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The convergence analysis of the projection methods for a system of generalized relaxed cocoercive variational inequalities in Hilbert spaces

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Abstract

In this paper, the approximate solvability for a system of generalized relaxed cocoercive nonlinear variational inequalities in Hilbert spaces is studied, based on the convergence of the projection methods. The results presented in this paper extend and improve the main results of Refs. (Verma in Comput. Math. Appl. 41:1025-1031, 2001; Verma in Int. J. Differ. Equ. Appl. 6:359-367, 2002; Verma in J. Optim. Theory Appl. 121(1):203-210, 2004; Verma in Appl. Math. Lett. 18(11):1286-1292, 2005; Chang, Lee and Chan in Appl. Math. Lett. 20:329-334, 2007).

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1 Introduction

Variational inequalities are one of the most interesting and intensively studied classes of mathematical problems and there exists a considerable amount of literature [1–15] on the approximate solvability of nonlinear variational inequalities. In this paper, we consider, based on the projection methods, the approximate solvability for a system of generalized relaxed cocoercive nonlinear variational inequalities in Hilbert spaces. The results presented in this paper extend and improve the main results in [1–5].

Throughout this paper, we assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and let P_C be the metric projection of H onto C. Let $T_i : C \times C \to H$, $g_i : C \to C$ and $f_i : C \to H$ be relaxed cocoercive mappings for each $i = 1, 2$. We consider a system of generalized nonlinear variational inequality (SGNVI) problem as follows: find an element $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda T_1(y^*, x^*) + g_1(x^*) - f_1(y^*), x - g_1(x^*)\rangle \geq 0, & \forall x \in C \text{ and } \lambda > 0, \\ \langle \mu T_2(x^*, y^*) + g_2(y^*) - f_2(x^*), x - g_2(y^*)\rangle \geq 0, & \forall x \in C \text{ and } \mu > 0. \end{cases} \tag{1.1}$$

SGNVI problem (1.1) is equivalent to the following projection problem:

$$\begin{cases} g_1(x^*) = P_C(f_1(y^*) - \lambda T_1(y^*, x^*)), & \forall \lambda > 0, \\ g_2(y^*) = P_C(f_2(x^*) - \mu T_2(x^*, y^*)), & \forall \mu > 0. \end{cases} \tag{1.2}$$

Next we consider some special cases of SGNVI problem (1.1), where I is the identity mapping.

(1) If $g_1 = g_2 = I$, then SGNVI problem (1.1) is reduced to the following: find an element $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda T_1(y^*, x^*) + x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in C \text{ and } \lambda > 0, \\ \langle \mu T_2(x^*, y^*) + y^* - f_2(x^*), x - y^* \rangle \geq 0, & \forall x \in C \text{ and } \mu > 0. \end{cases} \tag{1.3}$$

(2) If $f_1 = f_2 = I$, then SGNVI problem (1.1) is reduced to the following: find an element $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda T_1(y^*, x^*) + g_1(x^*) - y^*, x - g_1(x^*) \rangle \geq 0, & \forall x \in C \text{ and } \lambda > 0, \\ \langle \mu T_2(x^*, y^*) + g_2(y^*) - x^*, x - g_2(y^*) \rangle \geq 0, & \forall x \in C \text{ and } \mu > 0. \end{cases} \tag{1.4}$$

(3) If $g_1 = g_2 = f_1 = f_2 = I$, then SGNVI problem (1.1) is reduced to the following: find an element $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda T_1(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \text{ and } \lambda > 0, \\ \langle \mu T_2(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C \text{ and } \mu > 0. \end{cases} \tag{1.5}$$

(4) If T_1 and T_2 are univariate mappings, then SGNVI problem (1.1) is reduced to the following: find an element $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda T_1(y^*) + g_1(x^*) - f_1(y^*), x - g_1(x^*) \rangle \geq 0, & \forall x \in C \text{ and } \lambda > 0, \\ \langle \mu T_2(x^*) + g_2(y^*) - f_2(x^*), x - g_2(y^*) \rangle \geq 0, & \forall x \in C \text{ and } \mu > 0. \end{cases} \tag{1.6}$$

