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On the hierarchical variational inclusion problems in Hilbert spaces

Shih-sen Chang[1], Jong Kyu Kim[2*], HW Joseph Lee[3] and Chi Kin Chun[3]

*Correspondence: jongkyuk@kyungnam.ac.kr
[2]Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam 631-701, Korea
Full list of author information is available at the end of the article

Abstract
The purpose of this paper is by using Maingé's approach to study the existence and approximation problem of solutions for a class of hierarchical variational inclusion problems in the setting of Hilbert spaces. As applications, we solve the convex programming problems and quadratic minimization problems by using the main theorems. Our results extend and improve the corresponding recent results announced by many authors.
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1 Introduction

Throughout this paper, we assume that H is a real Hilbert space, C is a nonempty closed and convex subset of H and denote by $\mathrm{Fix}(T)$ the set of fixed points of a mapping $T : C \to C$.

Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a multi-valued mapping. The so-called *quasi-variational inclusion problem* (see [1–3]) is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u). \tag{1.1}$$

A number of problems arising in structural analysis, mechanics and economics can be considered in the framework of this kind of variational inclusions (see, for example, [4]).

The set of solutions of the variational inclusion (1.1) is denoted by Ω.

Special cases

(I) If $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathcal{R} \cup \{+\infty\}$ is a proper convex and lower semi-continuous function and $\partial\phi$ is the sub-differential of ϕ, then variational inclusion problem (1.1) is equivalent to finding $u \in H$ such that

$$\langle A(u), v - u\rangle + \phi(y) - \phi(u) \geq 0, \quad \forall y \in H, \tag{1.2}$$

which is called *the mixed quasi-variational inequality*.

Especially, if $A = 0$, then (1.2) is equivalent to the minimizing problem of ϕ on H, *i.e.*, to find $u \in H$ such that $\phi(u) = \inf_{y \in H} \phi(y)$.

(II) If $M = \partial\delta_C$, where C is a nonempty closed and convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, *i.e.*,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then variational inclusion problem (1.2) is equivalent to finding $u \in C$ such that

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

This problem is called *Hartman-Stampacchia variational inequality problem*.

(III) If $M = 0$ and $A = I - T$ where I is an identity mapping and $T : H \to H$ is a nonlinear mapping, then problem (1.1) is equivalent to the fixed point problem of T. That is, find $u \in H$ such that

$$u = Tu. \tag{1.4}$$

Recently, *hierarchical fixed point problems*, *hierarchical optimization problems* and *hierarchical minimization problems* have attracted many authors' attention due to their link with convex programming problems, optimization problems and monotone variational inequality problems *etc.* (see [5–21] and others).

The purpose of this paper is to introduce and study the following *bi-level hierarchical variational inclusion problem* in the setting of Hilbert spaces:

Find $(x^*, y^*) \in \Omega_1 \times \Omega_2$ such that for given positive real numbers ρ and η, the following inequalities hold:

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \end{cases} \tag{1.5}$$

where $F, A_1, A_2 : H \to H$ are mappings and $M_1, M_2 : H \to 2^H$ are multi-valued mappings, Ω_i is the set of solutions to variational inclusion problem (1.1) with $A = A_i$, $M = M_i$ for $i = 1, 2$.

Special examples

(I) If $M_i = 0$, $A_i = I - T_i$, where $T_i : H \to H$ is a nonlinear mapping for each $i = 1, 2$, then $\Omega_i = \mathrm{Fix}(T_i)$ and *bi-level hierarchical variational inclusion problem* (1.5) is equivalent to finding $(x^*, y^*) \in \mathrm{Fix}(T_1) \times \mathrm{Fix}(T_2)$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in \mathrm{Fix}(T_1), \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in \mathrm{Fix}(T_2). \end{cases} \tag{1.6}$$

This problem, which is called *bi-level hierarchical optimization problem*, was studied by Maingé [20] and Kraikaew *et al.* [21].

(II) In (1.6), if $T_i = P_{K_i}$ for each $i = 1,2$, where P_{K_i} is the metric projection from H onto a nonempty closed convex subset K_i, then it is clear that the $\Omega_i = \mathrm{Fix}(T_i) = K_i$ and bi-level hierarchical optimization problem (1.6) is equivalent to finding $(x^*, y^*) \in K_1 \times K_2$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in K_2. \end{cases} \tag{1.7}$$

This system forms a more general problem originated from Nash equilibrium points and it was treated from a theoretical viewpoint in [22–24].

