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# Some new common coupled fixed point results in two generalized metric spaces

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**Abstract**
The purpose of this paper is to extend some recent common coupled fixed point theorems in two $G$-metric spaces in an essentially different and more natural way. We also provide illustrative examples in support of our new results.
**MSC:** 47H10; 54H25

**Keywords:** common coupled fixed point; coupled coincidence point; $w^*$-compatible mapping pairs; generalized metric space

## 1 Introduction and preliminaries

In 2006, Mustafa and Sims [1] introduced a new structure of generalized metric spaces, which are called $G$-metric spaces, as follows.

**Definition 1.1** [1] Let $X$ be a nonempty set, and let $G : X \times X \times X \longrightarrow R^+$ be a function satisfying the following axioms:

(G1) $G(x,y,z)=0$ if $x = y = z$;
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function $G$ is called a generalized metric or a $G$-metric on $X$ and the pair $(X,G)$ is called a $G$-metric space.

It is known that the function $G(x,y,z)$ on a $G$-metric space $X$ is jointly continuous in all three of its variables, and $G(x,y,z) = 0$ if and only if $x = y = z$ (see [1]).

Based on the notion of generalized metric spaces, Mustafa *et al.* [1–6] obtained some fixed point results for mappings satisfying different contractive conditions. Chugh *et al.* [7] obtained some fixed point results for maps satisfying property $P$ in $G$-metric spaces. Shatanawi [8] obtained some fixed point results for contractive mappings satisfying $\Phi$-maps in $G$-metric spaces.

In 2009, Abbas and Rhoades [9] initiated the study of common fixed point theory in $G$-metric spaces. Since then, many common fixed point theorems for certain contractive conditions have been established in $G$-metric spaces (see [10–19]).

Bhaskar and Lakshmikantham [20] introduced the notion of coupled fixed point and proved some interesting coupled fixed point theorems for mappings satisfying the mixed monotone property. Later, Lakshmikantham and Ćirić [21] introduced the concept of mixed $g$-monotone mapping and proved coupled coincidence and coupled common fixed point theorems that extend theorems due to Bhaskar and Lakshmikantham [20].

In [22, 23], authors established coupled fixed point theorems in cone metric spaces. In 2011, Shatanawi [24] obtained some coupled fixed point results in $G$-metric spaces. Recently, in [25, 26] authors established some coupled fixed point and common coupled fixed point results in two $G$-metric spaces. Recently, coupled fixed point and common coupled fixed point problems have also been considered in partially ordered $G$-metric spaces (see [27–38]).

The aim of this article is to prove some new common coupled fixed point theorems for mappings defined on a set equipped with two generalized metrics.

First, we present some known definitions and propositions.

**Definition 1.2** [1] Let $(X, G)$ be a $G$-metric space, $\{x_n\} \subset X$ be a sequence. Then the sequence $\{x_n\}$ is called:

(i) a $G$-Cauchy sequence if, for any $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ (the set of natural numbers) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(ii) a $G$-convergent sequence if, for any $\varepsilon > 0$, there are an $x \in X$ and an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A $G$-metric space $(X, G)$ is said to be $G$-complete if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $X$. It is known that $\{x_n\}$ is $G$-convergent to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

**Proposition 1.3** [1] *Let $(X, G)$ be a $G$-metric space. Then the following are equivalent*:

(1) *$\{x_n\}$ is $G$-convergent to $x$.*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*

(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

**Proposition 1.4** [1] *Let $(X, G)$ be a $G$-metric space. Then, for any $x, y \in X$, we have $G(x, y, y) \leq 2G(y, x, x)$.*

**Definition 1.5** [20] An element $(x, y) \in X \times X$ is called:

($C_1$) a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$;

($C_2$) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$, and in this case, $(gx, gy)$ is called a coupled point of coincidence;

($C_3$) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

**Definition 1.6** [25] Mappings $F : X \times X \to X$ and $g : X \to X$ are called:

($W_1$) $w$-compatible if $gF(x, y) = F(gx, gy)$ whenever $F(x, y) = gx$ and $F(y, x) = gy$;

($W_2$) $w^*$-compatible if $gF(x, x) = F(gx, gx)$ whenever $F(x, x) = gx$.

Recently, Abbas, Khan and Nazir [25] extended some recent results of Abbas *et al.* [22] and Sabetghadam *et al.* [23] to the setting of two generalized metric spaces.

**Theorem 1.7** (see [25, Theorem 2.1]) *Let $G_1$ and $G_2$ be two $G$-metrics on $X$ such that $G_2(x, y, z) \leq G_1(x, y, z)$ for all $x, y, z \in X$, and let $F : X \times X \to X$, $g : X \to X$ be two mappings* *satisfying*

$$\begin{aligned} &G_1\big(F(x,y),F(u,v),F(s,t)\big) \\ &\quad \le a_1G_2(gx,gu,gs)+a_2G_2\big(F(x,y),gx,gx\big) \\ &\qquad + a_3G_2(gy,gv,gt)+a_4G_2\big(F(u,v),gu,gs\big) \\ &\qquad + a_5G_2\big(F(x,y),gu,gs\big)+a_6G_2\big(F(u,v),F(s,t),gx\big) \end{aligned} \tag{1.1}$$

*for all* $(x,y),(u,v),(s,t)\in X\times X$, *where* $a_i\ge 0$, *for* $i=1,2,\ldots,6$ *and* $a_1+a_3+a_5+2(a_2+a_4+a_6)<1$. *If* $F(X\times X)\subset g(X)$ *and* $g(X)$ *is a* $G_1$*-complete subspace of* $X$, *and* $F$ *and* $g$ *are* $w^*$*-compatible, then* $F$ *and* $g$ *have a unique common coupled fixed point.*

**Theorem 1.8** (see [25, Theorem 2.6]) *Let* $G_1$ *and* $G_2$ *be two G-metrics on* $X$ *such that* $G_2(x,y,z)\le G_1(x,y,z)$ *for all* $x,y,z\in X$, *and let* $F:X\times X\to X$, $g:X\to X$ *be two mappings satisfying*

$$\begin{aligned} &G_1\big(F(x,y),F(u,v),F(s,t)\big) \\ &\quad \le k\max\big\{G_2(gx,gu,gs),G_2(gy,gv,gt),G_2\big(F(x,y),gu,gs\big)\big\} \end{aligned} \tag{1.2}$$

*for all* $(x,y),(u,v),(s,t)\in X\times X$, *where* $0\le k<\frac{1}{2}$. *If* $F(X\times X)\subset g(X)$ *and* $g(X)$ *is a* $G_1$*-complete subspace of* $X$, *and* $F$ *and* $g$ *are* $w^*$*-compatible, then* $F$ *and* $g$ *have a unique common coupled fixed point.*

In this manuscript, we generalize, improve, enrich and extend the above coupled fixed point results. It is worth mentioning that our results do not rely on the continuity of mappings involved therein. We also state some examples to illustrate our results. This paper can be considered as a continuation of the remarkable works of Abbas *et al.* [22, 23] and Sabetghadam *et al.* [25].

## 2 Common coupled fixed points

We begin with an example to illustrate the weakness of Theorem 1.8 above.

**Example 2.1** Let $X=[0,1]$. Define $G_1,G_2:X\times X\times X\to[0,\infty)$ by

$$G_1(x,y,z)=|x-y|+|y-z|+|z-x| \quad\text{and}\quad G_2(x,y,z)=\frac{4}{5}\big(|x-y|+|y-z|+|z-x|\big)$$

for all $x,y,z\in X$. Then $(X,G_1)$ and $(X,G_2)$ are two $G$-metric spaces. Define a map $F:X\times X\to X$ by $F(x,y)=\frac{1}{16}x+\frac{5}{16}y$ and $gx=\frac{x}{2}$ for all $x,y\in X$. For $(x,y)=(u,v)=(2,0)$ and $(s,t)=(0,2)$, we have

$$\begin{aligned} G_1\big(F(x,y),F(u,v),F(s,t)\big) &= G_1\big(F(2,0),F(2,0),F(0,2)\big) \\ &= G_1\left(\frac{1}{8},\frac{1}{8},\frac{5}{8}\right) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned}
&\max\{G_2(gx,gu,gs),G_2(gy,gv,gt),G_2(F(x,y),gu,gs)\}\\
&\quad=\max\{G_2(g2,g2,g0),G_2(g0,g0,g2),G_2(F(2,0),g2,g0)\}\\
&\quad=\max\left\{G_2(1,1,0),G_2(0,0,1),G_2\left(\frac{1}{8},1,0\right)\right\}\\
&\quad=\frac{8}{5}.
\end{aligned}$$

Then it is easy to see that there is no $k\in[0,\frac{1}{2})$ such that

$$G_1(F(x,y),F(u,v),F(s,t))\le k\max\{G_2(gx,gu,gs),G_2(gy,gv,gt),G_2(F(x,y),gu,gs)\}$$

for all $(x,y),(u,v),(s,t)\in X\times X$. Thus, Theorem 1.8 cannot be applied to this example. However, it is easy to see that $(0,0)$ is the unique common coincidence point of $F$ and $g$. In fact, $F(0,0)=g(0)=0$.

Now we shall prove our main results.