(5) If $g_1 = g_2 = f_1 = f_2 = I$, T_1 and T_2 are univariate mappings, then SGNVI problem (1.1) is reduced to the following: find an element $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda T_1(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \text{ and } \lambda > 0, \\ \langle \mu T_2(x^*) + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C \text{ and } \mu > 0. \end{cases} \tag{1.7}$$

(6) If $g_1 = g_2 = f_1 = f_2 = I$ and $\mu = 0$, then SGNVI problem (1.1) is reduced to the following: find an element $x^* \in C$ such that

$$\bigl\langle T_1\bigl(x^*, x^*\bigr), x - x^* \bigr\rangle \geq 0, \quad \forall x \in C. \tag{1.8}$$

2 Preliminaries

In order to prove our main results in the next section, we recall several definitions and lemmas.

Definition 2.1 Let $T : C \to H$ be a mapping.

(1) T is said to be β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \leq \beta \|x - y\|, \quad \forall x, y \in C.$$

(2) T is said to be monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(3) T is said to be δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in C.$$

This implies that

$$\|Tx - Ty\| \geq \delta \|x - y\|, \quad \forall x, y \in C,$$

that is, T is δ-expansive.

(4) T is said to be γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Clearly, every γ-cocoercive mapping T is $\frac{1}{\gamma}$-Lipschitz continuous.

(5) T is said to be relaxed γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma) \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(6) T is said to be relaxed (γ, δ)-cocoercive if there exist two constants $\gamma, \delta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma) \|Tx - Ty\|^2 + \delta \|x - y\|^2, \quad \forall x, y \in C.$$

Definition 2.2 A mapping $T : C \times C \to H$ is said to be relaxed (γ, δ)-cocoercive if there exist two constants $\gamma, \delta > 0$ such that for all $x, x^* \in C$,

$$\langle T(x, y) - T(x^*, y^*), x - x^*\rangle \geq (-\gamma) \|T(x, y) - T(x^*, y^*)\|^2 + \delta \|x - x^*\|^2, \quad \forall y, y^* \in C.$$

Definition 2.3 A mapping $T : C \times C \to H$ is said to be β-Lipschitz continuous in the first variable if there exists a constant $\beta > 0$ such that for all $x, x^* \in C$,

$$\|T(x, y) - T(x^*, y^*)\| \leq \beta \|x - x^*\|, \quad \forall y, y^* \in C.$$

Definition 2.4 $P_C : H \to C$ is called a metric projection if for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Lemma 2.1 *$P_C : H \to C$ is a metric projection, then P_C is a nonexpansive mapping, i.e.,*

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Lemma 2.2 [5] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences such that*

$$a_{n+1} \le (1-\lambda_n)a_n + b_n + c_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we present the projection methods and give the convergence analysis of SGNVI problem (1.1) involving relaxed (γ,δ)-cocoercive and β-Lipschitz continuous mappings in Hilbert spaces.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \times C \to H$ be a relaxed (γ_i,δ_i)-cocoercive and α_i-Lipschitz continuous mapping in the first variable, let $g_i : C \to C$ be a relaxed (η_i,ρ_i)-cocoercive and β_i-Lipschitz continuous mapping, and let $f_i : C \to H$ be a relaxed $(\overline{\eta}_i,\overline{\rho}_i)$-cocoercive and $\overline{\beta}_i$-Lipschitz continuous mapping for each $i = 1,2$. Suppose that $(x^*,y^*) \in C \times C$ is a solution to SGNVI problem* (1.1). *For any $(x_0,y_0) \in C \times C$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by*

$$\begin{cases} t_{n+1} = (1-a_n)x_n + a_n(x_n - g_1(x_n) + P_C(f_1(y_n) - \lambda T_1(y_n,x_n))), & n \ge 0, \\ x_{n+1} = P_C t_{n+1}, & n \ge 0, \\ z_n = y_n - g_2(y_n) + P_C(f_2(x_n) - \mu T_2(x_n,y_n)), & n \ge 1, \\ y_n = (1-b_n)x_n + b_n P_C z_n, & n \ge 1, \end{cases} \tag{3.1}$$

