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An iterative approach to mixed equilibrium problems and fixed points problems

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Abstract

In the present paper, an iterative algorithm for solving mixed equilibrium problems and fixed points problems has been constructed. It is shown that under some mild conditions, the sequence generated by the presented algorithm converges strongly to the common solution of mixed equilibrium problems and fixed points problems. As an application, we can find the minimum norm element without involving projection.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. For a nonlinear mapping $A : C \to H$ and a bifunction $F : C \times C \to R$, the mixed equilibrium problem is to find $z \in C$ such that

$$F(z,y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by MEP. If $A = 0$, then (1.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of (1.2) is denoted by EP. If $F = 0$, then (1.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The solution set of (1.3) is denoted by VI. Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others. See, *e.g.*, [1–22].

For solving mixed equilibrium problem (1.1), Moudafi [9] introduced an iterative algorithm and proved a weak convergence theorem. Further, Takahashi and Takahashi [15] introduced the following iterative algorithm for finding an element of $F(S) \cap MEP$:

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + S(\alpha_n u + (1 - \beta_n) z_n) \end{cases} \tag{1.4}$$

for all $n \geq 0$, where $S : C \to C$ is a nonexpansive mapping. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to $z = \mathrm{Proj}_{F(S) \cap MEP}(u)$.

Recently, Yao and Shahzad [19] gave the following iteration process for nonexpansive mappings with perturbation: $x_1 \in C$ and

$$x_{n+1} = (1 - \beta_n) x_n + \beta_n \mathrm{Proj}_C \bigl(\alpha_n u_n + (1 - \alpha_n) T x_n \bigr), \quad n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and the sequence $\{u_n\}$ in H is a small perturbation for the n-step iteration satisfying $\|u_n\| \to 0$ as $n \to \infty$. In fact, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate.

Using the ideas in [19], Chuang *et al.* [4] introduced the following iteration process for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points for a quasi-nonexpansive mapping with perturbation: $q_1 \in H$ and

$$\begin{cases} x_n \in C \text{ such that } F(x_n, y) + \frac{1}{\lambda_n} \langle y - x_n, x_n - q_n \rangle \geq 0, \quad \forall y \in C, \\ q_{n+1} = \alpha_n u_n + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n) S x_n) \end{cases}$$

for all $n \geq 0$. They showed that the sequence $\{q_n\}$ converges strongly to $\mathrm{Proj}_{F(S) \cap EP}$.

Motivated and inspired by the above works, in the present paper, we construct an iterative algorithm for solving mixed equilibrium problems and fixed points problems. It is shown that under some mild conditions the sequence $\{x_n\}$ generated by the presented algorithm converges strongly to the common solution of mixed equilibrium problems and fixed points problems. As an application, we can find the minimum norm element without involving projection.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is clear that any α-inverse-strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. A mapping $S : C \to C$ is said to be *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. And a mapping $S : C \to C$ is said to be *strictly pseudo-contractive* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \bigl\| (I - S)x - (I - S)y \bigr\|^2, \quad \forall x, y \in C.$$

For such a case, we also say that S is a κ-strictly pseudo-contractive mapping.

Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x,x) = 0$ for all $x \in C$;
(H2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(H3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(H4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

We need the following lemmas for proving our main results.

Lemma 2.1 [7] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions* (H1)-(H4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if $T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then we have

(i) *T_r is single-valued and T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
(ii) *EP is closed and convex and $EP = F(T_r)$.*

Lemma 2.2 [19] *Let C, H, F and $T_r x$ be as in Lemma* 2.1. *Then we have*

$$\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.3 [19] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $r > 0$ be a constant. Then we have*

$$\big\|(I - rA)x - (I - rA)y\big\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.4 [23] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5 [24] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a λ-strict pseudo-contraction. Then we have*

(i) *$F(S) = \{x : Sx = x\}$ is closed convex;*
(ii) *$\kappa I + (1 - \kappa)S$ for $\kappa \in [\lambda, 1)$ is nonexpansive.*

Lemma 2.6 [25] *Let C be a nonempty closed and convex of a real Hilbert space H. Let $S : C \to C$ be a κ-strictly pseudo-contractive mapping. Then $I - S$ is demi-closed at* 0, *i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Sx_n \to 0$, then $x = Sx$.*

Lemma 2.7 [16] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n \gamma_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we prove our main results.

