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Some discussion on the existence of common fixed points for a pair of maps

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Abstract
In this paper, the concepts of conditionally sequential absorbing and pseudo-reciprocal continuous maps are introduced in connection to giving a brief discussion on the role of various types of commutativity (*e.g.*, weakly compatible, occasionally weakly compatible, subcompatible, pseudo-compatible, *etc.*) and continuity-type conditions (*e.g.*, reciprocal, weak reciprocal, g-reciprocal, conditionally reciprocal, subsequential and sequential continuity of type (A_g) and (A_f)) in the context of existence of common fixed points of a pair of maps. Here, the utility of newly introduced maps (*i.e.*, conditionally sequential absorbing and pseudo-reciprocal continuous) in view of common fixed points for a pair of maps satisfying contractive as well as nonexpansive Lipschitz-type conditions is shown.
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1 Introduction and preliminaries

The classical results of Banach [1] (see also [2]) and Edelstein [3] have been the inspiration for many researchers working in the area of metric fixed point theory. In 1976, Jungck [4] generalized the Banach contraction principle by introducing the idea of commuting maps and settled the historical open problem that a pair of commuting and continuous self-mappings on the unit interval $[0,1]$ need not have a common fixed point [5, 6]. This result of Jungck [4] made foundation to study and investigate common fixed points and their applications in various other branches of mathematical sciences in the last five decades. Since then many fixed point theorists have attempted to find weaker forms of commutativity and continuity that may ensure the existence of a common fixed point for a pair of self-mappings on a metric space. Systematic observations and comparison of commutativity-type mappings are available in [7].

Proving a common fixed point for mappings satisfying Banach-type contractive conditions involves the following steps: step one is to show that there exists a Cauchy sequence which converges to a point in X (where X is complete); the second step is to show the existence of a coincidence point by assuming suitable weaker forms of commutativity and continuity conditions; and step three automatically gives rise to the fact that this coincidence point is a unique common fixed point due to the contractive condition. Observing carefully step two, one finds that showing the existence of a coincidence point for involved maps is nothing but assuming the existence of a coincidence point itself by a suitable choice of weaker forms of commutativity and continuity conditions (see, for instance, [8–16]).

Keeping the above facts in mind, Jungck and Rhoades [17] utilized the notion of occasionally weakly compatible maps introduced in [18] (as a generalization of weakly compatible maps) for those pairs which do have at least one coincidence point where the maps commute (it is well known that a pair of maps without a coincidence point is always vacuously weakly compatible) and obtained fixed point theorems for such maps.

On the other hand, Singh and Mishra in [19] illustrated a technique to prove the existence of a coincidence point without assuming continuity and commutativity-type conditions. Whereas the result of Suzuki and Pathak [20] does not involve any continuity-type conditions to prove the existence of a coincidence point as well as a common fixed point for a pair of maps (but they used weaker forms of commutativity conditions). It is also worth mentioning that Suzuki and Pathak [20] did not provide any illustrative examples to discuss and highlight the above facts. It is also important to note that none of the results of Jungck [9], Singh and Mishra [19] and Suzuki and Pathak [20] can be obtained from each other due to their different characteristics. These facts are illustrated in this paper via Example 2.6 (p.10).

Motivated by the works of Jungck and Rhoades [17], Bouhadjera and Thobie [21] (respectively Hussain *et al.* [22] and Sintunavarat and Kumam [23]) introduced the notion of subcompatible maps (respectively the notions of occasionally weakly $\mathcal{JH}$ operator and occasionally weakly biased maps) as generalization of occasionally weakly compatible maps and obtained fixed point theorems for such maps. However, Dorić *et al.* in [24] (respectively Alghamdi *et al.* [25]) showed that in the event of a pair of single-valued maps, the notion of occasionally weakly compatible (respectively occasionally weakly $\mathcal{JH}$ operator and occasionally weakly biased maps) reduces to weak compatibility due to the unique coincidence point of the involved maps, which is always ensured by underlying contractive conditions. Hence weak compatibility remains the minimal commutativity condition for the existence of a common fixed point for a contractive pair of maps. In view of these, the various results for occasionally weakly compatible maps (occasionally weakly $\mathcal{JH}$ operator and occasionally weakly biased maps) obtained in [17, 21, 22, 26–30], which were used under contractive conditions, do not yield real generalizations (see also [31, 32]). Considering these facts, Pant and Pant [33] (see also [34]) redefined the concept of occasionally weakly compatibility by introducing the idea of conditionally commuting maps which constitute a proper setting in the context of studying non-unique common fixed points for a pair of maps.

Possibly the first common fixed point theorem (respectively fixed point theorem) without any continuity requirement was established by Pant [12, 35] when he introduced the idea of noncompatible and reciprocal continuous maps. (However, the origin of metric fixed point theory for a single mapping without continuity requirement can be traced back to Kannan [36].) Recently, Pant *et al.* [37] and Pant and Bisht [38] generalized the notion of reciprocal continuity by introducing weak reciprocal continuity and conditionally reciprocal continuity and utilized the same to obtain some common fixed point theorems. In this connection, the recent paper of Gopal *et al.* [39] is also readable.

Motivated by the results of Pant and Bisht [38, 40], we introduce the concept of conditionally sequential absorbing and pseudo-reciprocal continuous maps, which allows us to give a comparative study of various types of commutativity conditions (*e.g.*, compatible, weakly compatible, occasionally weakly compatible, conditionally commuting, pseudo-compatible) and continuity-type conditions (*e.g.*, reciprocal, weak reciprocal, g-reciprocal, conditionally reciprocal, subsequential and sequential continuity of type (A_g) and (A_f)) with these newly introduced notions in the context of existence of common fixed points of a pair of maps.