**Theorem 2.2** *Let $G_1$ and $G_2$ be two G-metrics on $X$ such that $G_2(x,y,z)\le G_1(x,y,z)$ for all $x,y,z\in X$, and let $F:X\times X\to X$, $g:X\to X$ be two mappings satisfying*

$$\begin{aligned}
&G_1(F(x,y),F(u,v),F(s,t))\\
&\quad\le a_1G_2(gx,gu,gs)+a_2G_2(gy,gv,gt)+a_3G_2(F(x,y),gx,gx)\\
&\qquad+a_4G_2(F(u,v),gu,gu)+a_5G_2(F(s,t),gs,gs)+a_6G_2(F(x,y),gu,gs)\\
&\qquad+a_7G_2(F(u,v),gs,gx)+a_8G_2(F(s,t),gx,gu)+a_9G_2(F(x,y),gx,gu)\\
&\qquad+a_{10}G_2(F(u,v),gu,gs)+a_{11}G_2(F(s,t),gs,gx)+a_{12}G_2(F(x,y),F(u,v),gs)\\
&\qquad+a_{13}G_2(F(u,v),F(s,t),gx)+a_{14}G_2(F(s,t),F(x,y),gu)
\end{aligned}\tag{2.1}$$

*for all $(x,y),(u,v),(s,t)\in X\times X$, where $a_i\ge 0$, for $i=1,2,\dots,14$ and*

$$a_1+a_2+a_6+a_9+2(a_3+a_4+a_5+a_{10}+a_{12}+a_{13}+a_{14})+3(a_7+a_8+a_{11})<1.\tag{2.2}$$

*If $F(X\times X)\subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of $X$, and $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point.*

*Proof* Let $x_0,y_0\in X$. Since $F(X\times X)\subset g(X)$, we can choose $x_1,y_1\in X$ such that $gx_1=F(x_0,y_0)$ and $gy_1=F(y_0,x_0)$. Similarly, we can choose $x_2,y_2\in X$ such that $gx_2=F(x_1,y_1)$ and $gy_2=F(y_1,x_1)$. Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$gx_{n+1}=F(x_n,y_n)\quad\text{and}\quad gy_{n+1}=F(y_n,x_n),\quad\forall n\ge 0.\tag{2.3}$$

It follows from (2.1), (2.3), (G5) and Proposition 1.4 that

$$\begin{aligned}
&G_1(gx_n, gx_{n+1}, gx_{n+1}) \\
&\quad = G_1\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n)\big) \\
&\quad \le a_1 G_2(gx_{n-1}, gx_n, gx_n) + a_2 G_2(gy_{n-1}, gy_n, gy_n) \\
&\qquad + a_3 G_2\big(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_{n-1}\big) \\
&\qquad + a_4 G_2\big(F(x_n, y_n), gx_n, gx_n\big) + a_5 G_2\big(F(x_n, y_n), gx_n, gx_n\big) \\
&\qquad + a_6 G_2\big(F(x_{n-1}, y_{n-1}), gx_n, gx_n\big) \\
&\qquad + a_7 G_2\big(F(x_n, y_n), gx_n, gx_{n-1}\big) + a_8 G_2\big(F(x_n, y_n), gx_{n-1}, gx_n\big) \\
&\qquad + a_9 G_2\big(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_n\big) + a_{10} G_2\big(F(x_n, y_n), gx_n, gx_n\big) \\
&\qquad + a_{11} G_2\big(F(x_n, y_n), gx_n, gx_{n-1}\big) + a_{12} G_2\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n), gx_n\big) \\
&\qquad + a_{13} G_2\big(F(x_n, y_n), F(x_n, y_n), gx_{n-1}\big) + a_{14} G_2\big(F(x_n, y_n), F(x_{n-1}, y_{n-1}), gx_n\big) \\
&\quad = a_1 G_2(gx_{n-1}, gx_n, gx_n) + a_2 G_2(gy_{n-1}, gy_n, gy_n) + a_3 G_2(gx_n, gx_{n-1}, gx_{n-1}) \\
&\qquad + a_4 G_2(gx_{n+1}, gx_n, gx_n) + a_5 G_2(gx_{n+1}, gx_n, gx_n) + a_6 G_2(gx_n, gx_n, gx_n) \\
&\qquad + a_7 G_2(gx_{n+1}, gx_n, gx_{n-1}) + a_8 G_2(gx_{n+1}, gx_{n-1}, gx_n) + a_9 G_2(gx_n, gx_{n-1}, gx_n) \\
&\qquad + a_{10} G_2(gx_{n+1}, gx_n, gx_n) + a_{11} G_2(gx_{n+1}, gx_n, gx_{n-1}) + a_{12} G_2(gx_n, gx_{n+1}, gx_n) \\
&\qquad + a_{13} G_2(gx_{n+1}, gx_{n+1}, gx_{n-1}) + a_{14} G_2(gx_{n+1}, gx_n, gx_n) \\
&\quad = (a_1 + a_9) G_2(gx_{n-1}, gx_n, gx_n) + a_2 G_2(gy_{n-1}, gy_n, gy_n) + a_3 G_2(gx_n, gx_{n-1}, gx_{n-1}) \\
&\qquad + (a_4 + a_5 + a_{10} + a_{12} + a_{14}) G_2(gx_{n+1}, gx_n, gx_n) \\
&\qquad + (a_7 + a_8 + a_{11}) G_2(gx_{n-1}, gx_n, gx_{n+1}) \\
&\qquad + a_{13} G_2(gx_{n+1}, gx_{n+1}, gx_{n-1}) \\
&\quad \le (a_1 + a_9) G_2(gx_{n-1}, gx_n, gx_n) + a_2 G_2(gy_{n-1}, gy_n, gy_n) + 2a_3 G_2(gx_{n-1}, gx_n, gx_n) \\
&\qquad + 2(a_4 + a_5 + a_{10} + a_{12} + a_{14}) G_2(gx_n, gx_{n+1}, gx_{n+1}) \\
&\qquad + (a_7 + a_8 + a_{11})\big[G_2(gx_{n-1}, gx_n, gx_n) + G_2(gx_n, gx_n, gx_{n+1})\big] \\
&\qquad + a_{13}\big[G_2(gx_{n-1}, gx_n, gx_n) + G_2(gx_n, gx_{n+1}, gx_{n+1})\big] \\
&\quad \le (a_1 + 2a_3 + a_7 + a_8 + a_9 + a_{11} + a_{13}) G_2(gx_{n-1}, gx_n, gx_n) + a_2 G_2(gy_{n-1}, gy_n, gy_n) \\
&\qquad + \big[2(a_4 + a_5 + a_7 + a_8 + a_{10} + a_{11} + a_{12} + a_{14}) + a_{13}\big] G_1(gx_n, gx_{n+1}, gx_{n+1}),
\end{aligned}$$

which implies that

$$\begin{aligned}
&G_1(gx_n, gx_{n+1}, gx_{n+1}) \\
&\quad \le \frac{(a_1 + 2a_3 + a_7 + a_8 + a_9 + a_{11} + a_{13}) G_2(gx_{n-1}, gx_n, gx_n) + a_2 G_2(gy_{n-1}, gy_n, gy_n)}{1 - 2(a_4 + a_5 + a_7 + a_8 + a_{10} + a_{11} + a_{12} + a_{14}) - a_{13}}.
\end{aligned} \tag{2.4}$$

Similarly, we can prove that

$$\begin{aligned} &G_1(gy_n, gy_{n+1}, gy_{n+1}) \\ &\quad \le \frac{(a_1 + 2a_3 + a_7 + a_8 + a_9 + a_{11} + a_{13})G_2(gy_{n-1}, gy_n, gy_n) + a_2 G_2(gx_{n-1}, gx_n, gx_n)}{1 - 2(a_4 + a_5 + a_7 + a_8 + a_{10} + a_{11} + a_{12} + a_{14}) - a_{13}}. \end{aligned} \tag{2.5}$$

By combining (2.4) and (2.5), we obtain

$$\begin{aligned} &G_1(gx_n, gx_{n+1}, gx_{n+1}) + G_1(gy_n, gy_{n+1}, gy_{n+1}) \\ &\quad \le \lambda\big[G_2(gx_{n-1}, gx_n, gx_n) + G_2(gy_{n-1}, gy_n, gy_n)\big], \end{aligned} \tag{2.6}$$

where $\lambda = \frac{a_1+a_2+2a_3+a_7+a_8+a_9+a_{11}+a_{13}}{1-2(a_4+a_5+a_7+a_8+a_{10}+a_{11}+a_{12}+a_{14})-a_{13}}$. Obviously, $0 \le \lambda < 1$.

Repeating the above inequality (2.6) $n$ times, we get

$$\begin{aligned} &G_1(gx_n, gx_{n+1}, gx_{n+1}) + G_1(gy_n, gy_{n+1}, gy_{n+1}) \\ &\quad \le \lambda\big[G_2(gx_{n-1}, gx_n, gx_n) + G_2(gy_{n-1}, gy_n, gy_n)\big] \\ &\quad \le \lambda\big[G_1(gx_{n-1}, gx_n, gx_n) + G_1(gy_{n-1}, gy_n, gy_n)\big] \\ &\quad \le \lambda^2\big[G_2(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G_2(gy_{n-2}, gy_{n-1}, gy_{n-1})\big] \\ &\quad \le \lambda^2\big[G_1(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G_1(gy_{n-2}, gy_{n-1}, gy_{n-1})\big] \\ &\quad \le \cdots \le \lambda^n\big[G_2(gx_0, gx_1, gx_1) + G_2(gy_0, gy_1, gy_1)\big]. \end{aligned} \tag{2.7}$$

Next, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are $G$-Cauchy sequences in $g(X)$.