(III) If $\eta = 0$, $\rho > 0$ and both sets Ω_1 and Ω_2 are nonempty closed and convex subsets of H, then *bi-level hierarchical variational inclusion problem* (1.5) reduces to the following *(one-level) hierarchical variational inclusion problem*:

Find $x^* \in \Omega_1$ such that for a given positive real number ρ, the following inequality holds:

$$\langle \rho F(y^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1. \tag{1.8}$$

(IV) If $K_1 = K_2 = K$ and $\eta = 0$, $\rho > 0$, then (1.7) reduces to the *classic variational inequality*, *i.e.*, the problem of finding $x^* \in K$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K. \tag{1.9}$$

In (1.5), it is worth noting that if Ω_1, Ω_2 are nonempty closed convex subsets in H, then the metric projections P_{Ω_1} and P_{Ω_2} from H onto Ω_1 and Ω_1, respectively, are well defined and problem (1.5) is equivalent to the problem of finding $(x^*, y^*) \in \Omega_1 \times \Omega_2$ such that

$$\begin{cases} x^* = P_{\Omega_1}[y^* - \rho F(y^*)], \\ y^* = P_{\Omega_2}[x^* - \eta F(x^*)]. \end{cases} \tag{1.10}$$

However, in practice, both solution sets Ω_1 and Ω_2 (and hence the two projections) are not given explicitly.

To overcome this drawback, inspired by the method studied by Yamada *et al.* [25, 26], Maingé [20] and Kraikaew *et al.* [21], we investigate a more general variant of the scheme proposed by Maingé [20], Kraikaew *et al.* [21] to replace the projection by some suitable mappings with a nice fixed point set. This strategy also suggests an effective approximation process. Our analysis and method allow us to prove the existence and approximation of solutions to problem (1.5). As applications, we utilize the main results to study the quadratic minimization problems and convex programming problems in Hilbert spaces. The results presented in the paper extend and improve the corresponding results in [20, 21, 25, 26] and others.

2 Preliminaries

For the sake of convenience, we first recall some definitions and lemmas for our main results.

Definition 2.1 A mapping $A : H \to H$ is said to be *α-inverse-strongly monotone* if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H.$$

A multi-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $u \in Mx$ and $v \in My$ imply that

$$\langle u - v, x - y\rangle \geq 0.$$

A multi-valued mapping $M : H \to 2^H$ is said to be *maximal monotone* if it is monotone and for any $(x, u) \in H \times H$,

$$\langle u - v, x - y\rangle \geq 0$$

for every $(y, v) \in \mathrm{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Lemma 2.2 [19] *Let $A : H \to H$ be an α-inverse-strongly monotone mapping. Then*

(i) *A is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping*;

(ii) *For any constant $\lambda > 0$, we have*

$$\big\|(I - \lambda A)x - (I - \lambda A)y\big\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2; \tag{2.1}$$

(iii) *If $\lambda \in (0, 2\alpha]$, then $I - \lambda A$ is a nonexpansive mapping, where I is the identity mapping on H.*

Let H be a real Hilbert space, C be a nonempty closed convex subset of H. For each $x \in H$, there exists a unique *nearest point* in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a mapping P_C from H onto C is called the *metric projection*.

Remark 2.3 It is well known that the metric projection P_C has the following properties:

(i) $P_C : H \to C$ is nonexpansive;

(ii) P_C *is firmly nonexpansive, i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H;$$

(iii) For each $x \in H$,

$$z = P_C(x) \quad \Leftrightarrow \quad \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Definition 2.4 Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H$$

is called *the resolvent operator associated with* M, where λ is any positive number and I is the identity mapping.

Proposition 2.5 [19] *Let* $M : H \to 2^H$ *be a multi-valued maximal monotone mapping, and let* $A : H \to H$ *be an* α*-inverse-strongly monotone mapping. Then the following conclusions hold.*

(i) *The resolvent operator* $J_{M,\lambda}$ *associated with* M *is single-valued and nonexpansive for all* $\lambda > 0$.

(ii) *The resolvent operator* $J_{M,\lambda}$ *is* 1*-inverse-strongly monotone, i.e.,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \le \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \quad \forall x, y \in H.$$

(iii) $u \in H$ *is a solution of the variational inclusion* (1.1) *if and only if* $u = J_{M,\lambda}(u - \lambda Au)$, $\forall \lambda > 0$, *i.e.,* u *is a fixed point of the mapping* $J_{M,\lambda}(I - \lambda A)$. *Therefore we have*

$$\Omega = \mathrm{Fix}\big(J_{M,\lambda}(I - \lambda A)\big), \quad \forall \lambda > 0, \tag{2.3}$$

where Ω *is the set of solutions of variational inclusion problem* (1.1).

(iv) *If* $\lambda \in (0, 2\alpha]$, *then* Ω *is a closed convex subset in* H.

In the sequel, we denote the strong and weak convergence of a sequence $\{x_n\}$ in H to an element $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Lemma 2.6 [27] *For* $x, y \in H$ *and* $\omega \in (0,1)$, *the following statements hold*:

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;

(ii) $\|(1 - \omega)x + \omega y\|^2 = (1 - \omega)\|x\|^2 + \omega\|y\|^2 - \omega(1 - \omega)\|x - y\|^2$.