Theorem 3.1 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $F : C \times C \to R$ *be a bifunction satisfying conditions* (H1)-(H4). *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping and let* $S : C \to C$ *be a* κ*-strictly pseudo-contractive mapping. Suppose that* $F(S) \cap MEP \neq \emptyset$. *Let* $x_0 \in C$, $\{z_n\}$ *and* $\{x_n\}$ *be sequences in* C *generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - (\alpha_n u_n + (1-\alpha_n)x_n)\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\gamma z_n + (1-\beta_n)(1-\gamma)Sz_n, \end{cases} \tag{3.1}$$

for all $n \ge 0$, *where* $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy*

(r1) $\lim_{n\to\infty} u_n = u$ *for some* $u \in H$;

(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(r3) $0 < c \le \beta_n \le d < 1$ *and* $\gamma \in [\kappa, 1)$;

(r4) $a(1-\alpha_n) \le \lambda_n \le b(1-\alpha_n)$, *where* $[a,b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ *generated by* (3.1) *converges strongly to* $\mathrm{Proj}_{F(S)\cap MEP}(u)$.

Proof Note that z_n can be rewritten as $z_n = T_{\lambda_n}(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n A x_n)$ for each n. Take $z \in F(S) \cap MEP$. It is obvious that $z = T_{\lambda_n}(z - \lambda_n A z) = T_{\lambda_n}(\alpha_n z + (1-\alpha_n)(z - \frac{\lambda_n A z}{1-\alpha_n}))$ for all $n \ge 0$. By using the nonexpansivity of T_{λ_n} and the convexity of $\|\cdot\|$, we derive

$$\begin{aligned} &\|z_n - z\|^2 \\ &\quad = \bigl\| T_{\lambda_n}\bigl(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n A x_n\bigr) - T_{\lambda_n}(z - \lambda_n A z)\bigr\|^2 \\ &\quad = \left\| T_{\lambda_n}\left(\alpha_n u_n + (1-\alpha_n)\left(x_n - \frac{\lambda_n A x_n}{1-\alpha_n}\right)\right) - T_{\lambda_n}\left(\alpha_n z + (1-\alpha_n)\left(z - \frac{\lambda_n A z}{1-\alpha_n}\right)\right)\right\|^2 \\ &\quad \le \left\| \left(\alpha_n u_n + (1-\alpha_n)\left(x_n - \frac{\lambda_n A x_n}{1-\alpha_n}\right)\right) - \left(\alpha_n z + (1-\alpha_n)\left(z - \frac{\lambda_n A z}{1-\alpha_n}\right)\right)\right\|^2 \\ &\quad = \left\| (1-\alpha_n)\left(\left(x_n - \frac{\lambda_n A x_n}{1-\alpha_n}\right) - \left(z - \frac{\lambda_n A z}{1-\alpha_n}\right)\right) + \alpha_n(u_n - z)\right\|^2 \\ &\quad \le (1-\alpha_n)\left\| \left(x_n - \frac{\lambda_n A x_n}{1-\alpha_n}\right) - \left(z - \frac{\lambda_n A z}{1-\alpha_n}\right)\right\|^2 + \alpha_n \|u_n - z\|^2. \end{aligned}$$

Since A is α-inverse strongly monotone, we know from Lemma 2.3 that

$$\begin{aligned} &\left\| \left(x_n - \frac{\lambda_n A x_n}{1-\alpha_n}\right) - \left(z - \frac{\lambda_n A z}{1-\alpha_n}\right)\right\|^2 \\ &\quad \le \|x_n - z\|^2 + \frac{\lambda_n(\lambda_n - 2(1-\alpha_n)\alpha)}{(1-\alpha_n)^2}\|Ax_n - Az\|^2. \end{aligned}$$