where $\lambda, \mu > 0$ and the following conditions are satisfied:

$$(1)\quad 0 < a_n, b_n \le 1; \qquad (2)\quad \sum_{n=0}^{\infty} a_n = \infty; \qquad (3)\quad \sum_{n=1}^{\infty}(1-b_n) < \infty;$$

$$(4)\quad 0 \le \theta_2, \theta_5 < 1; \qquad (5)\quad \theta_4 + \theta_6 \ge 1;$$

$$(6)\quad 0 < (\theta_1+\theta_3)(\theta_4+\theta_6) < (1-\theta_2)(1-\theta_5);$$

where

$$\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}, \qquad \theta_2 = \sqrt{1 - 2\rho_1 + \beta_1^2 + 2\eta_1\beta_1^2},$$

$$\theta_3 = \sqrt{1 - 2\overline{\rho}_1 + \overline{\beta}_1^2 + 2\overline{\eta}_1\overline{\beta}_1^2}, \qquad \theta_4 = \sqrt{1 - 2\mu\delta_2 + 2\mu\gamma_2\alpha_2^2 + \mu^2\alpha_2^2},$$

$$\theta_5 = \sqrt{1 - 2\rho_2 + \beta_2^2 + 2\eta_2\beta_2^2}, \qquad \theta_6 = \sqrt{1 - 2\overline{\rho}_2 + \overline{\beta}_2^2 + 2\overline{\eta}_2\overline{\beta}_2^2}.$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Proof Since $(x^*,y^*) \in C \times C$ is a solution to SGNVI problem (1.1), it follows that

$$\begin{cases} x^* = x^* - g_1(x^*) + P_C(f_1(y^*) - \lambda T_1(y^*,x^*)), \\ y^* = y^* - g_2(y^*) + P_C(f_2(x^*) - \mu T_2(x^*,y^*)). \end{cases}$$

For $n \geq 1$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P_C t_{n+1} - P_C x^*\| \leq \|t_{n+1} - x^*\| \\
&= \|(1-a_n)x_n + a_n(x_n - g_1(x_n) + P_C(f_1(y_n) - \lambda T_1(y_n, x_n))) - x^*\| \\
&\leq (1-a_n)\|x_n - x^*\| + a_n\|(x_n - g_1(x_n) + P_C(f_1(y_n) - \lambda T_1(y_n, x_n))) \\
&\quad - (x^* - g_1(x^*) + P_C(f_1(y^*) - \lambda T_1(y^*, x^*)))\| \\
&\leq (1-a_n)\|x_n - x^*\| + a_n\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \\
&\quad + a_n\|P_C(f_1(y_n) - \lambda T_1(y_n, x_n)) - P_C(f_1(y^*) - \lambda T_1(y^*, x^*))\| \\
&\leq (1-a_n)\|x_n - x^*\| + a_n\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \\
&\quad + a_n\|f_1(y_n) - f_1(y^*) - \lambda(T_1(y_n, x_n) - T_1(y^*, x^*))\| \\
&\leq (1-a_n)\|x_n - x^*\| + a_n\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \\
&\quad + a_n\|y_n - y^* - (f_1(y_n) - f_1(y^*))\| \\
&\quad + a_n\|y_n - y^* - \lambda(T_1(y_n, x_n) - T_1(y^*, x^*))\|.
\end{aligned} \tag{3.2}$$

