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Fixed points and strict fixed points for multivalued contractions of Reich type on metric spaces endowed with a graph

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Abstract

The purpose of this paper is to present some strict fixed point theorems for multivalued operators satisfying a Reich-type condition on a metric space endowed with a graph. The well-posedness of the fixed point problem is also studied.

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Keywords: fixed point; strict fixed point; metric space endowed with a graph; well-posed problem

1 Preliminaries

A new approach in the theory of fixed points was recently given by Jachymski [1] and Gwóźdź-Lukawska and Jachymski [2] by using the context of metric spaces endowed with a graph. Other recent results for single-valued and multivalued operators in such metric spaces are given by Nicolae, O'Regan and Petruşel in [3] and by Beg, Butt and Radojevic in [4].

Let (X,d) be a metric space and let Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_{n\in\{0,1,2,\ldots,k\}}$ of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1,2,\ldots,k\}$. Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges.

Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}. \qquad (*)$$

Since it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have that

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \qquad (**)$$

If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the symbol $[x]_G$ denotes the equivalence class of the relation $\Re$ defined on $V(G)$ by the rule:

$$y \Re z \quad \text{if there is a path in } G \text{ from } y \text{ to } z.$$

Let us consider the following families of subsets of a metric space (X, d):

$$P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}; \qquad P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\};$$
$$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}; \qquad P_{cp}(X) := \{Y \in P(X) \mid Y \text{ is compact}\}.$$

The gap functional between the sets A and B in the metric space (X, d) is given by

$$D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \qquad D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

The Pompeiu-Hausdorff functional is defined by

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$
$$H(A, B) = \max\Big\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(A, b)\Big\}.$$

The diameter generalized functional generated by d is given by

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$
$$\delta(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}.$$

In particular, we denote by $\delta(A) := \delta(A, A)$ the diameter of the set A.

Let (X, d) be a metric space. If $T : X \to P(X)$ is a multivalued operator, then $x \in X$ is called a fixed point for T if and only if $x \in T(x)$. The set $\mathrm{Fix}(T) := \{x \in X \mid x \in T(x)\}$ is called the fixed point set of T, while $S\mathrm{Fix}(T) = \{x \in X \mid \{x\} = Tx\}$ is called the strict fixed point set of T. $\mathrm{Graph}(T) := \{(x, y) \mid y \in T(x)\}$ denotes the graph of T.

Definition 1.1 Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping. Then φ is called a strong comparison function if the following assertions hold:

(i) φ is increasing;
(ii) $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t \in \mathbb{R}_+$;
(iii) $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t \in \mathbb{R}_+$.

Definition 1.2 Let (X, d) be a complete metric space, let G be a directed graph, and let $T : X \to P_b(X)$ be a multivalued operator. By definition, T is called a (δ, φ)-G-contraction if there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, a strong comparison function, such that

$$\delta\big(T(x), T(y)\big) \leq \varphi\big(d(x, y)\big) \quad \text{for all } (x, y) \in E(G).$$

In this paper, we present some fixed point and strict fixed point theorems for multivalued operators satisfying a contractive condition of Reich type involving the functional δ (see

[5, 6]). The equality between $\mathrm{Fix}(T)$ and $S\,\mathrm{Fix}(T)$ and the well-posedness of the fixed point problem are also studied.

Our results also generalize and extend some fixed point theorems in partially ordered complete metric spaces given in Harjani and Sadarangani [7], Nicolae *et al.* [3], Nieto and Rodríguez-López [8] and [9], Nieto *et al.* [10], O'Regan and Petruşel [11], Petruşel and Rus [12], and Ran and Reurings [13].

2 Fixed point and strict fixed point theorems

We begin this section by presenting a strict fixed point theorem for a Reich type contraction with respect to the functional δ.

Theorem 2.1 *Let (X,d) be a complete metric space and let G be a directed graph such that the triple (X,d,G) satisfies the following property*:

$$(P)\quad \begin{array}{l}\text{for any sequence } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } x_n \to x \text{ as } n \to \infty,\\ \text{there exists a subsequence } (x_{k_n})_{n\in\mathbb{N}} \text{ of } (x_n)_{n\in\mathbb{N}} \text{ such that } (x_{k_n},x) \in E(G).\end{array}$$

Let $T : X \to P_{\mathrm{b}}(X)$ be a multivalued operator. Suppose that the following assertions hold:

(i) *There exists $a, b, c \in \mathbb{R}_+$ with $b \neq 0$ and $a + b + c < 1$ such that*

$$\delta\big(T(x), T(y)\big) \le ad(x,y) + b\delta\big(x, T(x)\big) + c\delta\big(y, T(y)\big)$$

for all $(x,y) \in E(G)$.