In fact, for each $n, m \in \mathbb{N}$, $m > n$, from (G5) and (2.7), we have

$$\begin{aligned} &G_1(gx_n, gx_m, gx_m) + G_1(gy_n, gy_m, gy_m) \\ &\quad \le G_1(gx_n, gx_{n+1}, gx_{n+1}) + G_1(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\ &\qquad + G_1(gy_n, gy_{n+1}, gy_{n+1}) + G_1(gy_{n+1}, gy_{n+2}, gy_{n+2}) \\ &\qquad + \cdots + G_1(gx_{m-2}, gx_{m-1}, gx_{m-1}) + G_1(gx_{m-1}, gx_m, gx_m) \\ &\qquad + G_1(gy_{m-2}, gy_{m-1}, gy_{m-1}) + G_1(gy_{m-1}, gy_m, gy_m) \\ &\quad \le \big[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\big]\big[G_2(gx_0, gx_1, gx_1) + G_2(gy_0, gy_1, gy_1)\big] \\ &\quad \le \frac{\lambda^n}{1-\lambda}\big[G_2(gx_0, gx_1, gx_1) + G_2(gy_0, gy_1, gy_1)\big], \end{aligned}$$

which implies that

$$\lim_{n,m\to\infty}\big[G_1(gx_n, gx_m, gx_m) + G_1(gy_n, gy_m, gy_m)\big] = 0,$$

and so

$$\lim_{n,m\to\infty} G_1(gx_n, gx_m, gx_m) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} G_1(gy_n, gy_m, gy_m) = 0.$$

Hence $\{gx_n\}$ and $\{gy_n\}$ are $G_1$-Cauchy sequences in $g(X)$. By $G_1$-completeness of $g(X)$, there exist $gx, gy \in g(X)$ such that $\{gx_n\}$ and $\{gy_n\}$ converge to $gx$ and $gy$, respectively.

Now we prove that $F(x,y) = gx$ and $F(y,x) = gy$. In fact, it follows from (G5) and (2.1) that

$$\begin{aligned}
&G_1\bigl(F(x,y),gx,gx\bigr)\\
&\quad\le G_1\bigl(F(x,y),gx_{n+1},gx_{n+1}\bigr)+G_1(gx_{n+1},gx,gx)\\
&\quad= G_1\bigl(F(x,y),F(x_n,y_n),F(x_n,y_n)\bigr)+G_1(gx_{n+1},gx,gx)\\
&\quad\le a_1G_2(gx,gx_n,gx_n)+a_2G_2(gy,gy_n,gy_n)+a_3G_2\bigl(F(x,y),gx,gx\bigr)\\
&\qquad+a_4G_2\bigl(F(x_n,y_n),gx_n,gx_n\bigr)+a_5G_2\bigl(F(x_n,y_n),gx_n,gx_n\bigr)+a_6G_2\bigl(F(x,y),gx_n,gx_n\bigr)\\
&\qquad+a_7G_2\bigl(F(x_n,y_n),gx_n,gx\bigr)+a_8G_2\bigl(F(x_n,y_n),gx,gx_n\bigr)+a_9G_2\bigl(F(x,y),gx,gx_n\bigr)\\
&\qquad+a_{10}G_2\bigl(F(x_n,y_n),gx_n,gx_n\bigr)+a_{11}G_2\bigl(F(x_n,y_n),gx_n,gx\bigr)\\
&\qquad+a_{12}G_2\bigl(F(x,y),F(x_n,y_n),gx_n\bigr)\\
&\qquad+a_{13}G_2\bigl(F(x_n,y_n),F(x_n,y_n),gx\bigr)+a_{14}G_2\bigl(F(x_n,y_n),F(x,y),gx_n\bigr)+G_1(gx_{n+1},gx,gx)\\
&\quad\le a_1G_1(gx,gx_n,gx_n)+a_2G_1(gy,gy_n,gy_n)+a_3G_1\bigl(F(x,y),gx,gx\bigr)\\
&\qquad+a_4G_1(gx_{n+1},gx_n,gx_n)+a_5G_1(gx_{n+1},gx_n,gx_n)+a_6G_1\bigl(F(x,y),gx_n,gx_n\bigr)\\
&\qquad+a_7G_1(gx_{n+1},gx_n,gx)+a_8G_1(gx_{n+1},gx,gx_n)+a_9G_1\bigl(F(x,y),gx,gx_n\bigr)\\
&\qquad+a_{10}G_1(gx_{n+1},gx_n,gx_n)+a_{11}G_1(gx_{n+1},gx_n,gx)+a_{12}G_1\bigl(F(x,y),gx_{n+1},gx_n\bigr)\\
&\qquad+a_{13}G_1(gx_{n+1},gx_{n+1},gx)+a_{14}G_1\bigl(gx_{n+1},F(x,y),gx_n\bigr)+G_1(gx_{n+1},gx,gx).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$G_1\bigl(F(x,y),gx,gx\bigr)\le(a_3+a_6+a_9+a_{12}+a_{14})G_1\bigl(F(x,y),gx,gx\bigr). \tag{2.8}$$

By (2.2) we have that $a_3 + a_6 + a_9 + a_{12} + a_{14} < 1$. Hence, it follows from (2.8) that $G_1(F(x,y),gx,gx) = 0$, and so $F(x,y) = gx$. In the same way, we can show that $F(y,x) = gy$. Hence, $(gx,gy)$ is a coupled point of coincidence of mappings $F$ and $g$.

Next we prove that $gx = gy$. In fact, from (2.1) we have

$$\begin{aligned}
&G_1(gx,gy,gy)\\
&\quad= G_1\bigl(F(x,y),F(y,x),F(y,x)\bigr)\\
&\quad\le a_1G_2(gx,gy,gy)+a_2G_2(gy,gx,gx)+a_3G_2\bigl(F(x,y),gx,gx\bigr)\\
&\qquad+a_4G_2\bigl(F(y,x),gy,gy\bigr)+a_5G_2\bigl(F(y,x),gy,gy\bigr)+a_6G_2\bigl(F(x,y),gy,gy\bigr)\\
&\qquad+a_7G_2\bigl(F(y,x),gy,gx\bigr)+a_8G_2\bigl(F(y,x),gx,gy\bigr)+a_9G_2\bigl(F(x,y),gx,gy\bigr)\\
&\qquad+a_{10}G_2\bigl(F(y,x),gy,gy\bigr)+a_{11}G_2\bigl(F(y,x),gy,gx\bigr)+a_{12}G_2\bigl(F(x,y),F(y,x),gy\bigr)\\
&\qquad+a_{13}G_2\bigl(F(y,x),F(y,x),gx\bigr)+a_{14}G_2\bigl(F(y,x),F(x,y),gy\bigr)\\
&\quad= a_1G_2(gx,gy,gy)+a_2G_2(gy,gx,gx)+a_3G_2(gx,gx,gx)\\
&\qquad+a_4G_2(gy,gy,gy)+a_5G_2(gy,gy,gy)+a_6G_2(gx,gy,gy)
\end{aligned}$$

$$\begin{aligned}
&+ a_7 G_2(gy, gy, gx) + a_8 G_2(gy, gx, gy) + a_9 G_2(gx, gx, gy) \\
&+ a_{10} G_2(gy, gy, gy) + a_{11} G_2(gy, gy, gx) + a_{12} G_2(gx, gy, gy) \\
&+ a_{13} G_2(gy, gy, gx) + a_{14} G_2(gy, gx, gy) \\
&= (a_1 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14}) G_2(gx, gy, gy) \\
&+ (a_2 + a_9) G_2(gy, gx, gx) \\
&\leq (a_1 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14}) G_1(gx, gy, gy) \\
&+ (a_2 + a_9) G_1(gy, gx, gx),
\end{aligned}$$

which implies that

$$G_1(gx, gy, gy) \leq \frac{a_2 + a_9}{1 - (a_1 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14})} G_1(gy, gx, gx). \tag{2.9}$$

In a similar way, we can show that

$$G_1(gy, gx, gx) \leq \frac{a_2 + a_9}{1 - (a_1 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14})} G_1(gx, gy, gy). \tag{2.10}$$

Since $\frac{a_2+a_9}{1-(a_1+a_6+a_7+a_8+a_{11}+a_{12}+a_{13}+a_{14})} < 1$, from (2.9) and (2.10), we must have $G_1(gx, gy, gy) = 0$ so that $gx = gy$. Thus, $(gx, gx)$ is a coupled point of coincidence of mappings $F$ and $g$.