Since g_1 is a relaxed (η_1, ρ_1)-cocoercive and β_1-Lipschitz continuous mapping, we have

$$\begin{aligned}
&\|x_n - x^* - (g_1(x_n) - g_1(x^*))\|^2 \\
&\quad = \|x_n - x^*\|^2 - 2\langle g_1(x_n) - g_1(x^*), x_n - x^*\rangle + \|g_1(x_n) - g_1(x^*)\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - 2((-\eta_1)\|g_1(x_n) - g_1(x^*)\|^2 + \rho_1\|x_n - x^*\|^2) + \beta_1^2\|x_n - x^*\|^2 \\
&\quad = (1 - 2\rho_1 + \beta_1^2)\|x_n - x^*\|^2 + 2\eta_1\|g_1(x_n) - g_1(x^*)\|^2 \\
&\quad \leq (1 - 2\rho_1 + \beta_1^2)\|x_n - x^*\|^2 + 2\eta_1\beta_1^2\|x_n - x^*\|^2 \\
&\quad \leq \theta_2^2\|x_n - x^*\|^2,
\end{aligned} \tag{3.3}$$

where $\theta_2 = \sqrt{1 - 2\rho_1 + \beta_1^2 + 2\eta_1\beta_1^2}$. In a similar way, we can obtain that

$$\|y_n - y^* - (f_1(y_n) - f_1(y^*))\| \leq \theta_3\|y_n - y^*\|, \tag{3.4}$$

where $\theta_3 = \sqrt{1 - 2\overline{\rho}_1 + \overline{\beta}_1^2 + 2\overline{\eta}_1\overline{\beta}_1^2}$. By the assumption that T_1 is a relaxed (γ_1, δ_1)-cocoercive and α_1-Lipschitz continuous mapping in the first variable, we have

$$\begin{aligned}
&\|y_n - y^* - \lambda(T_1(y_n, x_n) - T_1(y^*, x^*))\|^2 \\
&\quad = \|y_n - y^*\|^2 - 2\lambda\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^*\rangle + \lambda^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
&\quad \leq \|y_n - y^*\|^2 - 2\lambda((-\gamma_1)\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 + \delta_1\|y_n - y^*\|^2) \\
&\qquad + \lambda^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
&\quad = (1 - 2\lambda\delta_1)\|y_n - y^*\|^2 + (2\lambda\gamma_1 + \lambda^2)\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
&\quad \leq (1 - 2\lambda\delta_1)\|y_n - y^*\|^2 + (2\lambda\gamma_1 + \lambda^2)\alpha_1^2\|y_n - y^*\|^2 \\
&\quad = \theta_1^2\|y_n - y^*\|^2,
\end{aligned} \tag{3.5}$$

where $\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}$. According to (3.2), (3.3), (3.4) and (3.5), we obtain that

$$\|x_{n+1} - x^*\| \leq \left[1 - a_n(1 - \theta_2)\right]\|x_n - x^*\| + a_n(\theta_1 + \theta_3)\|y_n - y^*\|. \tag{3.6}$$

Next, we estimate $\|y_n - y^*\|$. From (3.1), we see that

$$\begin{aligned}
\|y_n - y^*\| &= \|(1 - b_n)x_n + b_n P_C z_n - y^*\| \\
&\leq (1 - b_n)\|x_n - y^*\| + b_n\|P_C z_n - P_C y^*\| \\
&\leq (1 - b_n)\|x_n - y^*\| + b_n\|z_n - y^*\| \\
&= (1 - b_n)\|x_n - y^*\| + b_n\|y_n - g_2(y_n) + P_C(f_2(x_n) - \mu T_2(x_n, y_n)) \\
&\quad - (y^* - g_2(y^*) + P_C(f_2(x^*) - \mu T_2(x^*, y^*)))\| \\
&\leq (1 - b_n)\|x_n - y^*\| + b_n\|y_n - y^* - (g_2(y_n) - g_2(y^*))\| \\
&\quad + b_n\|P_C(f_2(x_n) - \mu T_2(x_n, y_n)) - P_C(f_2(x^*) - \mu T_2(x^*, y^*))\| \\
&\leq (1 - b_n)\|x_n - y^*\| + b_n\|y_n - y^* - (g_2(y_n) - g_2(y^*))\| \\
&\quad + b_n\|x_n - x^* - (f_2(x_n) - f_2(x^*))\| \\
&\quad + b_n\|x_n - x^* - \mu(T_2(x_n, y_n) - T_2(x^*, y^*))\|.
\end{aligned} \tag{3.7}$$