Lemma 2.7 [28] *Let* $\{a_n\}$ *be a sequence of real numbers, and there exists a subsequence* $\{a_{m_j}\}$ *of* $\{a_n\}$ *such that* $a_{m_j} < a_{m_j+1}$ *for all* $j \in N$, *where* N *is the set of all positive integers. Then there exists a nondecreasing sequence* $\{n_k\}$ *of* N *such that* $\lim_{k\to\infty} n_k = \infty$ *and the following properties are satisfied by all* (*sufficiently large*) *number* $k \in N$:

$$a_{n_k} \le a_{n_k+1} \quad \textit{and} \quad a_k \le a_{n_k+1}.$$

In fact, n_k *is the largest number* n *in the set* $\{1, 2, \ldots, k\}$ *such that* $a_n < a_{n+1}$ *holds.*

Lemma 2.8 [21] *Let* $\{a_n\} \subset [0, \infty)$, $\{\alpha_n\} \subset [0, 1)$, $\{b_n\} \subset (-\infty, +\infty)$, $\hat{\alpha} \in [0, 1)$ *be such that*

(i) $\{a_n\}$ *is a bounded sequence*;

(ii) $a_{n+1} \le (1 - \alpha_n)^2 a_n + 2\alpha_n\hat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$, $\forall n \ge 1$;

(iii) *whenever* $\{a_{n_k}\}$ *is a subsequence of* $\{a_n\}$ *satisfying*

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0,$$

it follows that $\limsup_{k\to\infty} b_{n_k} \le 0$;

(iv) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Definition 2.9

(i) A mapping $T : H \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

(ii) A mapping $T : H \to H$ is said to be *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in H, p \in \mathrm{Fix}(T).$$

It should be noted that T is quasi-nonexpansive if and only if $\forall x \in H$, $p \in \mathrm{Fix}(T)$

$$\langle x - Tx, x - p\rangle \geq \frac{1}{2}\|x - Tx\|^2. \tag{2.4}$$

(iii) A mapping $T : H \to H$ is said to be *strongly quasi-nonexpansive* if T is quasi-nonexpansive and

$$x_n - Tx_n \to 0 \tag{2.5}$$

whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in \mathrm{Fix}(T)$.

Lemma 2.10 *Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, $A : H \to H$ be an α-inverse-strongly monotone mapping and let Ω be the set of solutions of variational inclusion problem* (1.1) *and $\Omega \neq \emptyset$. Then the following statements hold.*

(i) *If $\lambda \in (0, 2\alpha]$, then the mapping $K : H \to H$ defined by*

$$K := J_{M,\lambda}(I - \lambda A) \tag{2.6}$$

is quasi-nonexpansive, where I is the identity mapping and $J_{M,\lambda}$ is the resolvent operator associated with M.

(ii) *The mapping $I - K : H \to H$ is demiclosed at zero, i.e., for any sequence $\{x_n\} \subset H$, if $x_n \rightharpoonup x$ and $(I - K)x_n \to 0$, then $x = Kx$.*

(iii) *For any $\beta \in (0, 1)$, the mapping K_β defined by*

$$K_\beta = (1 - \beta)I + \beta K \tag{2.7}$$

is a strongly quasi-nonexpansive mapping and $\mathrm{Fix}(K_\beta) = \mathrm{Fix}(K)$.

(iv) $I - K_\beta$, $\beta \in (0, 1)$ *is demiclosed at zero.*

Proof (i) Since $\lambda \in (0, 2\alpha]$, it follows from Lemma 2.2(iii) and Proposition 2.5 that the mapping K is nonexpansive and $\Omega = \mathrm{Fix}(K) \neq \emptyset$. This implies that K is quasi-nonexpansive.

(ii) Since K is a nonexpansive mapping on H, $I - K$ is demiclosed at zero.

(iii) It is obvious that $\mathrm{Fix}(K_\beta) = \mathrm{Fix}(K)$ and K_β is quasi-nonexpansive.

Next we prove that K_β, $\beta \in (0, 1)$ is a strongly quasi-nonexpansive mapping.

In fact, let $\{x_n\}$ be any bounded sequence in H and let $p \in \mathrm{Fix}(K_\beta)$ be a given point such that

$$\|x_n - p\| - \|K_\beta x_n - p\| \to 0. \tag{2.8}$$

Now we prove that $\|K_\beta x_n - x_n\| \to 0$.