The following relevant known definitions (and results) will be needed in our subsequent discussion. A pair (f,g) of self-mappings defined on a metric space (X,d) is said to be

(i) compatible [9] iff $\lim_n d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

It is clear from the above definition that f and g will be noncompatible [35] if there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X, but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or non-existent;

(ii) f-compatible [41] if $\lim_n d(fgx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X;

(iii) g-compatible [41] if $\lim_n (ffx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X;

(iv) weakly compatible [42] if the mappings commute at their coincidence points, *i.e.*, $fx = gx$ for some $x \in X$ implies $fgx = gfx$;

(v) occasionally weakly compatible [17] if there exists a point x in X that is a coincidence point of f and g at which f and g commute;

(vi) subcompatible [21] iff there exists a sequence $\{x_n\}$ in X such that $\lim_n d(fgx_n, gfx_n) = 0$ with $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$;

(vii) conditionally commuting [33] if they commute on a nonempty subset of the set of coincidence points whenever the set of their coincidence point is nonempty;

(viii) conditionally compatible [34] iff, whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ such that $\lim_n fy_n = \lim_n gy_n = t$ (say) and $\lim_n d(fgy_n, gfy_n) = 0$;

(ix) pseudo-compatible [40] iff, whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ such that $\lim_n fy_n = \lim_n gy_n = t$ (say), $\lim_n d(fgy_n, gfy_n) = 0$; and $\lim_n d(fgz_n, gfz_n) = 0$ for any associated sequence $\{z_n\}$ of $\{y_n\}$.

We also recall that a pair (f,g) of self-mappings defined on a metric space (X,d) is said to be

(i) reciprocally continuous [12, 43] iff $\lim_n fgx_n = ft$ and $\lim_n gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X;

(ii) weakly reciprocally continuous [37] if $\lim_n fgx_n = ft$ or $\lim_n gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X;

(iii) conditionally reciprocally continuous (CRC) [38] if, whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = \lim_n gy_n = t$ (say) such that $\lim_n fgy_n = ft$ and $\lim_n gfy_n = gt$;

(iv) g-reciprocally continuous [40] iff $\lim_n ffx_n = ft$ and $\lim_n gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X;

(v) sequentially continuous of type (A_g) [39] iff there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$ satisfying $\lim_n ffx_n = ft$ and $\lim_n gfx_n = gt$;

(vi) sequentially continuous of type (A_f) [39] iff there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$ satisfying $\lim_n fgx_n = ft$ and $\lim_n ggx_n = gt$;

(vii) subsequentially continuous [21] iff there exists a sequence x_n in X such that $\lim_n fgx_n = ft$ and $\lim_n gfx_n = gt$ with $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$.

Theorem 1.1 [40] *Let f and g be g-reciprocally continuous self-mappings of a complete metric space (X,d) such that*

(i) $fX \subseteq gX$;
(ii) $d(fx,fy) \leq kd(gx,gy)$, $k \in [0,1)$.

If f and g are pseudo-compatible, then f and g have a unique common fixed point.

Theorem 1.2 [40] *Let f and g be g-reciprocally continuous noncompatible self-mappings of a metric space (X,d) such that*

(i) $fX \subseteq gX$;
(ii) $d(fx,fy) < \max\{d(gx,gy), \frac{k[d(fx,gx)+d(fy,gy)]}{2}, \frac{d(fx,gy)+d(fy,gx)}{2}\}$, *where* $1 \leq k < 2$;
(iii) $d(x,fx) \neq \max\{d(x,gx), d(fx,gx)\}$,

whenever the right-hand side is nonzero. If f and g are pseudo-compatible, then f and g have a unique common fixed point.

Theorem 1.3 [34] *Let f and g be conditionally compatible self-mappings of a metric space (X,d) satisfying*

$$d(x,gx) \neq \max\{d(x,fx), d(gx,fx)\},$$

whenever the right-hand side is nonzero. If f and g are noncompatible and reciprocally continuous, then f and g have a common fixed point.

Theorem 1.4 [38] *Let f and g be conditionally reciprocal continuous self-mappings of a complete metric space (X,d) such that*

(i) $fX \subseteq gX$;
(ii) $d(fx,fy) \leq kd(gx,gy)$, $k \in [0,1)$.

If f and g are either compatible or g-compatible or f-compatible, then f and g have a unique common fixed point.

Theorem 1.5 [44] *Let (X,d) be a complete metric space, let f and g be two noncompatible self-mappings on X satisfying*

$$d(fx,fy) \leq \varphi\bigl(d(gx,gy)\bigr) \quad \text{for all } x,y \in X,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous from right and nondecreasing function such that $\varphi(t) < t$ for all $t > 0$. Assume that

(i) $\overline{f(X)} \subseteq g(X)$,
(ii) $\max\{d(ggx,fgx), d(ffx,gfx)\} \leq \varphi(d(fx,gx))$ *for all* $x \in X$ *and*
(iii) $\varphi(d(fx,f^2x)) \neq \varphi(\max\{d(gx,gfx), d(g^2x,gfx), d(fx,gx), d(f^2x,gfx), d(fx,gfx), d(gx,f^2x)\})$, *whenever* $fx \neq f^2x$.

Then f and g have a unique common fixed point. Also, f and g are discontinuous at the common fixed point.

2 Main results

We begin with the following example.

Example 2.1 Let $X = [2, 20]$ and d be the usual metric on X. Define self-mappings f and g on X as follows:

$$fx = \begin{cases} x+2 & \text{if } x \in (2,3], \\ 5 & \text{otherwise}, \end{cases} \qquad gx = \begin{cases} 2x & \text{if } x \in (2,3], \\ 20 & \text{otherwise}. \end{cases}$$

Then we can see that $f(X) = (4,5] \subseteq (4,6] \cup \{20\} = g(X)$ and the pair (f,g) is g-reciprocally continuous. It can be verified that $d(fx,fy) \leq kd(gx,gy)$ for all $x, y \in X$ with $k = \frac{1}{2}$. Thus, f and g satisfy all the conditions of Theorem 1.1 except pseudo-compatibility. For the pseudo-compatibility, consider the only existent sequence $x_n = y_n = 2 + \frac{1}{n}$, then we have $\lim_n fy_n = \lim_n gy_n = 4$, but $\lim_n fgy_n = \lim_n f(4 + \frac{2}{n}) = 5$, $\lim_n gfy_n = \lim_n (4 + \frac{1}{n}) = 20$, and so $\lim_n d(fgy_n, gfy_n) = 15 \neq 0$. Also note that the pair (f,g) is not compatible. Here, (f,g) has no coincidence point therefore it is also not an occasionally weakly compatible but vacuously weakly compatible pair.