Similarly, we obtain that

$$\|y_n - y^* - (g_2(y_n) - g_2(y^*))\| \leq \theta_5\|y_n - y^*\|, \tag{3.8}$$

where $\theta_5 = \sqrt{1 - 2\rho_2 + \beta_2^2 + 2\eta_2\beta_2^2}$, and

$$\|x_n - x^* - (f_2(x_n) - f_2(x^*))\| \leq \theta_6\|x_n - x^*\|, \tag{3.9}$$

where $\theta_6 = \sqrt{1 - 2\overline{\rho}_2 + \overline{\beta}_2^2 + 2\overline{\eta}_2\overline{\beta}_2^2}$, and

$$\|x_n - x^* - \mu(T_2(x_n, y_n) - T_2(x^*, y^*))\| \leq \theta_4\|x_n - x^*\|, \tag{3.10}$$

where $\theta_4 = \sqrt{1 - 2\mu\delta_2 + 2\mu\gamma_2\alpha_2^2 + \mu^2\alpha_2^2}$. According to (3.7), (3.8), (3.9) and (3.10), we obtain that

$$\begin{aligned}
\|y_n - y^*\| &\leq (1 - b_n)\|x_n - y^*\| + b_n\theta_5\|y_n - y^*\| \\
&\quad + b_n\theta_6\|x_n - x^*\| + b_n\theta_4\|x_n - x^*\| \\
&\leq (1 - b_n)\|x_n - x^*\| + (1 - b_n)\|x^* - y^*\| \\
&\quad + b_n\theta_5\|y_n - y^*\| + b_n(\theta_4 + \theta_6)\|x_n - x^*\|,
\end{aligned} \tag{3.11}$$

that is,

$$(1 - b_n\theta_5)\|y_n - y^*\| \leq (1 - b_n)\|x^* - y^*\| + \left[1 - b_n + b_n(\theta_4 + \theta_6)\right]\|x_n - x^*\|. \tag{3.12}$$

By conditions (1), (4) and (5), we have

$$\frac{1}{1-b_n\theta_5} \le \frac{1}{1-\theta_5} \quad \text{and} \quad 1-b_n+b_n(\theta_4+\theta_6) \le \theta_4+\theta_6. \tag{3.13}$$

Substituting (3.13) into (3.12), we have

$$\|y_n-y^*\| \le \frac{1-b_n}{1-\theta_5}\|x^*-y^*\| + \frac{\theta_4+\theta_6}{1-\theta_5}\|x_n-x^*\|. \tag{3.14}$$

According (3.6) and (3.14), for $n \ge 1$, we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le \left[1-a_n\left(1-\theta_2-\frac{(\theta_1+\theta_3)(\theta_4+\theta_6)}{1-\theta_5}\right)\right]\|x_n-x^*\| \\
&\quad + a_n\frac{(\theta_1+\theta_3)}{1-\theta_5}(1-b_n)\|x^*-y^*\| \\
&\le \left[1-a_n\left(1-\theta_2-\frac{(\theta_1+\theta_3)(\theta_4+\theta_6)}{1-\theta_5}\right)\right]\|x_n-x^*\| \\
&\quad + \frac{(\theta_1+\theta_3)}{1-\theta_5}(1-b_n)\|x^*-y^*\|.
\end{aligned} \tag{3.15}$$

From conditions (1), (2), (3) and (6), we get

$$a_n\left(1-\theta_2-\frac{(\theta_1+\theta_3)(\theta_4+\theta_6)}{1-\theta_5}\right) \in (0,1), \qquad \sum_{n=1}^{\infty} a_n\left(1-\theta_2-\frac{(\theta_1+\theta_3)(\theta_4+\theta_6)}{1-\theta_5}\right) = \infty$$

and

$$\sum_{n=1}^{\infty}\frac{(\theta_1+\theta_3)}{1-\theta_5}(1-b_n)\|x^*-y^*\| < \infty.$$

The conditions in Lemma 2.2 are satisfied, then $\|x_n-x^*\| \to 0$ (as $n\to\infty$), *i.e.*, $x_n \to x^*$ (as $n\to\infty$).