(ii) *For each $x \in X$, the set*

$$\tilde{X}_T(x) := \left\{ y \in T(x) : (x,y) \in E(G) \text{ and } \delta\big(x,T(x)\big) \le qd(x,y) \right.$$
$$\left. \text{for some } q \in \left]1, \frac{1-a-c}{b}\right[\right\}$$

is nonempty.

Then we have:

(a) $\mathrm{Fix}(T) = S\,\mathrm{Fix}(T) \neq \emptyset$;

(b) *If we additionally suppose that*

$$x^*, y^* \in \mathrm{Fix}(T) \quad\Rightarrow\quad \big(x^*, y^*\big) \in E(G),$$

then $\mathrm{Fix}(T) = S\,\mathrm{Fix}(T) = \{x^*\}$.

Proof (a) Let $x_0 \in X$. Since $\tilde{X}_T(x_0) \neq \emptyset$, there exists $x_1 \in T(x_0)$ and $1 < q < \frac{1-a-c}{b}$ such that $(x_0,x_1) \in E(G)$ and

$$\delta\big(x_0, T(x_0)\big) \le qd(x_0,x_1).$$

By (i) we have that

$$\begin{aligned}\delta\big(x_1, T(x_1)\big) &\le \delta\big(T(x_0), T(x_1)\big) \le ad(x_0,x_1) + b\delta\big(x_0, T(x_0)\big) + c\delta\big(x_1, T(x_1)\big)\\ &\le ad(x_0,x_1) + bqd(x_0,x_1) + c\delta\big(x_1, T(x_1)\big).\end{aligned}$$

Hence,

$$\delta\big(x_1, T(x_1)\big) \leq \frac{a + bq}{1 - c} d(x_0, x_1). \tag{2.1}$$

For $x_1 \in X$, since $\tilde{X}_T(x_1) \neq \emptyset$, we get again that there exists $x_2 \in T(x_1)$ such that $\delta(x_1, T(x_1)) \leq qd(x_1, x_2)$ and $(x_1, x_2) \in E(G)$. Then

$$d(x_1, x_2) \leq \delta\big(x_1, T(x_1)\big) \leq \frac{a + bq}{1 - c} d(x_0, x_1). \tag{2.2}$$

On the other hand, by (i), we have that

$$\begin{aligned}\delta\big(x_2, T(x_2)\big) &\leq \delta\big(T(x_1), T(x_2)\big) \leq ad(x_1, x_2) + b\delta\big(x_1, T(x_1)\big) + c\delta\big(x_2, T(x_2)\big) \\ &\leq ad(x_1, x_2) + bqd(x_1, x_2) + c\delta\big(x_2, T(x_2)\big).\end{aligned}$$

Using (2.2) we obtain

$$\delta\big(x_2, T(x_2)\big) \leq \frac{a + bq}{1 - c} d(x_1, x_2) \leq \left(\frac{a + bq}{1 - c}\right)^2 d(x_0, x_1). \tag{2.3}$$

For $x_2 \in X$, we have $\tilde{X}_T(x_2) \neq \emptyset$, and so there exists $x_3 \in T(x_2)$ such that $\delta(x_2, T(x_2)) \leq qd(x_2, x_3)$ and $(x_2, x_3) \in E(G)$.

Then

$$d(x_2, x_3) \leq \delta\big(x_2, T(x_2)\big) \leq \left(\frac{a + bq}{1 - c}\right)^2 d(x_0, x_1). \tag{2.4}$$

By these procedures, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

(1) $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$;

(2) $d(x_n, x_{n+1}) \leq (\frac{a+bq}{1-c})^n d(x_0, x_1)$ for each $n \in \mathbb{N}$;

(3) $\delta(x_n, T(x_n)) \leq (\frac{a+bq}{1-c})^n d(x_0, x_1)$ for each $n \in \mathbb{N}$.