In fact, it follows from (2.4) that

$$\begin{aligned}\|K_\beta x_n - p\|^2 &= \|x_n - p - \beta(x_n - Kx_n)\|^2 \\ &= \|x_n - p\|^2 - 2\beta\langle x_n - p, x_n - Kx_n\rangle + \beta^2\|x_n - Kx_n\|^2 \\ &\le \|x_n - p\|^2 - \beta(1-\beta)\|x_n - Kx_n\|^2.\end{aligned}$$

Hence from (2.8), we have

$$\beta(1-\beta)\|Kx_n - x_n\|^2 \le \|x_n - p\|^2 - \|K_\beta x_n - p\|^2 \to 0.$$

Since $\beta(1-\beta) > 0$, $\|Kx_n - x_n\| \to 0$, and so

$$\|K_\beta x_n - x_n\| = \beta\|Kx_n - x_n\| \to 0.$$

(iv) Since $I - K_\beta = \beta(I - K)$ and $I - K$ is demi-closed at zero, hence $I - K_\beta$ is demi-closed at zero. This completes the proof. □

3 Main results

Throughout this section we always assume that the following conditions are satisfied:

(C1) $M_i : H \to 2^H$, $i = 1, 2$, is a multi-valued maximal monotone mapping, $A_i : H \to H$ is an α-inverse-strongly monotone mapping and Ω_i is the set of solutions to variational inclusion problem (1.1) with $A = A_i$, $M = M_i$ and $\Omega_i \neq \emptyset$;

(C2) K_i and $K_{i\beta}$, $\beta \in (0,1)$, $i = 1, 2$, are the mappings defined by

$$\begin{cases} K_i := J_{M,\lambda}(I - \lambda A_i), & \lambda \in (0, 2\alpha], \\ K_{i,\beta} = (1-\beta)I + \beta K_i, & \beta \in (0,1), \end{cases} \tag{3.1}$$

respectively.

We have the following result.

Theorem 3.1 *Let A_i, M_i, Ω_i, K_i, $K_{i\beta}$, $i = 1, 2$, satisfy the conditions* (C1) *and* (C2), *and let $f, g : H \to H$ be contractions with a contractive constant $h \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences defined by*

$$\begin{cases} x_0, y_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f(K_{2,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{2,\beta}y_n + \alpha_n g(K_{1,\beta}x_n), & n = 0, 1, 2, \dots, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge to* x^* *and* y^*, *respectively, where* $(x^*, y^*) \in \Omega_1 \times \Omega_2$ *is the unique solution of the following (bi-level) hierarchical optimization problem*:

$$\begin{cases} \langle x^* - f(y^*), x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle y^* - g(x^*), y - y^* \rangle \geq 0, & \forall y \in \Omega_2. \end{cases} \tag{3.3}$$

Proof (I) First we prove that (3.3) has a unique solution $(x^*, y^*) \in \Omega_1 \times \Omega_2$.

Indeed, it follows from Proposition 2.5 and Lemma 2.10 that both sets Ω_1, Ω_2 are nonempty closed and convex and $\Omega_i = \operatorname{Fix}(K_i)$ for each $i = 1, 2$. Hence the metric projection P_{Ω_i} for each $i = 1, 2$ is well defined. It is clear that the mapping

$$P_{\Omega_1} \circ f \circ P_{\Omega_2} \circ g : H \to H$$

is a contraction. By the Banach contractive mapping principle, there exists a unique element $x^* \in H$ such that

$$x^* = (P_{\Omega_1} \circ f \circ P_{\Omega_2} \circ g)(x^*).$$

Letting $y^* = P_{\Omega_2} \circ g(x^*)$, then it is easy to see that

$$x^* = (P_{\Omega_1} \circ f)(y^*), \qquad y^* = (P_{\Omega_2} \circ g)(x^*)$$

are the unique solution of (3.3).

(II) Now we prove that $\{x_n\}$ and $\{y_n\}$ are bounded.

In fact, it follows from Lemma 2.10 that $K_{i,\beta}$, $i = 1, 2$, is strongly quasi-nonexpansive and $\operatorname{Fix}(K_{i,\beta}) = \operatorname{Fix}(K_i) = \Omega_i$. Since f is h-contractive and $x^* \in \operatorname{Fix}(K_{1,\beta})$, $y^* \in \operatorname{Fix}(K_{2,\beta})$, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|K_{1,\beta}x_n - x^*\| + \alpha_n \|f(K_{2,\beta}y_n) - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|f(K_{2,\beta}y_n) - f(y^*)\| + \alpha_n \|f(y^*) - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n h \|K_{2,\beta}y_n - y^*\| + \alpha_n \|f(y^*) - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n h \|y_n - y^*\| + \alpha_n \|f(y^*) - x^*\|. \end{aligned}$$

Similarly, we can also prove that

$$\|y_{n+1} - y^*\| \leq (1 - \alpha_n)\|y_n - y^*\| + \alpha_n h \|x_n - x^*\| + \alpha_n \|g(x^*) - y^*\|.$$