This suggests that pseudo-compatible is stronger than weakly compatible (and occasionally weakly compatible) in the context of Theorem 1.1 (such an observation is missing in [40]).

The above example motivated us to define the following.

Definition 2.1 Two self-mappings f and g of a metric space (X,d) are called conditionally sequential absorbing if, whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = \lim_n gy_n = t$ (say) such that $\lim_n d(fy_n, fgy_n) = 0$ and $\lim_n d(gy_n, gfy_n) = 0$.

Example 2.2 Let $X = [2, 10]$ and let d be the usual metric on X. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 2 & \text{if } x = 2 \text{ or } x > 5, \\ 6 & \text{if } x \in (2,5], \end{cases} \qquad gx = \begin{cases} 2 & \text{if } x = 2, \\ 2x & \text{if } x \in (2,5], \\ \frac{x+1}{3} & \text{if } x > 5. \end{cases}$$

Then the maps are conditionally sequential absorbing. To view this, consider the constant sequence $x_n = 2$. However, the pair (f,g) is not weakly compatible as they do not commute at their coincidence point $x = 3$. It may be noted that $x = 2$ and $x = 3$ are two coincidence points of f and g. But in respect of the unique coincidence point (common fixed point), conditionally sequential absorbing always implies weakly compatible and hence occasionally weakly compatible and pseudo-compatible, because the maps naturally commute at their unique coincidence point (common fixed point).

Example 2.3 Let $X = [0, 1]$ and let d be the usual metric on X. Define $f, g : X \to X$ as follows:

$$fx = 1 - x \quad \text{and} \quad gx = (1-x)^2 \quad \text{for all } x \in X.$$

Then f and g are weakly compatible but not conditionally sequential absorbing. Here, $x = 0$ and $x = 1$ are two coincidence points.

Remark 2.1 In Example 2.1, the pair (f,g) is vacuously weakly compatible but not conditionally sequential absorbing and not pseudo-compatible. Note that f and g do not have any coincidence point. In Example 2.2, the pair (f,g) is conditionally sequential absorbing but not weakly compatible. In Example 2.3, the pair (f,g) is weakly compatible but not conditionally sequential absorbing.

Thus, as definitions, weakly compatible, pseudo-compatible and conditionally sequential absorbing are very different. However, in the context of a unique coincidence point, conditionally sequential absorbing is stronger than weakly compatible, which will be shown in our Example 2.6.

Example 2.4 Let $X = [0,+\infty)$ and let d be the usual metric on X. Define $f,g : X \to X$ as follows:

$$fx = \begin{cases} \frac{x}{3} & \text{if } x \in [0,1], \\ 2x-1 & \text{if } x > 1, \end{cases} \qquad gx = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1], \\ 3x-2 & \text{if } x > 1. \end{cases}$$

Let us consider the sequence $x_n = \frac{1}{n}$ for $n = 1,2,\ldots$. Then

$$\begin{aligned} &\lim_n fx_n = \lim_n \frac{1}{3n} = 0, \qquad \lim_n gx_n = \lim_n \frac{1}{2n} = 0, \\ &\lim_n fgx_n = \lim_n \frac{1}{6n} = 0 = f(0), \\ &\lim_n gfx_n = \lim_n \frac{1}{6n} = 0 = g(0). \end{aligned}$$

Thus f and g are conditionally reciprocal continuous and subsequentially continuous. We can see that f and g are neither weak reciprocal continuous nor g-reciprocal continuous. To see this, consider the sequence $x_n = 1 + \frac{1}{n}$ for $n = 1,2,\ldots$, then

$$\begin{aligned} &\lim_n fx_n = \lim_n \left(1 + \frac{2}{n}\right) = 1, \qquad \lim_n gx_n = \lim_n \left(1 + \frac{3}{n}\right) = 1, \\ &\lim_n fgx_n = \lim_n \left(1 + \frac{6}{n}\right) = 1 \neq f(1), \\ &\lim_n gfx_n = \lim_n \left(1 + \frac{6}{n}\right) = 1 \neq g(1), \\ &\lim_n ffx_n = \lim_n \left(1 + \frac{4}{n}\right) = 1 \neq f(1), \\ &\lim_n ggx_n = \lim_n \left(1 + \frac{9}{n}\right) = 1 \neq g(1). \end{aligned}$$

Note that f and g do not have a coincidence point.

Example 2.5 Let $X = R$ and let d be the usual metric on X. Define $f, g : X \to X$ as follows:

$$fx = x \quad \text{and} \quad gx = x + 1 \quad \text{for all } x \in X.$$

Then it is easy to see that the pair (f, g) is reciprocal continuous, weak reciprocally continuous and conditionally reciprocally continuous but neither subsequentially continuous nor sequentially continuous of type (A_g) and (A_f). Note that the pair has no coincidence point.

In view of the above examples, we observe that in the event of no coincidence point, subsequential continuity as well as sequential continuity of type (A_g) and (A_f) are different from reciprocal continuity (respectively g-reciprocal and conditionally reciprocal continuity). However, in the context of a unique coincidence point (common fixed point), subsequential continuity as well as sequential continuity of type (A_g) and (A_f) are equivalent to conditionally reciprocal continuity.

The motivation of the following definition can be predicted from the proof of the last step in our Theorem 2.1.

Definition 2.2 Two self-mappings f and g of a metric space (X, d) are called pseudo-reciprocal continuous (PRC) (with respect to conditionally sequential absorbing) if, whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ (satisfying $\lim_n fy_n = \lim_n gy_n = t$ (say), $\lim_n d(fy_n, fgy_n) = 0$ and $\lim_n d(gy_n, gfy_n) = 0$) such that $\lim_n fgy_n = ft$ and $\lim_n gfy_n = gt$.