It follows that

$$\begin{aligned}\|z_n - z\|^2 &\leq (1-\alpha_n)\left(\|x_n - z\|^2 + \frac{\lambda_n(\lambda_n - 2(1-\alpha_n)\alpha)}{(1-\alpha_n)^2}\|Ax_n - Az\|^2\right) + \alpha_n\|u_n - z\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2. \end{aligned} \tag{3.2}$$

So, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \left\|\beta_n(x_n - z) + (1-\beta_n)\big((\gamma I + (1-\gamma)S)z_n - z\big)\right\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|z_n - z\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\big((1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2\big) \\ &= \big[1 - (1-\beta_n)\alpha_n\big]\|x_n - z\|^2 + (1-\beta_n)\alpha_n\|u_n - z\|^2 \\ &\leq \max\big\{\|x_n - z\|^2, \|u_n - z\|^2\big\}.\end{aligned}$$

Since $\lim_{n\to\infty} u_n = u$, $\{u_n\}$ is bounded. Therefore, by induction, we deduce that $\{x_n\}$ is bounded. Hence, $\{Ax_n\}$, $\{z_n\}$ and $\{Sz_n\}$ are also bounded.

Putting $y_n = \alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n$ for all n, we have

$$z_{n+1} - z_n = T_{\lambda_{n+1}}y_{n+1} - T_{\lambda_{n+1}}y_n + T_{\lambda_{n+1}}y_n - T_{\lambda_n}y_n.$$

It follows that

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \|T_{\lambda_{n+1}}y_{n+1} - T_{\lambda_{n+1}}y_n\| + \|T_{\lambda_{n+1}}y_n - T_{\lambda_n}y_n\| \\ &\leq \|y_{n+1} - y_n\| + \|T_{\lambda_{n+1}}y_n - T_{\lambda_n}y_n\|.\end{aligned} \tag{3.3}$$

From Lemma 2.3, we know that $I - \lambda A$ is nonexpansive for all $\lambda \in (0, 2\alpha)$. Thus, we have $I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A$ is nonexpansive for all n due to the fact that $\frac{\lambda_{n+1}}{1-\alpha_{n+1}} \in (0, 2\alpha)$. Then we get

$$\begin{aligned}\|y_{n+1} - y_n\| &= \big\|\alpha_{n+1}u_{n+1} + (1-\alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1} - \big(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n\big)\big\| \\ &\leq \left\|(1-\alpha_{n+1})\left(x_{n+1} - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}Ax_{n+1}\right) - (1-\alpha_n)\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right)\right\| \\ &\quad + \alpha_{n+1}\|u_{n+1}\| + \alpha_n\|u_n\| \\ &\leq (1-\alpha_{n+1})\left\|\left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_{n+1} - \left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_n\right\| \\ &\quad + \left\|(1-\alpha_{n+1})\left(x_n - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}Ax_n\right) - (1-\alpha_n)\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right)\right\| \\ &\quad + \alpha_{n+1}\|u_{n+1}\| + \alpha_n\|u_n\| \\ &\leq \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + \alpha_{n+1}\|u_{n+1}\| + \alpha_n\|u_n\|.\end{aligned} \tag{3.4}$$

By Lemma 2.2, we have

$$\|T_{\lambda_{n+1}}y_n - T_{\lambda_n}y_n\| \leq \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}y_n - y_n\|. \tag{3.5}$$

From (3.3)-(3.5), we obtain

$$\begin{aligned}\|z_{n+1}-z_n\| \le{}& \|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\|x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ &+ \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}y_n - y_n\| + \alpha_{n+1}\|u_{n+1}\| + \alpha_n\|u_n\|.\end{aligned}$$

Then

$$\begin{aligned}&\big\|\big(\gamma I + (1-\gamma)S\big)z_{n+1} - \big(\gamma I + (1-\gamma)S\big)z_n\big\| \\ &\quad\le \|z_{n+1}-z_n\| \\ &\quad\le \|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\|x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ &\qquad+ \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}y_n - y_n\| + \alpha_{n+1}\|u_{n+1}\| + \alpha_n\|u_n\|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\big\|\big(\gamma I + (1-\gamma)S\big)z_{n+1} - \big(\gamma I + (1-\gamma)S\big)z_n\big\| - \|x_{n+1}-x_n\| \\ &\quad\le |\alpha_{n+1}-\alpha_n|\|x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| + \alpha_{n+1}\|u_{n+1}\| + \alpha_n\|u_n\| \\ &\qquad+ \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}y_n - y_n\|.\end{aligned}$$