On the one hand, from condition (3) we know that $1-b_n \to 0$ (as $n\to\infty$). On the other hand, from (3.14) and the result that $x_n \to x^*$ (as $n\to\infty$), we can get $\|y_n-y^*\| \to 0$ (as $n\to\infty$), *i.e.*, $y_n \to y^*$ (as $n\to\infty$).

This completes the proof of Theorem 3.1. □

When $g_1=g_2=I, f_1=f_2=I$, then we have $\theta_2=\theta_5=0$, $\theta_3=\theta_6=0$, respectively. And from Theorem 3.1 we can get the following results immediately.

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C\times C \to H$ be a relaxed (γ_i,δ_i)-cocoercive and α_i-Lipschitz continuous mapping in the first variable, and let $f_i : C \to H$ be a relaxed $(\overline{\eta}_i,\overline{\rho}_i)$-cocoercive and $\overline{\beta}_i$-Lipschitz continuous mapping for each $i=1,2$. Suppose that $(x^*,y^*)\in C\times C$ is a solution to problem* (1.3). *For any $(x_0,y_0)\in C\times C$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by*

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_nP_C(f_1(y_n)-\lambda T_1(y_n,x_n)), & n\ge 0, \\ y_n = (1-b_n)x_n + b_nP_C(f_2(x_n)-\mu T_2(x_n,y_n)), & n\ge 1, \end{cases} \tag{3.16}$$

where $\lambda, \mu > 0$ *and the following conditions are satisfied*:

$$(1)\quad 0 < a_n, b_n \leq 1; \qquad (2)\quad \sum_{n=0}^{\infty} a_n = \infty; \qquad (3)\quad \sum_{n=1}^{\infty}(1 - b_n) < \infty;$$

$$(4)\quad \theta_4 + \theta_6 \geq 1; \qquad (5)\quad 0 < (\theta_1 + \theta_3)(\theta_4 + \theta_6) < 1;$$

where

$$\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}, \qquad \theta_3 = \sqrt{1 - 2\overline{\rho}_1 + \overline{\beta}_1^2 + 2\overline{\eta}_1\overline{\beta}_1^2},$$

$$\theta_4 = \sqrt{1 - 2\mu\delta_2 + 2\mu\gamma_2\alpha_2^2 + \mu^2\alpha_2^2}, \qquad \theta_6 = \sqrt{1 - 2\overline{\rho}_2 + \overline{\beta}_2^2 + 2\overline{\eta}_2\overline{\beta}_2^2}.$$

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* x^* *and* y^*, *respectively.*

Corollary 3.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T_i : C \times C \to H$ *be a relaxed* (γ_i, δ_i)*-cocoercive and* α_i*-Lipschitz continuous mapping in the first variable, and let* $g_i : C \to C$ *be a relaxed* (η_i, ρ_i)*-cocoercive and* β_i*-Lipschitz continuous mapping for each* $i = 1, 2$. *Suppose that* $(x^*, y^*) \in C \times C$ *is a solution to problem* (1.4). *For any* $(x_0, y_0) \in C \times C$, *the sequences* $\{x_n\}$ *and* $\{y_n\}$ *are generated by*

$$\begin{cases} t_{n+1} = (1 - a_n)x_n + a_n(x_n - g_1(x_n) + P_C(y_n - \lambda T_1(y_n, x_n))), \quad n \geq 0, \\ x_{n+1} = P_C t_{n+1}, \quad n \geq 0, \\ z_n = y_n - g_2(y_n) + P_C(x_n - \mu T_2(x_n, y_n)), \quad n \geq 1, \\ y_n = (1 - b_n)x_n + b_n P_C z_n, \quad n \geq 1, \end{cases} \tag{3.17}$$