Common fixed point theorems

Assume that $\phi, \psi : [0, \infty) \to [0, \infty)$ are two functions such that

(a) ϕ is nondecreasing, continuous and $\phi(0) = 0 < \phi(t)$ for every $t > 0$;

(b) ψ is nondecreasing, right-continuous, and $\psi(t) < t$ for every $t > 0$.

To prove our first result, we use the following lemma.

Lemma 2.1 [45] *For every function* $\psi : [0, \infty) \to [0, \infty)$, *let* ψ^n *be the nth iterate of* ψ. *Then the following hold*:

(i) *if* ψ *is nondecreasing, then for each* $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ *implies* $\psi(t) < t$;

(ii) *if* ψ *is right-continuous with* $\psi(t) < t$ *for* $t > 0$, *then* $\lim_{n\to\infty} \psi^n(t) = 0$.

Theorem 2.1 *Let f and g be two pseudo-reciprocal continuous* (*w.r.t. conditionally sequential absorbing*) *self-mappings of a complete metric space* (X, d) *such that* $fX \subseteq gX$, *and let* $\phi, \psi : [0, \infty) \to [0, \infty)$ *be two functions satisfying* (a) *and* (b). *If for all* $x, y \in X$,

$$\phi\bigl(d(fx, fy)\bigr) \le \psi\bigl(\phi\bigl(M(x, y)\bigr)\bigr), \tag{2.1}$$

where

$$M(x, y) = \max\left\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{d(gx, fy) + d(fx, gy)}{2}\right\},$$

then f and g have a unique common fixed point provided (f, g) *is conditionally sequential absorbing*.

Proof Let $x_0 \in X$ and since $fX \subseteq gX$, so we have a sequence $\{p_n\}$ defined by

$$p_n = fx_n = gx_{n+1}. \tag{2.2}$$

Now we show that $\{p_n\}$ is a Cauchy sequence. We have

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\biggl\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}), \\ &\quad \frac{d(gx_n, fx_{n+1}) + d(fx_n, gx_{n+1})}{2}\biggr\} \\ &= \max\biggl\{d(p_{n-1}, p_n), d(p_n, p_{n-1}), d(p_{n+1}, p_n), \\ &\quad \frac{d(p_{n-1}, p_{n+1}) + d(p_n, p_n)}{2}\biggr\} \\ &= \max\biggl\{d(p_{n-1}, p_n), d(p_{n+1}, p_n), \frac{d(p_{n-1}, p_{n+1})}{2}\biggr\} \\ &= \max\bigl\{d(p_{n-1}, p_n), d(p_{n+1}, p_n)\bigr\}. \end{aligned}$$

If we suppose $M(x_n, x_{n+1}) = d(p_n, p_{n+1})$, then

$$\begin{aligned} \phi\bigl(d(fx_n, fx_{n+1})\bigr) &\le \psi\bigl(\phi\bigl(M(x_n, x_{n+1})\bigr)\bigr) = \psi\bigl(\phi\bigl(d(p_n, p_{n+1})\bigr)\bigr) \\ &= \psi\bigl(\phi\bigl(d(fx_n, fx_{n+1})\bigr)\bigr) < \phi\bigl(d(fx_n, fx_{n+1})\bigr), \end{aligned}$$

which is a contradiction. Therefore

$$M(x_n, x_{n+1}) = d(p_{n-1}, p_n).$$

Similarly,

$$M(x_n, x_{n-1}) = d(p_{n-1}, p_{n-2}).$$

If for some n we have either $p_n = p_{n-1}$ or $p_n = p_{n+1}$, then by condition (2.1) we obtain that the sequence $\{p_n\}$ is definitely constant and thus it is a Cauchy sequence. Suppose $p_n \neq p_{n-1}$ for each n, then from condition (2.1) we have

$$\begin{aligned} \phi\bigl(d(p_n, p_{n-1})\bigr) = \phi\bigl(d(fx_n, fx_{n-1})\bigr) &\le \psi\bigl(\phi\bigl(M(x_n, x_{n-1})\bigr)\bigr) \\ &= \psi\bigl(\phi\bigl(d(p_{n-1}, p_{n-2})\bigr)\bigr) < \phi\bigl(d(p_{n-1}, p_{n-2})\bigr), \end{aligned}$$

and for all $n \in N$,

$$\phi\bigl(d(p_n, p_{n+1})\bigr) < \phi\bigl(d(p_{n-1}, p_n)\bigr).$$

Now, we have

$$\phi\bigl(d(p_{n+1}, p_n)\bigr) \le \psi\bigl(\phi\bigl(d(p_n, p_{n-1})\bigr)\bigr) \le \cdots \le \psi^n\bigl(\phi\bigl(d(p_0, p_1)\bigr)\bigr), \tag{2.3}$$

and then, by Lemma 2.1(ii),

$$\begin{aligned} &\lim_{n\to\infty} \psi^n\bigl(\phi\bigl(d(p_{n+1},p_n)\bigr)\bigr) = 0 \\ &\quad\Rightarrow \lim_{n\to\infty} \phi\bigl(d(p_{n+1},p_n)\bigr) = 0 \\ &\quad\Rightarrow \lim_{n\to\infty} d(p_{n+1},p_n) = 0. \end{aligned} \tag{2.4}$$

Now we prove that $\{p_n\}$ is Cauchy.