This implies that

$$\begin{aligned} &\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\ &\quad \leq \bigl(1 - \alpha_n(1 - h)\bigr)\bigl(\|x_n - x^*\| + \|y_n - y^*\|\bigr) \\ &\qquad + \alpha_n(1 - h)\frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1 - h} \\ &\quad \leq \max\left\{\bigl(\|x_n - x^*\| + \|y_n - y^*\|\bigr), \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1 - h}\right\}. \end{aligned}$$

By induction, we have

$$\begin{aligned}
&\|x_n - x^*\| + \|y_n - y^*\| \\
&\quad \le \max\left\{ \left(\|x_0 - x^*\| + \|y_0 - y^*\|\right), \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1-h} \right\}, \quad \forall n \ge 1.
\end{aligned}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, the sequences $\{K_{1,\beta}x_n\}$ and $\{K_{2,\beta}y_n\}$ both are bounded.

(III) Next we prove that for each $n \ge 1$ the following inequality holds.

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\
&\quad \le (1-\alpha_n)^2 \left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) \\
&\qquad + 2\alpha_n h \left(\|x_{n+1} - x^*\| \|y_n - y^*\| + \|x_n - x^*\| \|y_{n+1} - y^*\|\right) \\
&\qquad + 2\alpha_n \left(\langle f(y^*) - x^*, x_{n+1} - x^*\rangle + \langle g(x^*) - y^*, x_{n+1} - y^*\rangle\right). \qquad (3.4)
\end{aligned}$$

In fact, it follows from (3.2) and Lemma 2.6(i) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|(1-\alpha_n)\left(K_{1,\beta}x_n - x^*\right) + \alpha_n\left(f(K_{2,\beta}y_n) - x^*\right)\right\|^2 \\
&\le \left\|(1-\alpha_n)\left(K_{1,\beta}x_n - x^*\right)\right\|^2 + 2\alpha_n\left\langle \left(f(K_{2,\beta}y_n) - x^*\right), x_{n+1} - x^*\right\rangle \\
&= (1-\alpha_n)^2 \|K_{1,\beta}x_n - x^*\|^2 + 2\alpha_n\left\langle f(K_{2,\beta}y_n) - f(y^*), x_{n+1} - x^*\right\rangle \\
&\quad + 2\alpha_n\left\langle f(y^*) - x^*, x_{n+1} - x^*\right\rangle \\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \left\|f(K_{2,\beta}y_n) - f(y^*)\right\| \|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\left\langle f(y^*) - x^*, x_{n+1} - x^*\right\rangle \\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n h \|y_n - y^*\| \|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\left\langle f(y^*) - x^*, x_{n+1} - x^*\right\rangle.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|y_{n+1} - y^*\|^2 &\le (1-\alpha_n)^2 \|y_n - y^*\|^2 + 2\alpha_n h \|x_n - x^*\| \|y_{n+1} - y^*\| \\
&\quad + 2\alpha_n\left\langle g(x^*) - y^*, y_{n+1} - y^*\right\rangle.
\end{aligned}$$

Adding up the last two inequalities, the inequality (3.4) is proved.

(IV) Next we prove the following fact.

If there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1} - x^*\|^2 + \|y_{n_k+1} - y^*\|^2 - \left(\|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2\right)\right) \ge 0,$$

then

$$\limsup_{k\to\infty}\left(\langle f(y^*) - x^*, x_{n_k+1} - x^*\rangle + \langle g(x^*) - y^*, y_{n_k+1} - y^*\rangle\right) \le 0.$$

In fact, since the norm $\|\cdot\|^2$ is convex and $\lim_{n\to\infty}\alpha_n = 0$, from (3.2) we have that

$$\begin{aligned}
0 &\le \liminf_{k\to\infty}\bigl\{\|x_{n_k+1}-x^*\|^2+\|y_{n_k+1}-y^*\|^2-\bigl(\|x_{n_k}-x^*\|^2+\|y_{n_k}-y^*\|^2\bigr)\bigr\}\\
&\le \liminf_{k\to\infty}\bigl\{(1-\alpha_{n_k})\|K_{1,\beta}x_{n_k}-x^*\|^2+\alpha_{n_k}\|f(K_{2,\beta}y_{n_k})-x^*\|^2\\
&\quad+(1-\alpha_{n_k})\|K_{2,\beta}y_{n_k}-y^*\|^2+\alpha_{n_k}\|g(K_{1,\beta}x_{n_k})-y^*\|^2-\|x_{n_k}-x^*\|^2-\|y_{n_k}-y^*\|^2\bigr\}\\
&= \liminf_{k\to\infty}\bigl\{\bigl(\|K_{1,\beta}x_{n_k}-x^*\|^2-\|x_{n_k}-x^*\|^2\bigr)+\bigl(\|K_{2,\beta}y_{n_k}-y^*\|^2-\|y_{n_k}-y^*\|^2\bigr)\bigr\}\\
&\le \limsup_{k\to\infty}\bigl\{\bigl(\|K_{1,\beta}x_{n_k}-x^*\|^2-\|x_{n_k}-x^*\|^2\bigr)+\bigl(\|K_{2,\beta}y_{n_k}-y^*\|^2-\|y_{n_k}-y^*\|^2\bigr)\bigr\}\\
&= \limsup_{k\to\infty}\bigl\{\bigl(\|K_{1,\beta}x_{n_k}-x^*\|+\|x_{n_k}-x^*\|\bigr)\bigl(\|K_{1,\beta}x_{n_k}-x^*\|-\|x_{n_k}-x^*\|\bigr)\\
&\quad+\bigl(\|K_{2,\beta}y_{n_k}-y^*\|+\|y_{n_k}-y^*\|\bigr)\bigl(\|K_{2,\beta}y_{n_k}-y^*\|-\|y_{n_k}-y^*\|\bigr)\bigr\}\\
&\le 0.
\end{aligned}$$