Since $\alpha_n \to 0$, $\lambda_{n+1}-\lambda_n \to 0$ and $\liminf_{n\to\infty}\lambda_n > 0$, we obtain

$$\limsup_{n\to\infty}\big(\big\|\big(\gamma I + (1-\gamma)S\big)z_{n+1} - \big(\gamma I + (1-\gamma)S\big)z_n\big\| - \|x_{n+1}-x_n\|\big) \le 0.$$

This together with Lemma 2.4 implies that

$$\lim_{n\to\infty}\big\|\big(\gamma I + (1-\gamma)S\big)z_n - x_n\big\| = 0. \tag{3.6}$$

Consequently, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\big\|\big(\gamma I + (1-\gamma)S\big)z_n - x_n\big\| = 0.$$

From (3.1) and (3.2), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 \le{}& (1-\beta_n)\big\|\big(\gamma I + (1-\gamma)S\big)T_{\lambda_n}\big(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n\big) - z\big\|^2 \\ &+ \beta_n\|x_n - z\|^2 \\ \le{}& (1-\beta_n)\bigg\{(1-\alpha_n)\bigg(\|x_n-z\|^2 + \frac{\lambda_n}{(1-\alpha_n)^2}\big(\lambda_n - 2(1-\alpha_n)\alpha\big)\|Ax_n - Az\|^2\bigg) \\ &+ \alpha_n\|u_n - z\|^2\bigg\} + \beta_n\|x_n - z\|^2 \\ ={}& \big(1-(1-\beta_n)\alpha_n\big)\|x_n - z\|^2 + \frac{(1-\beta_n)\lambda_n}{1-\alpha_n}\big(\lambda_n - 2(1-\alpha_n)\alpha\big)\|Ax_n - Az\|^2 \\ &+ (1-\beta_n)\alpha_n\|u_n - z\|^2\end{aligned}$$

$$\begin{aligned}&\leq \|x_n - z\|^2 + \frac{(1-\beta_n)\lambda_n}{1-\alpha_n}\big(\lambda_n - 2(1-\alpha_n)\alpha\big)\|Ax_n - Az\|^2 \\ &\quad + (1-\beta_n)\alpha_n\|u_n - z\|^2.\end{aligned}$$

Then we obtain

$$\begin{aligned}&\frac{(1-\beta_n)\lambda_n}{1-\alpha_n}\big(2(1-\alpha_n)\alpha - \lambda_n\big)\|Ax_n - Az\|^2 \\ &\quad \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\beta_n)\alpha_n\|u_n - z\|^2 \\ &\quad \leq \big(\|x_n - z\| - \|x_{n+1} - z\|\big)\|x_{n+1} - x_n\| + (1-\beta_n)\alpha_n\|u_n - z\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\liminf_{n\to\infty}\frac{(1-\beta_n)\lambda_n}{1-\alpha_n}(2(1-\alpha_n)\alpha - \lambda_n) > 0$, we have

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.7}$$

Next, we show $\|x_n - z_n\| = \|x_n - T_{\lambda_n}y_n\| \to 0$. By using the firm nonexpansivity of T_{λ_n}, we have

$$\begin{aligned}\|T_{\lambda_n}y_n - z\|^2 &= \big\|T_{\lambda_n}y_n - T_{\lambda_n}(z - \lambda_n Az)\big\|^2 \\ &\leq \big\langle y_n - (z - \lambda_n Az), T_{\lambda_n}y_n - z\big\rangle \\ &= \frac{1}{2}\Big(\big\|y_n - (z - \lambda_n Az)\big\|^2 + \|T_{\lambda_n}y_n - z\|^2 \\ &\quad - \big\|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n(Ax_n - \lambda_n Az) - T_{\lambda_n}y_n\big\|^2\Big).\end{aligned}$$