Suppose not, then $\exists\epsilon > 0$ such that $d(p_n,p_m) \geq 2\epsilon$ for infinite value of m and n with $m < n$. This assumes that there exist two sequences $\{m_k\}$, $\{n_k\}$ of natural numbers with $m_k < n_k$ such that

$$d(p_{m_k},p_{n_k+1}) > \epsilon. \tag{2.5}$$

It is not restrictive to suppose that n_k is the least positive integer exceeding m_k and satisfying (2.5). We have

$$\begin{aligned} \epsilon &< d(p_{m_k},p_{n_k+1}) \\ &\leq d(p_{m_k},p_{n_k-1}) + d(p_{n_k-1},p_{n_k}) + d(p_{n_k},p_{n_k+1}) \\ &\leq \epsilon + d(p_{n_k-1},p_{n_k}) + d(p_{n_k},p_{n_k+1}). \end{aligned}$$

Then

$$d(p_{m_k},p_{n_k+1}) \to \epsilon \quad \text{as } k \to \infty. \tag{2.6}$$

We note

$$\begin{aligned} &d(p_{m_k},p_{n_k+1}) - d(p_{m_k},p_{m_k+1}) - d(p_{n_k+2},p_{n_k+1}) \\ &\quad\leq d(p_{m_k+1},p_{n_k+2}) \leq d(p_{m_k},p_{n_k+1}) - d(p_{m_k},p_{m_k+1}) - d(p_{n_k+2},p_{n_k+1}), \end{aligned}$$

and therefore

$$d(p_{m_k+1},p_{n_k+2}) \to \epsilon \quad \text{as } k \to \infty. \tag{2.7}$$

Now, we have

$$\begin{aligned} M(x_{n_k+2},x_{m_k+1}) = {} & \max\biggl\{d(p_{m_k},p_{n_k+1}), d(p_{n_k+1},p_{n_k+2}), d(p_{m_k},p_{m_k+1}), \\ & \frac{d(p_{m_k+1},p_{n_k+1}) + d(p_{m_k},p_{n_k+2})}{2}\biggr\} \\ = {} & d(p_{m_k},p_{n_k+1}) + d_k, \end{aligned} \tag{2.8}$$

where $d_k \to 0$ as $k \to +\infty$ and $d_k \geq 0$ for all k. Then from

$$\begin{aligned} \phi\bigl(d(p_{m_k+1},p_{n_k+2})\bigr) &= \phi\bigl(d(fx_{n_k+2},fx_{m_k+1})\bigr) \leq \psi\bigl(\phi\bigl(M(x_{n_k+2},x_{m_k+1})\bigr)\bigr) \\ &= \psi\bigl(\phi\bigl(d(p_{m_k},p_{n_k+1}) + d_k\bigr)\bigr), \end{aligned} \tag{2.9}$$

as $k \to +\infty$, ϕ being continuous and ψ right-continuous, we get

$$\phi(\epsilon) \le \psi\bigl(\phi(\epsilon)\bigr) < \phi(\epsilon).$$

This is a contradiction. Therefore $\{p_n\}$ is a Cauchy sequence. Since (X,d) is a complete metric space, therefore $\exists t \in X$ such that

$$p_n = fx_n = gx_{n+1} \to t \quad \text{as } n \to \infty. \tag{2.10}$$

Since the pair (f,g) is conditionally sequential absorbing, therefore there exists a sequence $\{y_n\}$ in X such that $fy_n \to u$, $gy_n \to u$ (say) satisfying

$$\lim_n d(fy_n, fgy_n) = 0 \quad \text{and} \quad \lim_n d(gy_n, gfy_n) = 0, \tag{2.11}$$

and by pseudo-reciprocal continuity (w.r.t. conditionally sequential absorbing) of (f,g), we have

$$\lim_n fgy_n = fu \quad \text{and} \quad \lim_n gfy_n = gu. \tag{2.12}$$

In view of (2.11) and (2.12), we get $fu = gu = u$, *i.e.*, u is a common fixed point of f and g. The uniqueness of a common fixed point follows easily from contractive condition (2.1). □

Example 2.6 Let $X = [0,1]$ with the usual metric d. Define self-maps f and g as follows:

$$fx = \begin{cases} \frac{1+x}{2} & \text{if } x \in [0,\frac{1}{2}), \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2},1], \end{cases} \qquad gx = \begin{cases} \frac{1}{2} + x & \text{if } x \in [0,\frac{1}{2}), \\ \frac{1}{2} & \text{if } x = \frac{1}{2}, \\ \frac{4}{5} & \text{if } x \in (\frac{1}{2},1]. \end{cases}$$

Then f and g satisfy all the conditions of Theorem 2.1 with $f(X) = [\frac{1}{2},\frac{3}{4}) \subseteq [\frac{1}{2},1) = g(X)$. Here, f and g are conditionally sequential absorbing and pseudo-reciprocal continuous (w.r.t. conditionally sequential absorbing) in respect of the constant sequence $x_n = \frac{1}{2}$. Let us consider the sequence $x_n = \frac{1}{n+2}$, then

$$\begin{aligned}
&\lim_n fx_n = \lim_n \left(\frac{1}{2} + \frac{1}{2n+4}\right) = \frac{1}{2} = \lim_n \left(\frac{1}{2} + \frac{1}{n+2}\right) = \lim_n gx_n, \\
&\lim_n fgx_n = \lim_n f\left(\frac{1}{2} + \frac{1}{n+2}\right) = \frac{1}{2} = f\left(\frac{1}{2}\right), \\
&\lim_n gfx_n = \lim_n g\left(\frac{1}{2} + \frac{1}{2n+4}\right) = \frac{4}{5} \ne g\left(\frac{1}{2}\right), \\
&\lim_n ffx_n = \lim_n f\left(\frac{1}{2} + \frac{1}{2n+4}\right) = \frac{1}{2} = f\left(\frac{1}{2}\right), \\
&\lim_n ggx_n = \lim_n g\left(\frac{1}{2} + \frac{1}{n+2}\right) = \frac{4}{5} \ne g\left(\frac{1}{2}\right).
\end{aligned}$$

Thus, (f,g) is not a reciprocal as well as not a g-reciprocal continuous pair. Also the pair (f,g) is neither compatible, f-compatible nor g-compatible.

If we take $\phi(t) = t$ and $\psi(t) = kt$, $k \in [0,1)$, then it can be verified that f and g satisfy contraction condition (2.1) with $k = \frac{5}{6}$. Here, $x = \frac{1}{2}$ is the unique common fixed point of f and g, which is also a point of discontinuity.