Now, we claim that a coupled point of coincidence is unique. Suppose that there is another $x^* \in X$ such $(gx^*, gx^*)$ is a coupled point of coincidence of mappings $F$ and $g$, then by (2.1) we have

$$\begin{aligned}
&G_1(gx, gx^*, gx^*) \\
&= G_1(F(x,x), F(x^*,x^*), F(x^*,x^*)) \\
&\leq a_1 G_2(gx, gx^*, gx^*) + a_2 G_2(gx, gx^*, gx^*) + a_3 G_2(F(x,x), gx, gx) \\
&+ a_4 G_2(F(x^*,x^*), gx^*, gx^*) + a_5 G_2(F(x^*,x^*), gx^*, gx^*) + a_6 G_2(F(x,x), gx^*, gx^*) \\
&+ a_7 G_2(F(x^*,x^*), gx^*, gx) + a_8 G_2(F(x^*,x^*), gx, gx^*) + a_9 G_2(F(x,x), gx, gx^*) \\
&+ a_{10} G_2(F(x^*,x^*), gx^*, gx^*) + a_{11} G_2(F(x^*,x^*), gx^*, gx) \\
&+ a_{12} G_2(F(x,x), F(x^*,x^*), gx^*) \\
&+ a_{13} G_2(F(x^*,x^*), F(x^*,x^*), gx) + a_{14} G_2(F(x^*,x^*), F(x,x), gx^*) \\
&= a_1 G_2(gx, gx^*, gx^*) + a_2 G_2(gx, gx^*, gx^*) + a_3 G_2(gx, gx, gx) \\
&+ a_4 G_2(gx^*, gx^*, gx^*) + a_5 G_2(gx^*, gx^*, gx^*) + a_6 G_2(gx, gx^*, gx^*) \\
&+ a_7 G_2(gx^*, gx^*, gx) + a_8 G_2(gx^*, gx, gx^*) + a_9 G_2(gx, gx, gx^*) \\
&+ a_{10} G_2(gx^*, gx^*, gx^*) + a_{11} G_2(gx^*, gx^*, gx) + a_{12} G_2(gx, gx^*, gx^*) \\
&+ a_{13} G_2(gx^*, gx^*, gx) + a_{14} G_2(gx^*, gx, gx^*) \\
&= (a_1 + a_2 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14}) G_2(gx, gx^*, gx^*) \\
&+ a_9 G_2(gx, gx, gx^*)
\end{aligned}$$

$$\begin{aligned}&\le (a_1 + a_2 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14})G_1\bigl(gx, gx^*, gx^*\bigr)\\&\quad + a_9 G_1\bigl(gx, gx, gx^*\bigr),\end{aligned}$$

which implies that

$$\begin{aligned}&G_1\bigl(gx, gx^*, gx^*\bigr)\\&\quad\le \frac{a_9}{1-(a_1 + a_2 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14})}G_1\bigl(gx, gx, gx^*\bigr).\end{aligned}\tag{2.11}$$

In the same way, we can show that

$$\begin{aligned}&G_1\bigl(gx^*, gx, gx\bigr)\\&\quad\le \frac{a_9}{1-(a_1 + a_2 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14})}G_1\bigl(gx^*, gx^*, gx\bigr).\end{aligned}\tag{2.12}$$

Since $\frac{a_9}{1-(a_1+a_2+a_6+a_7+a_8+a_{11}+a_{12}+a_{13}+a_{14})} < 1$, from (2.11) and (2.12), we must have $G_1(gx, gx^*, gx^*) = 0$ so that $gx = gx^*$. Hence, $(gx, gx)$ is a unique coupled point of coincidence of mappings $F$ and $g$.

Now we show that $F$ and $g$ have a unique common coupled fixed point. For this, let $gx = u$. Then we have $u = gx = F(x, x)$. By $w^*$-compatibility of $F$ and $g$, we have

$$gu = g(gx) = gF(x, x) = F(gx, gx) = F(u, u).$$

Thus, $(gu, gu)$ is a coupled point of coincidence of $F$ and $g$. By the uniqueness of a coupled point of coincidence, we have $gu = gx$. Therefore, $u = gu = F(u, u)$.

To prove the uniqueness, let $u^* \in X$ with $u^* \neq u$ such that

$$u^* = gu^* = F\bigl(u^*, u^*\bigr) \quad\text{and}\quad u = gu = F(u, u).$$

By using (2.1), following the same arguments as in the proof of (2.11) and (2.12), we obtain

$$\begin{aligned}&G_1\bigl(gu, gu^*, gu^*\bigr)\\&\quad\le \frac{a_9}{1-(a_1 + a_2 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14})}G_1\bigl(gu, gu, gu^*\bigr)\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}&G_1\bigl(gu^*, gu, gu\bigr)\\&\quad\le \frac{a_9}{1-(a_1 + a_2 + a_6 + a_7 + a_8 + a_{11} + a_{12} + a_{13} + a_{14})}G_1\bigl(gu^*, gu^*, gu\bigr).\end{aligned}\tag{2.14}$$

Since $\frac{a_9}{1-(a_1+a_2+a_6+a_7+a_8+a_{11}+a_{12}+a_{13}+a_{14})} < 1$, from (2.13) and (2.14), we must have $G_1(gu, gu^*, gu^*) = 0$ so that $u = gu = gu^* = u^*$. Thus, $F$ and $g$ have a unique common coupled fixed point. This completes the proof of Theorem 2.1. □

**Remark 2.3** Theorem 2.2 improves and extends Theorem 2.1 of Abbas *et al.* [25], the contractive condition defined by (1.1) is replaced by the new contractive condition defined by (2.1). Theorem 2.1 also improves and extends Theorem 2.4, Corollaries 2.5-2.8 and Theorem 2.11 of Abbas *et al.* [22]

Now, we introduce an example to support Theorem 2.2.

**Example 2.4** Let $X = [0,1]$, and let two $G$-metrics $G_1$, $G_2$ on $X$ be given as

$$G_1(x,y,z) = |x-y| + |y-z| + |z-x| \quad \text{and}$$
$$G_2(x,y,z) = \frac{1}{2}\bigl[|x-y| + |y-z| + |z-x|\bigr]$$

for all $x,y,z \in X$. Define $F : X \times X \to X$ and $g : X \to X$ as

$$F(x,y) = \frac{x+y}{32} \quad \text{and} \quad gx = \frac{x}{2}$$

for all $z,y \in X$.

Now, for $(x,y),(u,v),(s,t) \in X \times X$, we have

$$\begin{aligned}
&G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr)\\
&\quad = G_1\left(\frac{x+y}{32},\frac{u+v}{32},\frac{s+t}{32}\right)\\
&\quad = \frac{1}{32}\bigl[\bigl|x+y-(u+v)\bigr| + \bigl|u+v-(s+t)\bigr| + \bigl|s+t-(x+y)\bigr|\bigr]\\
&\quad \le \frac{1}{32}\bigl[|x-u| + |y-v| + |u-s| + |v-t| + |s-x| + |t-y|\bigr]\\
&\quad = \frac{1}{8}\left\{\frac{1}{4}\bigl[|x-u| + |y-v| + |u-s|\bigr] + \frac{1}{4}\bigl[|v-t| + |s-x| + |t-y|\bigr]\right\}\\
&\quad = \frac{1}{8}G_2(gx,gu,gs) + \frac{1}{8}G_2(gy,gv,gt)\\
&\quad \le \frac{1}{8}G_2(gx,gu,gs) + \frac{1}{8}G_2(gy,gv,gt) + \frac{1}{112}G_2\bigl(F(x,y),gx,gx\bigr)\\
&\qquad + \frac{1}{112}G_2\bigl(F(u,v),gu,gu\bigr) + \frac{1}{112}G_2\bigl(F(s,t),gs,gs\bigr) + \frac{1}{16}G_2\bigl(F(x,y),gu,gs\bigr)\\
&\qquad + \frac{1}{72}G_2\bigl(F(u,v),gs,gx\bigr) + \frac{1}{72}G_2\bigl(F(s,t),gx,gu\bigr) + \frac{1}{16}G_2\bigl(F(x,y),gx,gu\bigr)\\
&\qquad + \frac{1}{112}G_2\bigl(F(u,v),gu,gs\bigr) + \frac{1}{72}G_2\bigl(F(s,t),gs,gx\bigr) + \frac{1}{112}G_2\bigl(F(x,y),F(u,v),gs\bigr)\\
&\qquad + \frac{1}{112}G_2\bigl(F(u,v),F(s,t),gx\bigr) + \frac{1}{112}G_2\bigl(F(s,t),F(x,y),gu\bigr)
\end{aligned}$$

for all $(x,y),(u,v),(w,z) \in X \times X$. Thus, (2.1) is satisfied with $a_1 = a_2 = \frac{1}{8}$, $a_3 = a_4 = a_5 = a_{10} = a_{12} = a_{13} = a_{14} = \frac{1}{112}$, $a_6 = a_9 = \frac{1}{16}$ and $a_7 = a_8 = a_{11} = \frac{1}{72}$, where

$$a_1 + a_2 + a_6 + a_9 + 2(a_3 + a_4 + a_5 + a_{10} + a_{12} + a_{13} + a_{14}) + 3(a_7 + a_8 + a_{11}) = \frac{23}{48} < 1.$$

It is obvious that $F$ is $w^*$-compatible with $g$. Hence, all the conditions of Theorem 2.2 are satisfied. Moreover, $(0,0)$ is the unique common coupled fixed point of $F$ and $g$.