We note that

$$\big\|y_n - (z - \lambda_n Az)\big\|^2 \leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2.$$

Thus,

$$\begin{aligned}\|T_{\lambda_n}y_n - z\|^2 &\leq \frac{1}{2}\Big((1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2 + \|T_{\lambda_n}y_n - z\|^2 \\ &\quad - \big\|\alpha_n u_n + (1-\alpha_n)x_n - T_{\lambda_n}y_n - \lambda_n(Ax_n - \lambda_n Az)\big\|^2\Big).\end{aligned}$$

That is,

$$\begin{aligned}\|T_{\lambda_n}y_n - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2 \\ &\quad - \big\|\alpha_n u_n + (1-\alpha_n)x_n - T_{\lambda_n}y_n - \lambda_n(Ax_n - \lambda_n Az)\big\|^2 \\ &= (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2 - \big\|\alpha_n u_n + (1-\alpha_n)x_n - T_{\lambda_n}y_n\big\|^2 \\ &\quad + 2\lambda_n\big\langle\alpha_n u_n + (1-\alpha_n)x_n - T_{\lambda_n}y_n, Ax_n - Az\big\rangle - \lambda_n^2\|Ax_n - Az\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2 - \big\|\alpha_n u_n + (1-\alpha_n)x_n - T_{\lambda_n}y_n\big\|^2 \\ &\quad + 2\lambda_n\big\|\alpha_n u_n + (1-\alpha_n)x_n - T_{\lambda_n}y_n\big\|\|Ax_n - Az\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \beta_n\|x_n-z\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-z\|^2+(1-\beta_n)\alpha_n\|u_n-z\|^2\\&\quad-(1-\beta_n)\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|^2\\&\quad+2\lambda_n(1-\beta_n)\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|\|Ax_n-Az\|\\&=\bigl(1-(1-\beta_n)\alpha_n\bigr)\|x_n-z\|^2+(1-\beta_n)\alpha_n\|u_n-z\|^2\\&\quad-(1-\beta_n)\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|^2\\&\quad+2\lambda_n(1-\beta_n)\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|\|Ax_n-Az\|.\end{aligned}$$

Hence,

$$\begin{aligned}&(1-\beta_n)\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|^2\\&\quad\le\|x_n-z\|^2-\|x_{n+1}-z\|^2-(1-\beta_n)\alpha_n\|x_n-z\|^2\\&\qquad+(1-\beta_n)\alpha_n\|u_n-z\|^2+2\lambda_n(1-\beta_n)\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|\|Ax_n-Az\|\\&\quad\le\bigl(\|x_n-z\|+\|x_{n+1}-z\|\bigr)\|x_{n+1}-x_n\|+(1-\beta_n)\alpha_n\|u_n-z\|^2\\&\qquad+2\lambda_n(1-\beta_n)\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|\|Ax_n-Az\|.\end{aligned}$$

Since $\limsup_{n\to\infty}\beta_n<1$, $\|x_{n+1}-x_n\|\to 0$, $\alpha_n\to 0$ and $\|Ax_n-Az\|\to 0$, we deduce

$$\lim_{n\to\infty}\bigl\|\alpha_nu_n+(1-\alpha_n)x_n-T_{\lambda_n}y_n\bigr\|=0.$$

This implies that

$$\lim_{n\to\infty}\|x_n-z_n\|=\|x_n-T_{\lambda_n}y_n\|=0. \tag{3.8}$$

Put $\tilde{x}=\mathrm{Proj}_{F(S)\cap MEP}(u)$. We will finally show that $x_n\to\tilde{x}$.