On the other hand, notice that at $x = 1$, f and g do not satisfy the condition

$$\max\{d(ggx, fgx), d(ffx, gfx)\} \le \varphi(d(fx, gx))$$

used in Theorem 1.5. Here, it is worth noting that none of the earlier relevant theorems, for example, Theorem 1.1, Theorem 1.4 and Theorem 1.5, can be used in the context of this example. One more interesting part of this example is that neither $f(X)$ nor $g(X)$ is closed. Thus the result of Singh and Mishra [19] cannot be applicable in the context of this example.

Theorem 2.2 *Let f and g be pseudo-reciprocal continuous* (*w.r.t. conditionally sequential absorbing*) *and noncompatible self-mappings of a metric space* (X, d) *satisfying*

$$d(fx, fy) < \max\left\{d(gx, gy), \frac{k[d(fx, gx) + d(fy, gy)]}{2}, \frac{d(fx, gy) + d(fy, gx)}{2}\right\}, \tag{2.13}$$

where $1 \le k < 2$. If f and g are conditionally sequential absorbing, then f and g have a unique common fixed point.

Proof Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either $\lim_n d(fgx_n, gfx_n) \ne 0$ or the limit does not exist. Also, the pair (f, g) is conditionally sequential absorbing; therefore, there exists a sequence $\{y_n\}$ in X such that $\lim_n fy_n = \lim_n gy_n = u$ (say) with $\lim_n d(fy_n, fgy_n) = 0$ and $\lim_n d(gy_n, gfy_n) = 0$. Now, by the pseudo-reciprocal continuity (w.r.t. conditionally sequential absorbing) of the pair (f, g), we have $fgy_n \to fu$ and $gfy_n \to gu$. In view of these limits, we get u is a common fixed point of f and g.

Now, suppose that there exists another common fixed point w of f and g such that $w \ne u$. Then, on using (2.13), we have

$$d(fw, fu) < \max\left\{d(gw, gu), \frac{k[d(fw, gw) + d(fu, gu)]}{2}, \frac{d(fw, gu) + d(fu, gw)}{2}\right\}.$$

Thus, we have $d(w, u) < \frac{k}{2} d(w, u) < d(w, u)$, a contradiction and hence $w = u$. □

Example 2.7 Again consider Example 2.6 wherein the pair (f, g) satisfies all the conditions of Theorem 2.2 for all $k \in [1, 2)$. Note that at $x = 0$, f and g do not satisfy the condition

$$d(x, fx) \ne \max\{d(x, gx), d(fx, gx)\},$$

whenever the right-hand side is nonzero. Thus Theorem 2.2 is a genuine extension and improvement of Theorem 1.2 due to Pant and Bisht [40].

Observation The proof of Theorem 2.1, Theorem 2.2 and examples above immediately suggest us defining another type of continuity as follows.

Definition 2.3 Two self-mappings f and g of a metric space (X,d) are called pseudo-reciprocal continuous (PRC) (with respect to pseudo-compatible) if whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ (satisfying $\lim_n fy_n = \lim_n gy_n = t$ (say); $\lim_n d(fgy_n, gfy_n) = 0$; and $\lim_n d(fgz_n, gfz_n) = 0$ for any associated sequence $\{z_n\}$ of $\{y_n\}$) such that $\lim_n fgy_n = ft$ and $\lim_n gfy_n = gt$.

However, the notions of pseudo-compatibility and pseudo-reciprocal continuity (w.r.t. pseudo-compatibility) are no more applicable in the context of the existence of non-unique common fixed points for a pair of maps. This fact is illustrated in Example 2.11 below. At the same time, conditionally sequential absorbing and pseudo-reciprocal continuity (w.r.t. conditionally sequential absorbing) are easily applicable.

Theorem 2.3 *Let f and g be reciprocal (or g-reciprocal) continuous and noncompatible self-mappings of a metric space (X,d) satisfying* (2.13). *Then the pair (f,g) has a unique common fixed point provided it is conditionally sequential absorbing. Moreover, f and g are discontinuous at the common fixed point.*

Proof Since f and g are noncompatible, there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$, but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or not existent. Also, since f and g are conditionally sequential absorbing and $\lim_n fx_n = \lim_n gx_n = t$, there exists a sequence $\{y_n\}$ in X, satisfying $\lim_n fy_n = \lim_n gy_n = u$ (say), such that $\lim_n d(fy_n, fgy_n) = 0$ and $\lim_n d(gy_n, gfy_n) = 0$. The reciprocal continuity of the pair (f,g) implies that $\lim_n fgy_n = fu$ and $\lim_n gfy_n = gu$. Thus, in view of these limits, we obtain $fu = gu = u$. If we consider the pair (f,g) g-reciprocal continuous, then we have $\lim_n ffy_n = fu$ and $\lim_n gfy_n = gu$. Since $\lim_n d(gy_n, gfy_n) = 0$, so we have $gu = u$. Now, suppose $fu \neq u$. On using (2.13), we get $d(fu,u) < \frac{k}{2} d(fu,u) < d(fu,u)$, a contradiction and hence $fu = u$. Thus u is a common fixed point of f and g. Applying (2.13), we can show the uniqueness of the common fixed point.

We now show that f and g are discontinuous at the common fixed point u. If possible, suppose f is continuous at u. Then, considering the sequence $\{x_n\}$ of the present theorem and on using (2.13), we get $t = u$ and hence by the continuity of f, we have $ffx_n \to fu = u$ and $fgx_n \to fu = u$. Now, reciprocal (or g-reciprocal) continuity of the pair (f,g) implies that $gfx_n \to gu = u$. This further yields that $\lim_n d(fgx_n, gfx_n) = 0$, which contradicts the fact that $\lim_n d(fgx_n, gfx_n)$ is either nonzero or non-existent. Hence f is discontinuous at the fixed point.