In Theorem 2.2, take $\alpha_1 = a_1$, $\alpha_2 = a_2$, $\alpha_3 = a_6$, $\alpha_4 = a_3$, $\alpha_5 = a_{10}$, $\alpha_6 = a_{13}$ and $a_4 = a_5 = a_7 = a_8 = a_9 = a_{11} = a_{12} = a_{14} = 0$, to obtain Theorem 2.1 of Abbas *et al.* [25] as the following corollary.

**Corollary 2.5** *Let $G_1$ and $G_2$ be two G-metrics on $X$ such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x,y,z \in X$, and let $F : X \times X \to X$, $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} &G_1\big(F(x,y),F(u,v),F(s,t)\big) \\ &\quad \leq \alpha_1 G_2(gx,gu,gs) + \alpha_2 G_2(gy,gv,gt) + \alpha_3 G_2\big(F(x,y),gu,gs\big) \\ &\qquad + \alpha_4 G_2\big(F(x,y),gx,gx\big) + \alpha_5 G_2\big(F(u,v),gu,gs\big) + \alpha_6 G_2\big(F(u,v),F(s,t),gx\big) \end{aligned} \tag{2.15}$$

*for all $(x,y),(u,v),(s,t) \in X \times X$, where $\alpha_i \geq 0$, for $i = 1,2,\ldots,6$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5 + \alpha_6) < 1$. If $F(X \times X) \subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of $X$, and $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point.*

In Theorem 2.2, take $s = u$ and $t = v$ to obtain the following corollary, which extends and generalizes the corresponding results of [22, 23, 25].

**Corollary 2.6** *Let $G_1$ and $G_2$ be two G-metrics on $X$ such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x,y,z \in X$, and let $F : X \times X \to X$, $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} &G_1\big(F(x,y),F(u,v),F(u,v)\big) \\ &\quad \leq a_1 G_2(gx,gu,gu) + a_2 G_2(gy,gv,gv) + a_3 G_2\big(F(x,y),gx,gx\big) \\ &\qquad + a_4 G_2\big(F(u,v),gu,gu\big) + a_5 G_2\big(F(u,v),gu,gu\big) + a_6 G_2\big(F(x,y),gu,gu\big) \\ &\qquad + a_7 G_2\big(F(u,v),gu,gx\big) + a_8 G_2\big(F(u,v),gx,gu\big) + a_9 G_2\big(F(x,y),gx,gu\big) \\ &\qquad + a_{10} G_2\big(F(u,v),gu,gu\big) + a_{11} G_2\big(F(u,v),gu,gx\big) + a_{12} G_2\big(F(x,y),F(u,v),gu\big) \\ &\qquad + a_{13} G_2\big(F(u,v),F(u,v),gx\big) + a_{14} G_2\big(F(u,v),F(x,y),gu\big) \end{aligned} \tag{2.16}$$

*for all $(x,y),(u,v) \in X \times X$, where $a_i \geq 0$, for $i = 1,2,\ldots,14$ and*

$$a_1 + a_2 + a_6 + a_9 + 2(a_3 + a_4 + a_5 + a_{10} + a_{12} + a_{13} + a_{14}) + 3(a_7 + a_8 + a_{11}) < 1.$$

*If $F(X \times X) \subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of $X$, and $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point.*

If we take $\alpha = a_1$, $\beta = a_2$, $\gamma = a_6$ and $a_3 = a_4 = a_5 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = 0$ in Theorem 2.2, then the following corollary, which extends and generalizes the comparable results of [22, 23], is obtained.

**Corollary 2.7** *Let $G_1$ and $G_2$ be two G-metrics on $X$ such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x,y,z \in X$, and let $F : X \times X \to X$, $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} &G_1\big(F(x,y),F(u,v),F(s,t)\big) \\ &\quad \leq \alpha G_2(gx,gu,gs) + \beta G_2(gy,gv,gt) + \gamma G_2\big(F(x,y),gu,gs\big) \end{aligned} \tag{2.17}$$

*for all* $(x,y),(u,v),(s,t) \in X \times X$, *where* $\alpha,\beta,\gamma \geq 0$ *and* $\alpha+\beta+\gamma<1$. *If* $F(X\times X)\subset g(X)$ *and* $g(X)$ *is a* $G_1$*-complete subspace of* $X$, *and* $F$ *and* $g$ *are* $w^*$*-compatible, then* $F$ *and* $g$ *have a unique common coupled fixed point.*

If we take $\alpha = a_1$, $\beta = a_2$, $\gamma = a_6$, $\delta = a_9$ and $a_3 = a_4 = a_5 = a_7 = a_8 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = 0$ in Theorem 2.2, then the following corollary is obtained.

**Corollary 2.8** *Let* $G_1$ *and* $G_2$ *be two G-metrics on* $X$ *such that* $G_2(x,y,z) \leq G_1(x,y,z)$ *for all* $x,y,z \in X$, *and let* $F: X\times X \to X$, $g: X\to X$ *be two mappings satisfying*

$$\begin{aligned} G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr) &\leq \alpha G_2(gx,gu,gs) + \beta G_2(gy,gv,gt) \\ &\quad + \gamma G_2\bigl(F(x,y),gu,gs\bigr) + \delta G_2\bigl(F(x,y),gx,gu\bigr) \end{aligned} \tag{2.18}$$

*for all* $(x,y),(u,v),(s,t) \in X \times X$, *where* $\alpha,\beta,\gamma,\delta \geq 0$ *and* $\alpha+\beta+\gamma+\delta<1$. *If* $F(X\times X)\subset g(X)$ *and* $g(X)$ *is a* $G_1$*-complete subspace of* $X$, *and* $F$ *and* $g$ *are* $w^*$*-compatible, then* $F$ *and* $g$ *have a unique common coupled fixed point.*

If we take $\alpha_1 = a_3$, $\alpha_2 = a_4$, $\alpha_3 = a_5$, $\alpha_4 = a_{10}$, $\alpha_5 = a_{12}$, $\alpha_6 = a_{13}$, $\alpha_7 = a_{14}$ and $a_1 = a_2 = a_6 = a_7 = a_8 = a_9 = a_{11} = 0$ in Theorem 2.2, then the following corollary is obtained.

**Corollary 2.9** *Let* $G_1$ *and* $G_2$ *be two G-metrics on* $X$ *such that* $G_2(x,y,z) \leq G_1(x,y,z)$ *for all* $x,y,z \in X$, *and let* $F: X\times X \to X$, $g: X\to X$ *be two mappings satisfying*

$$\begin{aligned} &G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr) \\ &\quad \leq \alpha_1 G_2\bigl(F(x,y),gx,gx\bigr) + \alpha_2 G_2\bigl(F(u,v),gu,gu\bigr) + \alpha_3 G_2\bigl(F(s,t),gs,gs\bigr) \\ &\qquad + \alpha_4 G_2\bigl(F(u,v),gu,gs\bigr) + \alpha_5 G_2\bigl(F(x,y),F(u,v),gs\bigr) \\ &\qquad + \alpha_6 G_2\bigl(F(u,v),F(s,t),gx\bigr) + \alpha_7 G_2\bigl(F(s,t),F(x,y),gu\bigr) \end{aligned} \tag{2.19}$$

*for all* $(x,y),(u,v),(s,t) \in X \times X$, *where* $\alpha_i \geq 0$, *for* $i = 1,2,\ldots,7$ *and*

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+a_7<\frac{1}{2}.$$

*If* $F(X\times X)\subset g(X)$ *and* $g(X)$ *is a* $G_1$*-complete subspace of* $X$, *and* $F$ *and* $g$ *are* $w^*$*-compatible, then* $F$ *and* $g$ *have a unique common coupled fixed point.*

If we take $\alpha = a_7$, $\beta = a_8$, $\gamma = a_{11}$ and $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_9 = a_{10} = a_{12} = a_{13} = a_{14} = 0$ in Theorem 2.2, then the following corollary is obtained.

**Corollary 2.10** *Let* $G_1$ *and* $G_2$ *be two G-metrics on* $X$ *such that* $G_2(x,y,z) \leq G_1(x,y,z)$ *for all* $x,y,z \in X$, *and let* $F: X\times X \to X$, $g: X\to X$ *be two mappings satisfying*

$$\begin{aligned} &G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr) \\ &\quad \leq \alpha G_2\bigl(F(u,v),gs,gx\bigr) + \beta G_2\bigl(F(s,t),gx,gu\bigr) + \gamma G_2\bigl(F(s,t),gs,gx\bigr) \end{aligned} \tag{2.20}$$

*for all $(x,y),(u,v),(s,t) \in X \times X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+\gamma < \frac{1}{3}$. If $F(X \times X) \subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of $X$, and $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point.*

**Theorem 2.11** *Let $G_1$ and $G_2$ be two $G$-metrics on $X$ such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x,y,z \in X$, and let $F : X \times X \to X$, $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} &G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr) \\ &\quad \leq k \max \left\{ \begin{array}{c} G_2(gx,gu,gs), G_2(gy,gv,gt), G_2(F(x,y),gu,gs), \\ G_2(F(x,y),gx,gu), \frac{1}{2}G_2((F(x,y),gx,gx), \frac{1}{2}G_2(F(u,v),gu,gu), \\ \frac{1}{2}G_2(F(s,t),gs,gs), \frac{1}{2}G_2(F(u,v),gu,gs), \\ \frac{1}{2}G_2(F(x,y),F(u,v),gs), \frac{1}{2}G_2(F(s,t),F(x,y),gu) \end{array} \right\} \end{aligned} \tag{2.21}$$

*for all $(x,y),(u,v),(s,t) \in X \times X$, where $0 \leq k < 1$. If $F(X \times X) \subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of $X$, and $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point.*

*Proof* Let $x_0, y_0 \in X$. We choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$, this can be done in view of $F(X \times X) \subset g(X)$. Similarly, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1,y_1)$ and $gy_2 = F(y_1,x_1)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $gx_{n+1} = F(x_n,y_n)$ and $gy_{n+1} = F(y_n,x_n)$.