Setting $v_n=x_n-\frac{\lambda_n}{1-\alpha_n}(Ax_n-A\tilde{x})$ for all n. Taking $z=\tilde{x}$ in (3.7) to get $\|Ax_n-A\tilde{x}\|\to 0$. First, we prove $\limsup_{n\to\infty}\langle u-\tilde{x},v_n-\tilde{x}\rangle\le 0$. We take a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle u-\tilde{x},v_n-\tilde{x}\rangle=\lim_{i\to\infty}\langle u-\tilde{x},v_{n_i}-\tilde{x}\rangle.$$

It is clear that $\{v_{n_i}\}$ is bounded due to the boundedness of $\{x_n\}$ and $\|Ax_n-A\tilde{x}\|\to 0$. Then there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to some point $w\in C$. Hence, $\{x_{n_{i_j}}\}$ also converges weakly to w. At the same time, from (3.6) and (3.8), we have

$$\lim_{j\to\infty}\bigl\|x_{n_{i_j}}-\bigl(\gamma I+(1-\gamma)S\bigr)x_{n_{i_j}}\bigr\|=0. \tag{3.9}$$

By the demi-closedness principle (see Lemma 2.6) and (3.9), we deduce $w\in F(S)$.

Further, we show that w is also in MEP. From (3.1), we have

$$F(z_n,y)+\langle Ax_n,y-z_n\rangle+\frac{1}{\lambda_n}\bigl\langle y-z_n,z_n-\bigl(\alpha_nu_n+(1-\alpha_n)x_n\bigr)\bigr\rangle\ge 0.$$

From (H2), we have

$$\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\big\langle y - z_n, z_n - \big(\alpha_n u_n + (1-\alpha_n)x_n\big)\big\rangle \geq F(y, z_n). \tag{3.10}$$

Put $x_t = ty + (1-t)w$ for all $t \in (0, 1 - \frac{\lambda}{2\alpha})$ and $y \in C$. Then we have $x_t \in C$. So, from (3.10), we have

$$\begin{aligned}\langle x_t - z_n, Ax_t\rangle \geq{}& \langle x_t - z_n, Ax_t\rangle - \langle x_t - z_n, Ax_n\rangle \\ & - \frac{1}{\lambda_n}\big\langle x_t - z_n, z_n - \big(\alpha_n u_n + (1-\alpha_n)x_n\big)\big\rangle + F(x_t, z_n) \\ ={}& \langle x_t - z_n, Ax_t - Az_n\rangle + \langle x_t - z_n, Az_n - Ax_n\rangle \\ & - \frac{1}{\lambda_n}\big\langle x_t - z_n, z_n - \big(\alpha_n u_n + (1-\alpha_n)x_n\big)\big\rangle + F(x_t, z_n).\end{aligned}$$

Since $\|z_n - x_n\| \to 0$, we have $\|Az_n - Ax_n\| \to 0$. Further, from monotonicity of A, we have $\langle x_t - z_n, Ax_t - Az_n\rangle \geq 0$. So, from (H4), we have

$$\langle x_t - w, Ax_t\rangle \geq F(x_t, w), \quad \text{as } n \to \infty. \tag{3.11}$$

From (H1), (H4) and (3.11), we also have

$$\begin{aligned}0 &= F(x_t, x_t) \\ &\leq tF(x_t, y) + (1-t)F(x_t, w) \\ &\leq tF(x_t, y) + (1-t)\langle x_t - w, Ax_t\rangle \\ &= tF(x_t, y) + (1-t)t\langle y - w, Ax_t\rangle\end{aligned}$$

and hence

$$0 \leq F(x_t, y) + (1-t)\langle y - w, Ax_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq F(w, y) + \langle y - w, Aw\rangle.$$

This implies $w \in MEP$. Hence, we have $w \in F(S) \cap MEP$. This implies that

$$\limsup_{n\to\infty}\langle u - \tilde{x}, v_n - \tilde{x}\rangle = \lim_{j\to\infty}\langle u - \tilde{x}, v_{n_{i_j}} - \tilde{x}\rangle = \langle u - \tilde{x}, w - \tilde{x}\rangle.$$