The above conclusion can be proved as follows.

Indeed, since the sequences $\{\|K_{1,\beta}x_{n_k}-x^*\|+\|x_{n_k}-x^*\|\}$ and $\{\|K_{2,\beta}y_{n_k}-y^*\|+\|y_{n_k}-y^*\|\}$ are bounded, and $K_{i,\beta}$, $i=1,2$, is quasi-nonexpansive, we have

$$\begin{aligned}
&\|K_{1,\beta}x_{n_k}-x^*\|\le\|x_{n_k}-x^*\|,\\
&\|K_{2,\beta}y_{n_k}-y^*\|\le\|y_{n_k}-y^*\|.
\end{aligned}$$

The conclusion is proved. Therefore we have that

$$\lim_{k\to\infty}\bigl(\|K_{1,\beta}x_{n_k}-x^*\|-\|x_{n_k}-x^*\|\bigr)=\lim_{k\to\infty}\bigl(\|K_{2,\beta}y_{n_k}-y^*\|-\|y_{n_k}-y^*\|\bigr)=0. \tag{3.5}$$

By Lemma 2.10(iii), the mapping $K_{i,\beta}$, $i=1,2$, is strongly quasi-nonexpansive. Hence from (3.5) we have that

$$K_{1,\beta}x_{n_k}-x_{n_k}\to 0,\qquad K_{2,\beta}y_{n_k}-y_{n_k}\to 0. \tag{3.6}$$

This together with (3.2) shows that

$$x_{n_k+1}-x_{n_k}\to 0\quad\text{and}\quad y_{n_k+1}-y_{n_k}\to 0.$$

Since $\{x_{n_k}\}$ is bounded and H is reflexive, there exists a subsequence $\{x_{n_{k_l}}\}\subset\{x_{n_k}\}$ such that $x_{n_{k_l}}\rightharpoonup u$ and

$$\lim_{l\to\infty}\langle f(y^*)-x^*,x_{n_{k_l}}-x^*\rangle=\limsup_{k\to\infty}\langle f(y^*)-x^*,x_{n_k}-x^*\rangle=\limsup_{k\to\infty}\langle f(y^*)-x^*,x_{n_k+1}-x^*\rangle.$$

On the other hand, by virtue of Lemma 2.10(iv), $I-K_{1,\beta}$ is demiclosed at zero, and so $u\in\mathrm{Fix}(K_{1,\beta})=\Omega_1$. Hence from (3.3) we have

$$\lim_{l\to\infty}\langle f(y^*)-x^*,x_{n_{k_l}}-x^*\rangle=\langle f(y^*)-x^*,u-x^*\rangle\le 0.$$

Consequently,

$$\limsup_{k\to\infty}\langle f(y^*) - x^*, x_{n_k+1} - x^*\rangle \le 0.$$

Similarly, by using the same argument, we have

$$\limsup_{k\to\infty}\langle g(x^*) - y^*, y_{n_k+1} - y^*\rangle \le 0.$$

The desired inequality is proved.

(V) Finally we prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.2) converge to x^* and y^*, respectively.

It is easy to see that

$$\begin{aligned}&\|x_{n+1} - x^*\|\|y_n - y^*\| + \|x_n - x^*\|\|y_{n+1} - y^*\|\\&\quad\le \left(\|y_n - y^*\|^2 + \|x_n - x^*\|^2\right)^{\frac{1}{2}}\left(\|y_{n+1} - y^*\|^2 + \|x_{n+1} - x^*\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.4), simplifying and putting

$$\begin{aligned}&a_n := \|x_n - x^*\|^2 + \|y_n - y^*\|^2,\\&b_n := 2\left(\langle f(y^*) - x^*, x_{n+1} - x^*\rangle + \langle g(x^*) - y^*, x_{n+1} - y^*\rangle\right),\end{aligned}$$

then we have the following conclusions:

(i) By (II), $\{a_n\}$ is a bounded sequence;

(ii) From (3.4), $a_{n+1} \le (1-\alpha_n)^2 a_n + 2\alpha_n h\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$, $\forall n \ge 1$;

(iii) By (IV), for any subsequence $\{a_{n_k}\} \subset \{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0,$$

it follows that $\limsup_{k\to\infty} b_{n_k} \le 0$.