Next, suppose that g is continuous at u. Then, for the sequence $\{x_n\}$, we get $gfx_n \to gu = u$ and $ggx_n \to gu = u$. If (f,g) is reciprocal continuous, then we have $fgx_n \to fu = u$, and if it is g-reciprocal continuous, then on using (2.13), we get $fgx_n \to fu = u$. Thus, we obtain $\lim_n d(fgx_n, gfx_n) = 0$, a contradiction. Therefore f and g are discontinuous at their common fixed point. □

Example 2.8 Let $X = [2,20]$ with the usual metric d. Define $f,g : X \to X$ as follows:

$$fx = \begin{cases} 2 & \text{if } x = 2, x > 5, \\ 4 & \text{if } 2 < x \leq 5, \end{cases} \qquad gx = \begin{cases} 2 & \text{if } x = 2, x \geq 8, \\ 20 & \text{if } 2 < x \leq 5, \\ \frac{x+1}{3} & \text{if } 5 < x < 8. \end{cases}$$

Then f and g satisfy all the conditions of Theorem 2.3. It can be verified in this example that f and g satisfy contractive condition (2.13) for all $k \in [1,2)$. To see that f and g are noncompatible, consider the sequence $\{x_n\}$ in X such that $x_n = 5 + \epsilon_n$, then $fx_n \to 2$, $gx_n = (2 + \frac{\epsilon_n}{3}) \to 2$, $fgx_n \to 4$, $gfx_n \to 2$, and so $\lim_n d(fgx_n, gfx_n) \neq 0$. Also here the pair (f,g) is g-reciprocal continuous. To see this, let $\{x_n\}$ be a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. Then $t = 2$, $x_n = 2$ or $x_n = 5 + \epsilon_n$, $ffx_n \to 2 = f(2)$ and $gfx_n \to 2 = g(2)$. The pair (f,g) is conditionally sequential absorbing in respect of the constant sequence $\{y_n\}$ given by $y_n = 2$. Here, $x = 2$ is the unique common fixed point where f and g are discontinuous.

Note that at $x = 8$, f and g do not satisfy the condition

$$d(x,fx) \neq \max\{d(x,gx), d(fx,gx)\},$$

whenever the right-hand side is nonzero. Also notice that at $x = 3$, f and g do not satisfy

$$\max\{d(ggx,fgx), d(ffx,gfx)\} \leq \varphi\bigl(d(fx,gx)\bigr).$$

Thus, Theorem 2.3 is a genuine extension and improvement of Theorem 1.2 due to Pant and Bisht [40] and Theorem 1.5 due to Rezapour and Shahzad [44].

In the absence of contractive condition (2.13), the following corollaries are straightforward from Theorems 2.2 and 2.3.

Corollary 2.1 *Let f and g be pseudo-reciprocal continuous* (*w.r.t. conditionally sequential absorbing*) *and noncompatible self-mappings of a metric space* (X,d). *Then f and g have a common fixed point provided it is conditionally sequential absorbing.*

Corollary 2.2 *Let f and g be reciprocal* (*or g-reciprocal*) *continuous and noncompatible self-mappings of a metric space* (X,d). *Then the pair (f,g) has a common fixed point provided it is conditionally sequential absorbing.*

The following examples illustrate the above corollaries.

Example 2.9 Consider $X = [2,23]$ and let d be the usual metric on X. Define $f,g : X \to X$ as

$$fx = \begin{cases} 2 & \text{if } x \in \{2\} \cup (5,7) \cup (7,10) \cup (10,11) \cup (11,12) \\ & \qquad \cup (12,13) \cup (13,21) \cup (21,23), \\ \frac{x+5}{2} & \text{if } 2 < x \leq 5, \\ 7 & \text{if } x = 7, 23, \\ 12 & \text{if } x = 10, \\ 11 & \text{if } x = 11, 13, \\ 11.5 & \text{if } x = 12, \\ 10 & \text{if } x = 21, \end{cases}$$

$$gx = \begin{cases} 2 & \text{if } x \in \{2\} \cup [7,10) \cup (10,11) \cup (11,12) \cup (12,13) \\ & \qquad \cup (13,21) \cup (21,22) \cup (22,23), \\ 6 & \text{if } 2 < x \leq 5, \\ \frac{x+1}{3} & \text{if } x \in (5,7), \\ 11 & \text{if } x = 10, 11, 13, 22, \\ 11.6 & \text{if } x = 12, \\ 10 & \text{if } x = 21, \\ 7 & \text{if } x = 23. \end{cases}$$

Here f and g satisfy all the conditions of Corollary 2.1. In view of the constant sequence $x_n = 2$ or $x_n = 11$, the pair (f, g) is conditionally sequential absorbing and pseudo-reciprocal continuous (w.r.t. conditionally sequential absorbing). For noncompatibility as well as non-reciprocal continuity, let us consider the sequence $x_n = 5 + (\frac{1}{n})$, then we have

$$\begin{aligned} &fx_n \to 2, \qquad gx_n = \left(2 + \frac{1}{3n}\right) \to 2 \\ &\lim_n fgx_n = \lim_n \left(\frac{7}{2} + \frac{1}{6n}\right) = \frac{7}{2} \neq f(2), \\ &\lim_n gfx_n = 2 = g(2), \end{aligned}$$

and so $\lim_n d(fgx_n, gfx_n) \neq 0$. Here, 2 and 11 are two common fixed points of f and g. Also the pair is not weakly compatible as f and g do not commute at their coincidence point $x = 23$.

Note that at $x = 13$ and $y = 22$ the present example does not satisfy condition (2.13) for any $k \in [1, 2)$ and also Lipschitz-type condition used in [33] for any $k \geq 0$. Also notice that at $x = 21$, the involved maps do not satisfy any of the conditions:

(i) $d(x, fx) \neq \max\{d(x, gx), d(fx, gx)\}$,
(ii) $d(x, gx) \neq \max\{d(x, gx), d(gx, fx)\}$,
(iii) $d(x, gx) \neq \max\{d(x, fx), d(gx, fx)\}$, and
(iv) $d(fx, f^2x) \neq \max\{d(gx, gfx), d(fx, gx), d(f^2x, gfx), d(fx, gfx), d(gx, f^2x)\}$,

whenever the right-hand side is nonzero. Here, it is worth noting that none of the Theorem 1.3 due to Pant and Bisht [34] and the main results contained in Pant and Pant [33] and Gopal *et al.* [46] can be used in the context of Corollary 2.1.