By using (2.21) and Proposition 1.4, we obtain

$$\begin{aligned} &G_1(gx_n,gx_{n+1},gx_{n+1}) \\ &\quad = G_1\bigl(F(x_{n-1},y_{n-1}),F(x_n,y_n),F(x_n,y_n)\bigr) \\ &\quad \leq k \max \left\{ \begin{array}{c} G_2(gx_{n-1},gx_n,gx_n), G_2(gy_{n-1},gy_n,gy_n), \\ G_2(F(x_{n-1},y_{n-1}),gx_n,gx_n), G_2(F(x_{n-1},y_{n-1}),gx_{n-1},gx_n), \\ \frac{1}{2}G_2(F(x_{n-1},y_{n-1}),gx_{n-1},gx_{n-1}), \frac{1}{2}G_2(F(x_n,y_n),gx_n,gx_n), \\ \frac{1}{2}G_2(F(x_n,y_n),gx_n,gx_n), \frac{1}{2}G_2(F(x_n,y_n),gx_n,gx_n), \\ \frac{1}{2}G_2(F(x_{n-1},y_{n-1}),F(x_n,y_n),gx_n), \frac{1}{2}G_2(F(x_n,y_n),F(x_{n-1},y_{n-1}),gx_n) \end{array} \right\} \\ &\quad = k \max \left\{ \begin{array}{c} G_2(gx_{n-1},gx_n,gx_n), G_2(gy_{n-1},gy_n,gy_n), \\ G_2(gx_n,gx_n,gx_n), G_2(gx_n,gx_{n-1},gx_n), \\ \frac{1}{2}G_2(gx_n,gx_{n-1},gx_{n-1}), \frac{1}{2}G_2(gx_{n+1},gx_n,gx_n), \\ \frac{1}{2}G_2(gx_{n+1},gx_n,gx_n), \frac{1}{2}G_2(gx_{n+1},gx_n,gx_n), \\ \frac{1}{2}G_2(gx_n,gx_{n+1},gx_n), \frac{1}{2}G_2(gx_{n+1},gx_n,gx_n) \end{array} \right\} \\ &\quad = k \max \left\{ \begin{array}{c} G_2(gx_{n-1},gx_n,gx_n), G_2(gy_{n-1},gy_n,gy_n), \\ \frac{1}{2}G_2(gx_n,gx_{n-1},gx_{n-1}), \frac{1}{2}G_2(gx_{n+1},gx_n,gx_n) \end{array} \right\} \\ &\quad \leq k \max \left\{ \begin{array}{c} G_2(gx_{n-1},gx_n,gx_n), G_2(gy_{n-1},gy_n,gy_n), \\ G_2(gx_{n-1},gx_n,gx_n), G_2(gx_n,gx_{n+1},gx_{n+1}) \end{array} \right\} \\ &\quad = k \max\bigl\{G_2(gx_{n-1},gx_n,gx_n), G_2(gy_{n-1},gy_n,gy_n), G_2(gx_n,gx_{n+1},gx_{n+1})\bigr\} \\ &\quad \leq k \max\bigl\{G_1(gx_{n-1},gx_n,gx_n), G_1(gy_{n-1},gy_n,gy_n), G_1(gx_n,gx_{n+1},gx_{n+1})\bigr\}. \end{aligned} \tag{2.22}$$

If

$$\begin{aligned}&\max\bigl\{G_1(gx_{n-1},gx_n,gx_n),G_1(gy_{n-1},gy_n,gy_n),G_1(gx_n,gx_{n+1},gx_{n+1})\bigr\}\\&\quad=G_1(gx_n,gx_{n+1},gx_{n+1}),\end{aligned}$$

then inequality (2.22) becomes

$$G_1(gx_n,gx_{n+1},gx_{n+1})\le kG_1(gx_n,gx_{n+1},gx_{n+1}),$$

which is a contradiction. So that

$$\begin{aligned}&\max\bigl\{G_1(gx_{n-1},gx_n,gx_n),G_1(gy_{n-1},gy_n,gy_n),G_1(gx_n,gx_{n+1},gx_{n+1})\bigr\}\\&\quad=\max\bigl\{G_1(gx_{n-1},gx_n,gx_n),G_1(gy_{n-1},gy_n,gy_n)\bigr\}.\end{aligned}$$

This implies that

$$G_1(gx_n,gx_{n+1},gx_{n+1})\le k\max\bigl\{G_1(gx_{n-1},gx_n,gx_n),G_1(gy_{n-1},gy_n,gy_n)\bigr\}.\tag{2.23}$$

In a similar way, we obtain

$$G_1(gy_n,gy_{n+1},gy_{n+1})\le k\max\bigl\{G_1(gy_{n-1},gy_n,gy_n),G_1(gx_{n-1},gx_n,gx_n)\bigr\}.\tag{2.24}$$

Repeating inequalities (2.23) and (2.24), we obtain

$$\begin{aligned}G_1(gx_n,gx_{n+1},gx_{n+1})&\le k\max\bigl\{G_1(gx_{n-1},gx_n,gx_n),G_1(gy_{n-1},gy_n,gy_n)\bigr\}\\&\le k^2\max\bigl\{G_1(gx_{n-2},gx_{n-1},gx_{n-1}),G_1(gy_{n-2},gy_{n-1},gy_{n-1})\bigr\}\\&\le k^3\max\bigl\{G_1(gx_{n-3},gx_{n-2},gx_{n-2}),G_1(gy_{n-3},gy_{n-2},gy_{n-2})\bigr\}\\&\le\cdots\\&\le k^n\max\bigl\{G_1(gx_0,gx_1,gx_1),G_1(gy_0,gy_1,gy_1)\bigr\}\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}G_1(gy_n,gy_{n+1},gy_{n+1})&\le k\max\bigl\{G_1(gy_{n-1},gy_n,gy_n),G_1(gx_{n-1},gx_n,gx_n)\bigr\}\\&\le k^2\max\bigl\{G_1(gy_{n-2},gy_{n-1},gy_{n-1}),G_1(gx_{n-2},gx_{n-1},gx_{n-1})\bigr\}\\&\le k^3\max\bigl\{G_1(gy_{n-3},gy_{n-2},gy_{n-2}),G_1(gx_{n-3},gx_{n-2},gx_{n-2})\bigr\}\\&\le\cdots\\&\le k^n\max\bigl\{G_1(gy_0,gy_1,gy_1),G_1(gx_0,gx_1,gx_1)\bigr\}.\end{aligned}\tag{2.26}$$

By virtue of inequalities (2.25) and (2.26), for each $m,n\in\mathbb{N}$, $m>n$, repeated use (G5) of a $G$-metric gives

$$\begin{aligned}G_1(gx_n,gx_m,gx_m)&\le G_1(gx_n,gx_{n+1},gx_{n+1})+G_1(gx_{n+1},gx_{n+2},gx_{n+2})\\&\quad+\cdots+G_1(gx_{m-2},gx_{m-1},gx_{m-1})+G_1(gx_{m-1},gx_m,gx_m)\end{aligned}$$

$$
\begin{aligned}
&\le \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) \max\left\{G_1(gx_0, gx_1, gx_1), G_1(gy_0, gy_1, gy_1)\right\} \\
&\le \frac{k^n}{1-k} \max\left\{G_1(gx_0, gx_1, gx_1), G_1(gy_0, gy_1, gy_1)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
G_1(gy_n, gy_m, gy_m) &\le G_1(gy_n, gy_{n+1}, gy_{n+1}) + G_1(gy_{n+1}, gy_{n+2}, gy_{n+2}) \\
&\quad + \cdots + G_1(gy_{m-2}, gy_{m-1}, gy_{m-1}) + G_1(gy_{m-1}, gy_m, gy_m) \\
&\le \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) \max\left\{G_1(gx_0, gx_1, gx_1), G_1(gy_0, gy_1, gy_1)\right\} \\
&\le \frac{k^n}{1-k} \max\left\{G_1(gx_0, gx_1, gx_1), G_1(gy_0, gy_1, gy_1)\right\},
\end{aligned}
$$

which implies that

$$
\lim_{n,m\to\infty} G_1(gx_n, gx_m, gx_m) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} G_1(gy_n, gy_m, gy_m) = 0.
$$

Hence $\{gx_n\}$ and $\{gy_n\}$ are $G_1$-Cauchy sequences in $g(X)$. By $G_1$-completeness of $g(X)$, there exist $gx, gy \in g(X)$ such that $\{gx_n\}$ and $\{gy_n\}$ converge to $gx$ and $gy$, respectively.