Hence it follows from Lemma 2.8 that $x_n \to x^*$ and $y_n \to y^*$. This completes the proof of Theorem 3.1. □

Definition 3.2 A mapping $F : H \to H$ is said to be *μ-Lipschitzian and r-strongly monotone*, if there exist constants $\mu > 0$ and $r > 0$ such that

$$\|Fx - Fy\| \le \mu\|x - y\|, \qquad \langle Fx - Fy, x - y\rangle \ge r\|x - y\|^2, \quad \forall x, y \in H.$$

Remark 3.3 It is easy to prove that if $F : H \to H$ is a μ-Lipschitzian and r-strongly monotone mapping and if $\rho \in (0, \frac{2r}{\mu^2})$, then the mapping $f := I - \rho F : H \to H$ is a contraction.

Now we are in a position to prove the following main result.

Theorem 3.4 *Let A_i, M_i, Ω_i, K_i, $K_{i\beta}$, $i = 1, 2$, be the same as in Theorem* 3.1. *Let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$ and $\{y_n\}$ be the sequences* *defined by*

$$\begin{cases} x_0, y_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f(K_{2,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{2,\beta}y_n + \alpha_n g(K_{1,\beta}x_n), \end{cases} \tag{3.8}$$

where $f := I - \rho F$, $g := I - \eta F$ *with* $\rho, \eta \in (0, \frac{2r}{\mu^2})$, $\beta \in (0,1)$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying the following conditions*:

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then the sequence $(\{x_n\}, \{y_n\})$ *converges strongly to the unique solution* (x^*, y^*) *of bi-level hierarchical variational inclusion problem* (1.5).

Proof Indeed, it follows from Remark 3.3 that both mappings $f, g : H \to H$ are contractive. Therefore all the conditions in Theorem 3.1 are satisfied. By Theorem 3.1, the sequence $(\{x_n\}, \{y_n\})$ converges strongly to $(x^*, y^*) \in \Omega_1 \times \Omega_2$, which is the unique solution of the following bi-level hierarchical optimization problem:

$$\begin{cases} \langle x^* - f(y^*), x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle y^* - g(x^*), y - y^* \rangle \geq 0, & \forall y \in \Omega_2. \end{cases} \tag{3.9}$$

Since $f = I - \rho F$ and $g = I - \eta F$, we have

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in \Omega_2. \end{cases} \tag{3.10}$$

This implies that the sequence $(\{x_n\}, \{y_n\})$ converges strongly to $(x^*, y^*) \in \Omega_1 \times \Omega_2$, which is the unique solution of bi-level hierarchical variational inclusion problem (1.5). This completes the proof of Theorem 3.4. □

4 Some applications

In this section, we shall utilize Theorem 3.1 and Theorem 3.4 to study the convex mathematical programming problem and quadratic minimization problem.

(I) Applications to convex mathematical programming problems.

Let $\psi : H \to \mathcal{R}$ be a convex and lower semi-continuous function with $\nabla\psi$ being μ-Lipschitzian and r-strongly monotone, *i.e.*, it satisfies the following conditions:

$$\|\nabla\psi(x) - \nabla\psi(y)\| \leq \mu\|x - y\|, \quad \forall x, y \in H, \mu > 0, \tag{4.1}$$

and

$$\langle \nabla\psi(x) - \nabla\psi(y), x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H, r > 0. \tag{4.2}$$

In (1.5) taking $\eta = 0$, $\rho \in (0, \frac{2r}{\mu^2})$ and $F = \nabla\psi$, then hierarchical variational inclusion problem (1.5) reduces to the following problem:

Find a point $x^* \in \Omega_1$ such that

$$\langle \nabla\psi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1. \tag{4.3}$$

By using the subdifferential inequality, this implies that

$$\psi(x) - \psi(x^*) \geq \langle \nabla\psi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1.$$

Therefore we have

$$\psi(x) - \psi(x^*) \geq 0, \quad \forall x \in \Omega_1. \tag{4.4}$$

Thus problem (4.3) reduces to the *convex mathematical programming problem on* Ω_1:

Find a point $x^* \in \Omega_1$ such that

$$\min_{x \in \Omega_1} \psi(x). \tag{4.5}$$

Hence, we have the following result.