where $\lambda, \mu > 0$ *and the following conditions are satisfied*:

$$(1)\quad 0 < a_n, b_n \leq 1; \qquad (2)\quad \sum_{n=0}^{\infty} a_n = \infty; \qquad (3)\quad \sum_{n=1}^{\infty}(1 - b_n) < \infty;$$

$$(4)\quad 0 \leq \theta_2, \theta_5 < 1; \qquad (5)\quad \theta_4 \geq 1; \qquad (6)\quad 0 < \theta_1\theta_4 < (1 - \theta_2)(1 - \theta_5);$$

where

$$\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}, \qquad \theta_2 = \sqrt{1 - 2\rho_1 + \beta_1^2 + 2\eta_1\beta_1^2},$$

$$\theta_4 = \sqrt{1 - 2\mu\delta_2 + 2\mu\gamma_2\alpha_2^2 + \mu^2\alpha_2^2}, \qquad \theta_5 = \sqrt{1 - 2\rho_2 + \beta_2^2 + 2\eta_2\beta_2^2}.$$

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* x^* *and* y^*, *respectively.*

Corollary 3.3 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T_i : C \times C \to H$ *be a relaxed* (γ_i, δ_i)*-cocoercive and* α_i*-Lipschitz continuous mapping in the first variable for each* $i = 1, 2$. *Suppose that* $(x^*, y^*) \in C \times C$ *is a solution to problem* (1.5). *For any* $(x_0, y_0) \in C \times C$, *the sequences* $\{x_n\}$ *and* $\{y_n\}$ *are generated by*

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n P_C(y_n - \lambda T_1(y_n, x_n)), \quad n \geq 0, \\ y_n = (1 - b_n)x_n + b_n P_C(x_n - \mu T_2(x_n, y_n)), \quad n \geq 1, \end{cases} \tag{3.18}$$

where $\lambda, \mu > 0$ *and the following conditions are satisfied*:

$$(1)\quad 0 < a_n, b_n \leq 1; \qquad (2)\quad \sum_{n=0}^{\infty} a_n = \infty; \qquad (3)\quad \sum_{n=1}^{\infty} (1 - b_n) < \infty;$$

$$(4)\quad \theta_4 \geq 1; \qquad (5)\quad 0 < \theta_1\theta_4 < 1;$$

where

$$\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}, \qquad \theta_4 = \sqrt{1 - 2\mu\delta_2 + 2\mu\gamma_2\alpha_2^2 + \mu^2\alpha_2^2}.$$

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* x^* *and* y^*, *respectively.*

Corollary 3.4 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T_i : C \to H$ *be a relaxed* (γ_i, δ_i)-*cocoercive and* α_i-*Lipschitz continuous mapping, let* $g_i : C \to C$ *be a relaxed* (η_i, ρ_i)-*cocoercive and* β_i-*Lipschitz continuous mapping, and let* $f_i : C \to H$ *be a relaxed* $(\overline{\eta}_i, \overline{\rho}_i)$-*cocoercive and* $\overline{\beta}_i$-*Lipschitz continuous mapping for each* $i = 1, 2$. *Suppose that* $(x^*, y^*) \in C \times C$ *is a solution to problem* (1.6). *For any* $(x_0, y_0) \in C \times C$, *the sequences* $\{x_n\}$ *and* $\{y_n\}$ *are generated by*

$$\begin{cases} t_{n+1} = (1 - a_n)x_n + a_n(x_n - g_1(x_n) + P_C(f_1(y_n) - \lambda T_1(y_n))), \quad n \geq 0, \\ x_{n+1} = P_C t_{n+1}, \quad n \geq 0, \\ z_n = y_n - g_2(y_n) + P_C(f_2(x_n) - \mu T_2(x_n)), \quad n \geq 1, \\ y_n = (1 - b_n)x_n + b_n P_C z_n, \quad n \geq 1, \end{cases} \tag{3.19}$$

where $\lambda, \mu > 0$ *and the following conditions are satisfied*:

