1 are satisfied. It follows from Theorem 2.1 that the sequence $x_n \to q \in F :=$ $\bigcap_{t \geq 0} F(T(t))$ as $n \to \infty$.

This completes the proof of Theorem 2.2. □

Remark 2.3 Theorems 2.1 and 2.2 extend and improve the corresponding results of Chang *et al.* [11], Shioji and Takahashi [1], Suzuki [2], Xu [3], Aleyner and Reich [4] and others.

Open problem It may be interesting to post the following open problem: Can Theorem 2.1 be generalized to a finite family of semigroups of mappings S which are representation, so commutative or left reversible (see Liu and Zhang [13] and related references there), which is not necessarily the positive real number with addition?

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