Now, we prove that $F(x,y) = gx$ and $F(y,x) = gy$. For this, using (G5) and (2.21), we have

$$
\begin{aligned}
&G_1\left(F(x,y), gx, gx\right) \\
&\quad \le G_1\left(F(x,y), gx_{n+1}, gx_{n+1}\right) + G_1(gx_{n+1}, gx, gx) \\
&\quad = G_1\left(F(x,y), F(x_n,y_n), F(x_n,y_n)\right) + G_1(gx_{n+1}, gx, gx) \\
&\quad \le k \max\left\{\begin{array}{c}
G_2(gx, gx_n, gx_n), G_2(gy, gy_n, gy_n), \\
G_2(F(x,y), gx_n, gx_n), G_2(F(x,y), gx, gx_n), \\
\frac{1}{2}G_2(F(x,y), gx, gx), \frac{1}{2}G_2(F(x_n,y_n), gx_n, gx_n), \\
\frac{1}{2}G_2(F(x_n,y_n), gx_n, gx_n), \frac{1}{2}G_2(F(x_n,y_n), gx_n, gx_n), \\
\frac{1}{2}G_2(F(x,y), F(x_n,y_n), gx_n), \frac{1}{2}G_2(F(x_n,y_n), F(x,y), gx_n)
\end{array}\right\} \\
&\qquad + G_1(gx_{n+1}, gx, gx) \\
&\quad = k \max\left\{\begin{array}{c}
G_2(gx, gx_n, gx_n), G_2(gy, gy_n, gy_n), \\
G_2(F(x,y), gx_n, gx_n), G_2(F(x,y), gx, gx_n), \\
\frac{1}{2}G_2(F(x,y), gx, gx), \frac{1}{2}G_2(gx_{n+1}, gx_n, gx_n), \\
\frac{1}{2}G_2(gx_{n+1}, gx_n, gx_n), \frac{1}{2}G_2(gx_{n+1}, gx_n, gx_n), \\
\frac{1}{2}G_2(F(x,y), gx_{n+1}, gx_n), \frac{1}{2}G_2(gx_{n+1}, F(x,y), gx_n)
\end{array}\right\} \\
&\qquad + G_1(gx_{n+1}, gx, gx) \\
&\quad = k \max\left\{\begin{array}{c}
G_2(gx, gx_n, gx_n), G_2(gy, gy_n, gy_n), \\
G_2(F(x,y), gx_n, gx_n), G_2(F(x,y), gx, gx_n), \\
\frac{1}{2}G_2(F(x,y), gx, gx), \frac{1}{2}G_2(gx_{n+1}, gx_n, gx_n), \\
\frac{1}{2}G_2(F(x,y), gx_{n+1}, gx_n), \frac{1}{2}G_2(gx_{n+1}, F(x,y), gx_n)
\end{array}\right\} \\
&\qquad + G_1(gx_{n+1}, gx, gx).
\end{aligned}
\tag{2.27}
$$

On taking the limit as $n \to \infty$, we obtain that

$$G_1\bigl(F(x,y),gx,gx\bigr) \le kG_2\bigl(F(x,y),gx,gx\bigr) \le kG_1\bigl(F(x,y),gx,gx\bigr), \tag{2.28}$$

which implies that $G_1(F(x,y),gx,gx) = 0$, and so $F(x,y) = gx$. In a similar way, we can show that $F(y,x) = gy$. Hence, $(gx,gy)$ is a coupled point of coincidence of the mappings $F$ and $g$.

Now, we shall show that $gx = gy$. In fact, from (2.21) we have

$$\begin{aligned} &G_1(gx,gy,gy) \\ &\quad = G_1\bigl(F(x,y),F(y,x),F(y,x)\bigr) \\ &\quad \le k\max\left\{\begin{array}{c} G_2(gx,gy,gy), G_2(gy,gx,gx), G_2(F(x,y),gy,gy), G_2(F(x,y),gx,gy), \\ \frac{1}{2}G_2((F(x,y),gx,gx), \frac{1}{2}G_2(F(y,x),gy,gy), \frac{1}{2}G_2(F(y,x),gy,gy), \\ \frac{1}{2}G_2(F(y,x),gy,gy), \frac{1}{2}G_2(F(x,y),F(y,x),gy), \frac{1}{2}G_2(F(y,x),F(x,y),gy) \end{array}\right\} \\ &\quad = k\max\bigl\{G_2(gx,gy,gy), G_2(gy,gx,gx)\bigr\} \\ &\quad \le k\max\bigl\{G_1(gx,gy,gy), G_1(gy,gx,gx)\bigr\}. \end{aligned} \tag{2.29}$$

In the same way, we can show that

$$G_1(gy,gx,gx) \le k\bigl\{G_1(gy,gx,gx), G_1(gx,gy,gy)\bigr\}. \tag{2.30}$$

If

$$\max\bigl\{G_1(gx,gy,gy), G_1(gy,gx,gx)\bigr\} = G_1(gx,gy,gy),$$

then by (2.29) we have $G_1(gx,gy,gy) \le kG_1(gx,gy,gy)$. This implies that $G_1(gx,gy,gy) = 0$, so that $gx = gy$. If

$$\max\bigl\{G_1(gx,gy,gy), G_1(gy,gx,gx)\bigr\} = G_1(gy,gx,gx),$$

then from (2.30) we obtain $G_1(gy,gx,gx) \le kG_1(gy,gx,gx)$, which implies that $G_1(gy,gx,gx) = 0$, so that $gx = gy$.

Therefore, $(gx,gx)$ is a coupled point of coincidence of mappings $F$ and $g$.

If there is another $x^* \in X$ such that $(gx^*,gx^*)$ is a coupled point of coincidence of mappings $F$ and $g$, then by (2.21) we get

$$\begin{aligned} &G_1\bigl(gx,gx^*,gx^*\bigr) \\ &\quad = G_1\bigl(F(x,x),F\bigl(x^*,x^*\bigr),F\bigl(x^*,x^*\bigr)\bigr) \\ &\quad \le k\max\left\{\begin{array}{c} G_2(gx,gx^*,gx^*), G_2(gx,gx^*,gx^*), G_2(F(x,x),gx^*,gx^*), \\ G_2(F(x,x),gx,gx^*), \frac{1}{2}G_2((F(x,x),gx,gx), \frac{1}{2}G_2(F(x^*,x^*),gx^*,gx^*), \\ \frac{1}{2}G_2(F(x^*,x^*),gx^*,gx^*), \frac{1}{2}G_2(F(x^*,x^*),gx^*,gx^*), \\ \frac{1}{2}G_2(F(x,x),F(x^*,x^*),gx^*), \frac{1}{2}G_2(F(x^*,x^*),F(x,x),gx^*) \end{array}\right\} \\ &\quad = k\max\bigl\{G_2\bigl(gx,gx^*,gx^*\bigr), G_2\bigl(gx^*,gx,gx\bigr)\bigr\} \\ &\quad \le k\max\bigl\{G_1\bigl(gx,gx^*,gx^*\bigr), G_1\bigl(gx^*,gx,gx\bigr)\bigr\}. \end{aligned} \tag{2.31}$$

In the same way, we can show that

$$G_1(gx^*, gx, gx) \le k\{G_1(gx^*, gx, gx), G_1(gx, gx^*, gx^*)\}. \tag{2.32}$$

If

$$\max\{G_1(gx, gx^*, gx^*), G_1(gx^*, gx, gx)\} = G_1(gx, gx^*, gx^*),$$

then by (2.31) we have $G_1(gx, gx^*, gx^*) \le kG_1(gx, gx^*, gx^*)$. This implies that $G_1(gx, gx^*, gx^*) = 0$, so that $gx = gx^*$. If

$$\max\{G_1(gx, gx^*, gx^*), G_1(gx^*, gx, gx)\} = G_1(gx^*, gx, gx),$$

then from (2.32) we obtain $G_1(gx^*, gx, gx) \le kG_1(gx^*, gx, gx)$, which implies that $G_1(gx^*, gx, gx) = 0$, so that $gx = gx^*$.

Thus, $(gx, gx)$ is a unique coupled point of coincidence of mappings $F$ and $g$.

Now we show that $F$ and $g$ have a unique common coupled fixed point. For this, let $gx = u$. Then we have $u = gx = F(x, x)$. By $w^*$-compatibility of $F$ and $g$, we have

$$gu = g(gx) = gF(x, x) = F(gx, gx) = F(u, u).$$

Thus, $(gu, gu)$ is a coupled point of coincidence of $F$ and $g$. By the uniqueness of a coupled point of coincidence, we have $gu = gx$. Therefore, $u = gu = F(u, u)$, that is, $(u, u)$ is the common coupled fixed point of $F$ and $g$.