$$(1)\quad 0 < a_n, b_n \leq 1; \qquad (2)\quad \sum_{n=0}^{\infty} a_n = \infty; \qquad (3)\quad \sum_{n=1}^{\infty} (1 - b_n) < \infty;$$

$$(4)\quad 0 \leq \theta_2, \theta_5 < 1; \qquad (5)\quad \theta_4 + \theta_6 \geq 1;$$

$$(6)\quad 0 < (\theta_1 + \theta_3)(\theta_4 + \theta_6) < (1 - \theta_2)(1 - \theta_5);$$

where

$$\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}, \qquad \theta_2 = \sqrt{1 - 2\rho_1 + \beta_1^2 + 2\eta_1\beta_1^2},$$
$$\theta_3 = \sqrt{1 - 2\overline{\rho}_1 + \overline{\beta}_1^2 + 2\overline{\eta}_1\overline{\beta}_1^2}, \qquad \theta_4 = \sqrt{1 - 2\mu\delta_2 + 2\mu\gamma_2\alpha_2^2 + \mu^2\alpha_2^2},$$
$$\theta_5 = \sqrt{1 - 2\rho_2 + \beta_2^2 + 2\eta_2\beta_2^2}, \qquad \theta_6 = \sqrt{1 - 2\overline{\rho}_2 + \overline{\beta}_2^2 + 2\overline{\eta}_2\overline{\beta}_2^2}.$$

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* x^* *and* y^*, *respectively.*

Corollary 3.5 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T_i : C \to H$ *be a relaxed* (γ_i, δ_i)-*cocoercive and* α_i-*Lipschitz continuous mapping for each* $i = 1, 2$. *Suppose that* $(x^*, y^*) \in C \times C$ *is a solution to problem* (1.7). *For any* $(x_0, y_0) \in C \times C$, *the sequences* $\{x_n\}$ *and* $\{y_n\}$ *are generated by*

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n P_C(y_n - \lambda T_1(y_n)), & n \geq 0, \\ y_n = (1-b_n)x_n + b_n P_C(x_n - \mu T_2(x_n)), & n \geq 1, \end{cases} \tag{3.20}$$

where $\lambda, \mu > 0$ *and the following conditions are satisfied*:

$$(1)\quad 0 < a_n, b_n \leq 1; \qquad (2)\quad \sum_{n=0}^{\infty} a_n = \infty; \qquad (3)\quad \sum_{n=1}^{\infty}(1-b_n) < \infty;$$

$$(4)\quad \theta_4 \geq 1; \qquad (5)\quad 0 < \theta_1\theta_4 < 1;$$

where

$$\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}, \qquad \theta_4 = \sqrt{1 - 2\mu\delta_2 + 2\mu\gamma_2\alpha_2^2 + \mu^2\alpha_2^2}.$$

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* x^* *and* y^*, *respectively.*

For $\mu = 0$, $b_n = 1$ in Corollary 3.3, we arrive at the following result.

Corollary 3.6 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T_1 : C \times C \to H$ *be a relaxed* (γ_1, δ_1)*-cocoercive and* α_1*-Lipschitz continuous mapping in the first variable. Suppose that* $x^* \in C$ *is a solution to problem* (1.8). *For any* $x_0 \in C$, *the sequence* $\{x_n\}$ *is generated by*

$$x_{n+1} = (1-a_n)x_n + a_n P_C\big(x_n - \lambda T_1(x_n, x_n)\big), \quad n \geq 0, \tag{3.21}$$

where $\lambda > 0$ *and the following conditions are satisfied*:

$$(1)\quad 0 < a_n \leq 1; \qquad (2)\quad \sum_{n=0}^{\infty} a_n = \infty; \qquad (3)\quad 0 < \theta_1 < 1;$$

where

$$\theta_1 = \sqrt{1 - 2\lambda\delta_1 + 2\lambda\gamma_1\alpha_1^2 + \lambda^2\alpha_1^2}.$$

Then the sequence $\{x_n\}$ *converges strongly to* x^*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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