Note that $\tilde{x} = \mathrm{Proj}_{F(S)\cap MEP}(u)$. Then $\langle u - \tilde{x}, w - \tilde{x}\rangle \leq 0$, $w \in F(S) \cap MEP$. Therefore,

$$\limsup_{n\to\infty}\langle u - \tilde{x}, v_n - \tilde{x}\rangle \leq 0.$$

Since $u_n \to u$, we have

$$\limsup_{n\to\infty}\langle u_n - \tilde{x}, v_n - \tilde{x}\rangle \leq 0.$$

From (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
&\quad \le \beta_n \|x_n - \tilde{x}\|^2 + (1-\beta_n)\big\|\big(\gamma I + (1-\gamma)S\big)T_{\lambda_n} y_n - \tilde{x}\big\|^2 \\
&\quad \le \beta_n \|x_n - \tilde{x}\|^2 + (1-\beta_n)\|T_{\lambda_n} y_n - \tilde{x}\|^2 \\
&\quad = \beta_n \|x_n - \tilde{x}\|^2 + (1-\beta_n)\big\|T_{\lambda_n} y_n - T_{\lambda_n}(\tilde{x} - \lambda_n A\tilde{x})\big\|^2 \\
&\quad \le \beta_n \|x_n - \tilde{x}\|^2 + (1-\beta_n)\big\|y_n - (\tilde{x} - \lambda_n A\tilde{x})\big\|^2 \\
&\quad = \beta_n \|x_n - \tilde{x}\|^2 + (1-\beta_n)\big\|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n A x_n - (\tilde{x} - \lambda_n A\tilde{x})\big\|^2 \\
&\quad = (1-\beta_n)\left\|(1-\alpha_n)\left(\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x}\right)\right) + \alpha_n(u_n - \tilde{x})\right\|^2 \\
&\qquad + \beta_n \|x_n - \tilde{x}\|^2 \\
&\quad = (1-\beta_n)\left((1-\alpha_n)^2\left\|\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x}\right)\right\|^2\right. \\
&\qquad + 2\alpha_n(1-\alpha_n)\left\langle u_n - \tilde{x}, \left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x}\right)\right\rangle \\
&\qquad \left. + \alpha_n^2\|u_n - \tilde{x}\|^2\right) + \beta_n\|x_n - \tilde{x}\|^2 \\
&\quad \le \beta_n \|x_n - \tilde{x}\|^2 + (1-\beta_n)\left((1-\alpha_n)^2\|x_n - \tilde{x}\|^2\right. \\
&\qquad \left. + 2\alpha_n(1-\alpha_n)\left\langle u_n - \tilde{x}, x_n - \frac{\lambda_n}{1-\alpha_n}(Ax_n - A\tilde{x}) - \tilde{x}\right\rangle + \alpha_n^2\|u_n - \tilde{x}\|^2\right) \\
&\quad \le \big(1 - (1-\beta_n)\alpha_n\big)\|x_n - \tilde{x}\|^2 \\
&\qquad + (1-\beta_n)\alpha_n\big\{2(1-\alpha_n)\langle u_n - \tilde{x}, v_n - \tilde{x}\rangle + \alpha_n\|u_n - \tilde{x}\|^2\big\}.
\end{aligned}$$

It is clear that $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$ and $\limsup_{n\to\infty}(2(1-\alpha_n)\langle u_n - \tilde{x}, v_n - \tilde{x}\rangle + \alpha_n\|u_n - \tilde{x}\|^2) \le 0$. We can therefore apply Lemma 2.7 to conclude that $x_n \to \tilde{x}$. This completes the proof. □

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions* (H1)-(H4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping. Suppose that $F(S) \cap MEP \neq \emptyset$. Let $x_0 \in C$, $\{z_n\}$ and $\{x_n\}$ be sequences in C generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - (\alpha_n u_n + (1-\alpha_n)x_n)\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\gamma z_n + (1-\beta_n)(1-\gamma)Sz_n \end{cases} \tag{3.12}$$

for all $n \ge 0$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(r1) $\lim_{n\to\infty} u_n = u$ *for some* $u \in H$;
(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(r3) $0 < c \le \beta_n \le d < 1$ *and* $\gamma \in (0,1)$;
(r4) $a(1-\alpha_n) \le \lambda_n \le b(1-\alpha_n)$, *where* $[a,b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.12) *converges strongly to* $\mathrm{Proj}_{F(S)\cap MEP}(u)$.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions* (H1)-(H4). *Let $S : C \to C$ be a κ-strictly pseudo-contractive mapping. Suppose that $F(S) \cap EP \neq \emptyset$. Let $x_0 \in C$, $\{z_n\}$ and $\{x_n\}$ be sequences in C generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (\alpha_n u_n + (1-\alpha_n)x_n)\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\gamma z_n + (1-\beta_n)(1-\gamma)Sz_n \end{cases} \tag{3.13}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(r1) $\lim_{n\to\infty} u_n = u$ *for some* $u \in H$;