Theorem 4.1 *Let* A_1, M_1, Ω_1, K_1, $K_{1,\beta}$, $\{\alpha_n\}$ *be the same as in Theorem* 3.4. *Let* $\{x_n\}$ *be the iterative sequence defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n(I - \rho F)(K_{1,\beta}x_n), \end{cases} \tag{4.6}$$

where $\rho \in (0, \frac{2r}{\mu^2})$, $\beta \in (0,1)$. *Then* $\{x_n\}$ *converges strongly to* $x^* \in \Omega_1$, *which is the unique solution of convex mathematical programming problem* (4.5).

(II) Applications to quadratic minimization problems.

Recall that a linear bounded operator $T : H \to H$ is said to be ξ*-strongly positive* if there exists a positive constant ξ such that

$$\langle Tx, x \rangle \geq \xi \|x\|^2, \quad \forall x \in H.$$

Lemma 4.2 *Let* $T : H \to H$ *be a* ξ*-strongly positive linear operator and let* γ *be a positive number with* $\gamma < \frac{1}{\|T\|}$, *where* $\|T\|$ *is the norm of* T *defined by*

$$\|T\| = \sup\{\langle Tu, u \rangle : u \in H, \|u\| = 1\}.$$

Then we have

(1) *The linear operator* $F := I + \gamma T : H \to H$ *is* μ*-Lipschitzian and* r*-strongly monotone, where* $\mu = (1 + \gamma\|T\|)$ *and* $r = 1 + \gamma\xi$.

(2) *If* $\rho \in (0, \frac{1}{1+\gamma\xi})$, *then the linear operator* $(I - \rho(I + \gamma T))$ *is contractive with a contractive constant* $h := 1 - \rho(1 + \gamma\xi)$.

Proof (1) In fact, for any $x, y \in H$, we have

$$\|(I+\gamma T)(x-y)\| \le (1+\gamma\|T\|)\|x-y\| = \mu\|x-y\|.$$

Again, since $T : H \to H$ is a ξ-strongly positive linear operator, we have

$$\langle (I+\gamma T)(x-y), x-y\rangle \ge (1+\gamma\xi)\|x-y\|^2 = r\|x-y\|^2.$$

Conclusion (1) is proved.

(2) By the definition of the norm of the bounded linear operator $(I - \rho(I+\gamma T))$, we have

$$\begin{aligned} \|I-\rho(I+\gamma T)\| &= \sup\{\langle (I-\rho(I+\gamma T))u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{(1-\rho-\rho\gamma)\langle Tu, u\rangle : u \in H, \|u\| = 1\} \\ &\le 1-\rho-\rho\gamma\xi, \quad \forall \rho \in \left(0, \frac{1}{1+\gamma\xi}\right). \end{aligned}$$

Therefore, $(I - \rho(I+\gamma T))$ is contractive with a contractive constant $1 - \rho(1+\gamma\xi)$. This completes the proof. □

From Theorem 3.4 and Lemma 4.2 we have the following result.

Theorem 4.3 *Let A, M, K, K_β, Ω and $\{\alpha_n\}$ satisfy the same conditions as given in Theorem 3.4. Let the linear mappings T and F satisfy the same conditions as in Lemma 4.2. Then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_\beta x_n + \alpha_n(I-\rho F)(K_\beta x_n), \end{cases} \tag{4.7}$$

where $\rho \in (0, \frac{1}{1+\gamma\xi})$, $\beta \in (0,1)$, converges strongly to $x^* \in \Omega_1$, which is the unique solution of the hierarchical variational inclusion problem:

$$\langle \rho(I+\gamma T)x^*, x-x^*\rangle \ge 0, \quad \forall x \in \Omega,$$

that is,

$$\langle (I+\gamma T)x^*, x-x^*\rangle \ge 0, \quad \forall x \in \Omega. \tag{4.8}$$

Letting $g(x) := \frac{\gamma}{2}\langle Tx, x\rangle + \frac{1}{2}\|x\|^2$, then it is easy to know that $g : H \to R^+$ is a continuous and convex functional and $\partial g(x^*) = (I+\gamma T)(x^*)$. By the subdifferential inequality of g, we have

$$g(x) - g(x^*) \ge \langle (I+\gamma T)(x^*), x-x^*\rangle \ge 0, \quad \forall x \in \Omega.$$

This implies that x^* solves the following quadratic minimization problem:

$$\min_{x\in\Omega}\left\{\frac{\gamma}{2}\langle Tx, x\rangle + \frac{1}{2}\|x\|^2\right\} \tag{4.9}$$

and $x_n \to x^*$. This completes the proof.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by JKK. JKK and SC prepared the manuscript initially and performed all the steps of the proof in this research. All authors read and approved the final manuscript.

Author details
[1]College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China. [2]Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam 631-701, Korea. [3]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, P.R. China.

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