To prove the uniqueness, let $v \in X$ with $v \neq u$ such that

$$v = gv = F(v, v) \quad \text{and} \quad u = gu = F(u, u).$$

By using (2.21), following the same arguments as in the proof of (2.31) and (2.32), we obtain

$$\begin{aligned} G_1(u, v, v) = G_1(gu, gv, gv) &\le k\{G_1(gu, gv, gv), G_1(gv, gu, gu)\} \\ &= k\{G_1(u, v, v), G_1(v, u, u)\} \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} G_1(v, u, u) = G_1(gv, gu, gu) &\le k\{G_1(gv, gu, gu), G_1(gu, gv, gxv)\} \\ &= k\{G_1(v, u, u), G_1(u, v, v)\}. \end{aligned} \tag{2.34}$$

If $\max\{G_1(u, v, v), G_1(v, u, u)\} = G_1(u, v, v)$, then by (2.33) we have $G_1(u, v, v) \le kG_1(u, v, v)$, which implies that $G_1(u, v, v) = 0$, so that $u = v$. If $\max\{G_1(u, v, v), G_1(v, u, u)\} = G_1(v, u, u)$, then from (2.34) we obtain $G_1(v, u, u) \le kG_1(v, u, u)$, which implies that $G_1(v, u, u) = 0$, so that $u = v$.

Thus, $(u, u)$ is a unique common coupled fixed point of mappings $F$ and $g$. This completes the proof of Theorem 2.11. □

**Remark 2.12** Theorem 2.11 improves and extends Theorem 2.6 of Abbas *et al.* [25] in the following aspects:

(1) The contractive condition defined by (1.2) is replaced by the new contractive condition defined by (2.21).
(2) The condition $0 \leq k < \frac{1}{2}$ is replaced by the new condition $0 \leq k < 1$.

**Corollary 2.13** *Let $G_1$ and $G_2$ be two G-metrics on $X$ such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x,y,z \in X$, and let $F : X \times X \to X$, $g : X \to X$ be two mappings satisfying*

$$G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr) \\ \leq k \max\bigl\{ G_2(gx,gu,gs), G_2(gy,gv,gt), G_2\bigl(F(x,y),gu,gs\bigr) \bigr\} \tag{2.35}$$

*for all $(x,y),(u,v),(s,t) \in X \times X$, where $0 \leq k < 1$. If $F(X \times X) \subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of $X$, and $F$ and $g$ are $w^*$-compatible, then $F$ and $g$ have a unique common coupled fixed point.*

**Remark 2.14** Corollary 2.13 improves and extends Theorem 2.6 of Abbas *et al.* [25], the condition $0 \leq k < \frac{1}{2}$ is replaced by the new condition $0 \leq k < 1$.

Next, we introduce two examples to support Corollary 2.13.

**Example 2.15** Let us reconsider Example 2.1. For all $(x,y),(u,v),(s,t) \in X \times X$, we have

$$\begin{aligned} &G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr) \\ &\quad = G_1\left(\frac{1}{16}x + \frac{5}{16}y, \frac{1}{16}u + \frac{5}{16}v, \frac{1}{16}s + \frac{5}{16}t\right) \\ &\quad \leq \frac{1}{16}\bigl(|x-u| + |u-s| + |s-x|\bigr) + \frac{5}{16}\bigl(|y-v| + |v-t| + |t-y|\bigr) \\ &\quad = \frac{5}{32} \cdot \frac{4}{5}\bigl(|gx-gu| + |gu-gs| + |gs-gx|\bigr) + \frac{25}{32} \cdot \frac{4}{5}\bigl(|gy-gv| + |gv-gt| + |gt-gy|\bigr) \\ &\quad = \frac{5}{32}G_2(gx,gu,gs) + \frac{25}{32}(gy,gv,gt) \\ &\quad \leq \left(\frac{5}{32} + \frac{25}{32}\right)\max\bigl\{G_2(gx,gu,gs),(gy,gv,gt)\bigr\} \\ &\quad \leq \frac{15}{16}\max\bigl\{G_2(gx,gu,gs),G_2(gy,gv,gt),G_2\bigl(F(x,y),gu,gs\bigr)\bigr\}. \end{aligned}$$

Then the statement (2.35) of Corollary 2.13 is satisfied for $k = \frac{15}{16}$. Other assumptions of Corollary 2.13 are easy to verify. So, by Corollary 2.13, there exists a unique $x \in X$ such that $gx = F(x,x) = x$. In fact, it is easy to see that $(0,0)$ is the unique common coupled fixed point of $F$ and $g$.

**Example 2.16** Let $X = [0,1]$. Define $G_1, G_2 : X \times X \times X \to [0,\infty)$ by

$$G_1(x,y,z) = |x-y| + |y-z| + |z-x| \quad \text{and} \quad G_2(x,y,z) = \frac{4}{5}\bigl(|x-y| + |y-z| + |z-x|\bigr)$$

for all $x, y, z \in X$. Then $(X, G_1)$ and $(X, G_2)$ are two $G$-metric spaces. Define a map $F : X \times X \to X$ by $F(x,y) = 1 - \frac{1}{8}x - \frac{5}{8}y$ and $gx = x$ for all $x, y \in X$. We have

$$\begin{aligned} &G_1\bigl(F(x,y), F(u,v), F(s,t)\bigr) \\ &\quad = G_1\left(1 - \frac{1}{8}x - \frac{5}{8}y, 1 - \frac{1}{8}u - \frac{5}{8}v, 1 - \frac{1}{8}s - \frac{5}{8}t\right) \\ &\quad \le \frac{1}{8}\bigl(|x-u| + |u-s| + |s-x|\bigr) + \frac{5}{8}\bigl(|y-v| + |v-t| + |t-y|\bigr) \\ &\quad = \frac{5}{32}G_2(gx, gu, gs) + \frac{25}{32}(gy, gv, gt) \\ &\quad \le \left(\frac{5}{32} + \frac{25}{32}\right)\max\bigl\{G_2(gx, gu, gs), (gy, gv, gt)\bigr\} \\ &\quad \le \frac{15}{16}\max\bigl\{G_2(gx, gu, gs), G_2(gy, gv, gt), G_2\bigl(F(x,y), gu, gs\bigr)\bigr\}. \end{aligned}$$

Then the statement (2.35) of Corollary 2.13 is satisfied for $k = \frac{15}{16}$. Other assumptions of Corollary 2.13 are easy to verify. So, by Corollary 2.13, there exists a unique $x \in X$ such that $gx = F(x,x) = x$. In fact, $g(\frac{4}{7}) = F(\frac{4}{7}, \frac{4}{7}) = \frac{4}{7}$.

**Remark 2.17** Theorem 1.8 cannot be applied to Example 2.16 since (1.2) does not hold. In fact, if (1.2) holds for some $k \in [0, \frac{1}{2})$, then

$$\begin{aligned} 1 &= G_1\left(\frac{3}{8}, \frac{7}{8}, \frac{7}{8}\right) \\ &= G_1\bigl(F(0,1), F(1,0), F(1,0)\bigr) \\ &\le k\max\bigl\{G_2(g0, g1, g1), G_2(g1, g0, g0), G_2\bigl(F(0,1), g1, g1\bigr)\bigr\} \\ &= k\max\left\{G_2(0,1,1), G_2(1,0,0), G_2\left(\frac{3}{8}, 1, 1\right)\right\} \\ &= k\max\left\{\frac{8}{5}, \frac{8}{5}, \frac{4}{5}\cdot\frac{5}{4}\right\} \\ &= \frac{8}{5}k < \frac{4}{5}, \end{aligned}$$

which is a contradiction.

**Corollary 2.18** *Let $G_1$ and $G_2$ be two G-metrics on X such that $G_2(x,y,z) \le G_1(x,y,z)$ for all $x, y, z \in X$, and let $F : X \times X \to X$, $g : X \to X$ be two mappings satisfying*

$$G_1\bigl(F(x,y), F(u,v), F(s,t)\bigr) \le k\max\left\{\begin{array}{c} G_2(gx, gu, gs), G_2(gy, gv, gt), \\ G_2(F(x,y), gu, gs), G_2(F(x,y), gx, gu) \end{array}\right\} \tag{2.36}$$

*for all $(x,y), (u,v), (s,t) \in X \times X$, where $0 \le k < 1$. If $F(X \times X) \subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of X, and F and g are $w^*$-compatible, then F and g have a unique common coupled fixed point.*

**Corollary 2.19** *Let $G_1$ and $G_2$ be two G-metrics on X such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x,y,z \in X$, and let $F: X \times X \to X$, $g: X \to X$ be two mappings satisfying*

$$G_1\bigl(F(x,y),F(u,v),F(s,t)\bigr) \leq k\max\left\{\begin{array}{c} G_2((F(x,y),gx,gx),G_2(F(u,v),gu,gu), \\ G_2(F(s,t),gs,gs),G_2(F(u,v),gu,gs), \\ G_2(F(x,y),F(u,v),gs),G_2(F(s,t),F(x,y),gu) \end{array}\right\} \tag{2.37}$$

*for all $(x,y),(u,v),(s,t) \in X \times X$, where $0 \leq k < \frac{1}{2}$. If $F(X \times X) \subset g(X)$ and $g(X)$ is a $G_1$-complete subspace of X, and F and g are $w^*$-compatible, then F and g have a unique common coupled fixed point.*

**Remark 2.20** Theorem 2.2 and Corollaries 2.5-2.10 improve and extend Theorems 2.2, 2.5, 2.6, Corollary 2.3, 2.7 and 2.8 of Sabetghadam *et al.* [23].

**Competing interests**
The author declares that they have no competing interests.

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