(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(r3) $0 < c \leq \beta_n \leq d < 1$ *and* $\gamma \in [\kappa, 1)$;

(r4) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$, *where* $[a,b] \subset (0,1)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.13) *converges strongly to* $\mathrm{Proj}_{F(S)\cap EP}(u)$.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions* (H1)-(H4). *Let $S : C \to C$ be a nonexpansive mapping. Suppose that $F(S) \cap EP \neq \emptyset$. Let $x_0 \in C$, $\{z_n\}$ and $\{x_n\}$ be sequences in C generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (\alpha_n u_n + (1-\alpha_n)x_n)\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\gamma z_n + (1-\beta_n)(1-\gamma)Sz_n \end{cases} \tag{3.14}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(r1) $\lim_{n\to\infty} u_n = u$ *for some* $u \in H$;

(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(r3) $0 < c \leq \beta_n \leq d < 1$ *and* $\gamma \in (0, 1)$;

(r4) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$, *where* $[a,b] \subset (0,1)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.14) *converges strongly to* $\mathrm{Proj}_{F(S)\cap EP}(u)$.

Corollary 3.5 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions* (H1)-(H4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a κ-strictly pseudo-contractive mapping. Suppose that $F(S) \cap MEP \neq \emptyset$. Let $x_0 \in C$, $\{z_n\}$ and $\{x_n\}$ be sequences in C generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - (1-\alpha_n)x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\gamma z_n + (1-\beta_n)(1-\gamma)Sz_n \end{cases} \tag{3.15}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(r3) $0 < c \leq \beta_n \leq d < 1$ *and* $\gamma \in [\kappa, 1)$;

(r4) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$, *where* $[a,b] \subset (0,2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.15) *converges strongly to* $\mathrm{Proj}_{F(S)\cap MEP}(0)$, *which is the minimum norm element in* $F(S) \cap MEP$.

Corollary 3.6 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions* (H1)-(H4). *Let $S : C \to C$ be a κ-strictly pseudo-contractive mapping. Suppose that $F(S) \cap EP \neq \emptyset$. Let $x_0 \in C$, $\{z_n\}$ and $\{x_n\}$ be sequences in C generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (1-\alpha_n)x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\gamma z_n + (1-\beta_n)(1-\gamma)Sz_n \end{cases} \tag{3.16}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(r3) $0 < c \leq \beta_n \leq d < 1$ *and* $\gamma \in [\kappa, 1)$;

(r4) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$, *where* $[a,b] \subset (0,1)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.16) *converges strongly to* $\mathrm{Proj}_{F(S)\cap EP}(0)$, *which is the minimum norm element in $F(S) \cap EP$.*

Corollary 3.7 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions* (H1)-(H4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Suppose that $MEP \neq \emptyset$. Let $x_0 \in C$, $\{z_n\}$ and $\{x_n\}$ be sequences in C generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - (1-\alpha_n)x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)z_n \end{cases} \tag{3.17}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(r3) $0 < c \leq \beta_n \leq d < 1$;

(r4) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$, *where* $[a,b] \subset (0,2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.17) *converges strongly to* $\mathrm{Proj}_{MEP}(0)$, *which is the minimum norm element in MEP.*

Corollary 3.8 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions* (H1)-(H4). *Suppose $EP \neq \emptyset$. Let $x_0 \in C$, $\{z_n\}$ and $\{x_n\}$ be sequences in C generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (1-\alpha_n)x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)z_n \end{cases} \tag{3.18}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(r2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(r3) $0 < c \leq \beta_n \leq d < 1$;

(r4) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$, *where* $[a,b] \subset (0,1)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.18) *converges strongly to* $\mathrm{Proj}_{EP}(0)$, *which is the minimum norm element in EP.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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