Example 2.10 Consider $X = [2, 23]$ and let d be the usual metric on X. Define $f, g : X \to X$ as

$$fx = \begin{cases} 2 & \text{if } x \in [2,7) \cup (7,10) \cup (10,11) \cup (11,12) \\ & \qquad \cup (12,13) \cup (13,21) \cup (21,23), \\ 7 & \text{if } x = 7, 23, \\ 12 & \text{if } x = 10, \\ 11 & \text{if } x = 11, 13, \\ 11.5 & \text{if } x = 12, \\ 10 & \text{if } x = 21, \end{cases}$$

$$gx = \begin{cases} 2 & \text{if } x \in \{2\} \cup (5,10) \cup (10,11) \cup (11,12) \cup (12,13) \\ & \qquad \cup (13,21) \cup (21,22) \cup (22,23), \\ 6 & \text{if } x \in (2,5], \\ 11 & \text{if } x = 10, 11, 13, 22, \\ 11.6 & \text{if } x = 12, \\ 10 & \text{if } x = 21, \\ 7 & \text{if } x = 23. \end{cases}$$

In this example the pair (f,g) is noncompatible as well as reciprocal continuous and satisfies all the conditions of Corollary 2.2. Let us consider the sequence $x_n = 23$, then $fx_n \to 7$, $gx_n \to 7$ and

$$fgx_n \to 7 = f(7),$$
$$gfx_n \to 2 = g(7),$$

therefore $\lim_n d(fgx_n, gfx_n) \neq 0$, and so (f,g) is noncompatible. Here, 2 and 11 are two common fixed points of f and g.

Finally, we present an example which shows that the requirement of conditionally sequential absorbing property is necessary for producing common fixed points of mappings satisfying non-expansive or Lipschitz-type conditions besides exhibiting the limitations of commuting properties of the pairs utilized in earlier related results of Pant and Bisht [34], Pant and Pant [33] and Jungck and Rhoades [17].

Example 2.11 Let $X = [2,20]$ endowed with the usual metric d and $f,g : X \to X$ by

$$fx = \begin{cases} 6 & \text{if } 2 \le x < 6 \text{ or } x > 6, \\ \frac{13}{2} & \text{if } x = 6, \end{cases} \qquad gx = \begin{cases} 5 & \text{if } 2 \le x \le 5, \\ \frac{x+7}{2} & \text{if } 5 < x \le 6, \\ 10 & \text{if } 6 < x < 13/2 \text{ or } x > 13/2, \\ 6 & \text{if } x = 13/2. \end{cases}$$

Then by a routine calculation, it can be verified that $\overline{f(X)} \subseteq g(X)$ and $d(fx,fy) \le kd(gx,gy)$ for all $x,y \in X$, where $k \ge 0$. Also, f and g are a noncompatible and weakly commuting (and hence occasionally weakly compatible and conditionally commuting) pair. In order to show that (f,g) is noncompatible, the sequence $x_n = 5 + 1/n$; $n > 1$, $n \in N$ satisfies the requirements. Also, it is straightforward to verify that the pair (f,g) is pseudo-compatible as well as pseudo-reciprocal continuous (w.r.t. pseudo-compatible), but the pair is not conditionally sequential absorbing in respect of $x_n = 6$ or $5 + 1/n$. On the other hand, at $x = 6$, it can be verified that the mappings f and g do not satisfy any one of the conditions described by (i), (ii), (iii) or (iv) mentioned earlier. Notice that the estimated pair has no common fixed point.

Observations The following definitions can be considered as variants of conditionally sequential absorbing. Two self-mappings f and g of a metric space (X,d) are called conditionally sequential absorbing

(i) *of type* (A) *if, whenever the set of sequences* $\{x_n\}$ *satisfying* $\lim_n fx_n = \lim_n gx_n$ *is nonempty, there exists a sequence* $\{y_n\}$ *satisfying* $\lim_n fy_n = \lim_n gy_n = t$ *(say) such that* $\lim_n d(fy_n, ffy_n) = 0$ *and* $\lim_n d(gy_n, gfy_n) = 0$*;*

(ii) *of type* (B) *if, whenever the set of sequences* $\{x_n\}$ *satisfying* $\lim_n fx_n = \lim_n gx_n$ *is nonempty, there exists a sequence* $\{y_n\}$ *satisfying* $\lim_n fy_n = \lim_n gy_n = t$ *(say) such that* $\lim_n d(fy_n, fgy_n) = 0$ *and* $\lim_n d(gy_n, ggy_n) = 0$*;*

(iii) *of type* (C) *if, whenever the set of sequences* $\{x_n\}$ *satisfying* $\lim_n fx_n = \lim_n gx_n$ *is nonempty, there exists a sequence* $\{y_n\}$ *satisfying* $\lim_n fy_n = \lim_n gy_n = t$ *(say) such that* $\lim_n d(fy_n, ffy_n) = 0$ *and* $\lim_n d(gy_n, ggy_n) = 0$.

We can have some more variants by interchanging the place of f and g. In respect of these variants, we can also define the corresponding pseudo-reciprocal continuity, for example, *two self-mappings* f *and* g *of a metric space* (X, d) *are called pseudo-reciprocal continuous of type (A) if, whenever the set of sequences* $\{x_n\}$ *satisfying* $\lim_n fx_n = \lim_n gx_n$ *is nonempty, there exists a sequence* $\{y_n\}$ (*satisfying* $\lim_n fy_n = \lim_n gy_n = t$ *(say),* $\lim_n d(fy_n, ffy_n) = 0$ *and* $\lim_n d(gy_n, gfy_n) = 0$) *such that* $\lim_n ffy_n = ft$ *and* $\lim_n gfy_n = gt$.

Remark 2.2 The conclusion of our previous results will remain true if we replace the conditionally sequential absorbing and pseudo-reciprocal continuity by any one of the above variants of conditionally sequential absorbing and corresponding pseudo-reciprocal continuity. However, in the context of a unique coincidence or common fixed point, all these variants coincide with each others.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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