From (2) we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since the metric space X is complete, we get that the sequence is convergent, *i.e.*, $x_n \to x^*$ as $n \to \infty$. By the property (P), there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{k_n}, x^*) \in E(G)$ for each $n \in \mathbb{N}$.

We have

$$\begin{aligned}\delta\big(x^*, T(x^*)\big) &\leq d\big(x^*, x_{k_{n+1}}\big) + \delta\big(x_{k_{n+1}}, T(x^*)\big) \leq d\big(x^*, x_{k_{n+1}}\big) + \delta\big(T(x_{k_n}), T(x^*)\big) \\ &\leq d\big(x^*, x_{k_{n+1}}\big) + ad\big(x_{k_n}, x^*\big) + b\delta\big(x_{k_n}, T(x_{k_n})\big) + c\delta\big(x^*, T(x^*)\big) \\ &\leq d\big(x^*, x_{k_{n+1}}\big) + ad\big(x_{k_n}, x^*\big) + b\left(\frac{a + bq}{1 - c}\right)^{k_n} d(x_0, x_1) + c\delta\big(x^*, T(x^*)\big),\end{aligned}$$

$$\delta\big(x^*, T(x^*)\big) \leq \frac{1}{1 - c} d\big(x^*, x_{k_{n+1}}\big) + \frac{a}{1 - c} d\big(x_{k_n}, x^*\big) + \frac{b}{1 - c}\left(\frac{a + bq}{1 - c}\right)^{k_n} d(x_0, x_1). \tag{2.5}$$

But $d(x^*, x_{k_{n+1}}) \to 0$ as $n \to \infty$ and $d(x_{k_n}, x^*) \to 0$ as $n \to \infty$. Hence, $\delta(x^*, T(x^*)) = 0$, which implies that $x^* \in S\operatorname{Fix}(T)$. Thus $S\operatorname{Fix}(T) \neq \emptyset$.

We shall prove now that $\operatorname{Fix}(T) = S\operatorname{Fix}(T)$.

Because $S\operatorname{Fix}(T) \subset \operatorname{Fix}(T)$, we need to show that $\operatorname{Fix}(T) \subset S\operatorname{Fix}(T)$.

Let $x^* \in \operatorname{Fix}(T) \Rightarrow x^* \in T(x^*)$. Because $\Delta \subset E(G)$, we have that $(x^*, x^*) \in E(G)$. Using (ii) with $x = y = x^*$, we obtain

$$\delta(T(x^*)) \leq ad(x^*, x^*) + b\delta(x^*, T(x^*)) + c\delta(x^*, T(x^*)).$$

So, $\delta(T(x^*)) \leq (b+c)\delta(x^*, T(x^*))$. Because $x^* \in T(x^*)$, we get that $\delta(x^*, T(x^*)) \leq \delta(T(x^*))$. Hence, we have

$$\delta(T(x^*)) \leq (b+c)\delta(T(x^*)). \tag{2.6}$$

Suppose that $\operatorname{card}(T(x^*)) > 1$. This implies that $\delta(T(x^*)) > 0$. Thus from (2.6) we obtain that $b + c > 1$, which contradicts the hypothesis $a + b + c < 1$.

Thus $\delta(T(x^*)) = 0 \Rightarrow T(x^*) = \{x^*\}$, *i.e.*, $x^* \in S\operatorname{Fix}(T)$ and $\operatorname{Fix}(T) \subset S\operatorname{Fix}(T)$.

Hence, $\operatorname{Fix}(T) = S\operatorname{Fix}(T) \neq \emptyset$.

(b) Suppose that there exist $x^*, y^* \in \operatorname{Fix}(T) = S\operatorname{Fix}(T)$ with $x^* \neq y^*$. We have that

- $x^* \in S\operatorname{Fix}(T) \Rightarrow \delta(x^*, T(x^*)) = 0$;
- $y^* \in S\operatorname{Fix}(T) \Rightarrow \delta(y^*, T(y^*)) = 0$;
- $(x^*, y^*) \in E(G)$.

Using (i) we obtain

$$d(x^*, y^*) = \delta(T(x^*), T(y^*)) \leq ad(x^*, y^*) + b\delta(x^*, T(x^*)) + c\delta(y^*, T(y^*)).$$

Thus, $d(x^*, y^*) \leq ad(x^*, y^*)$, which implies that $a \geq 1$, which is a contradiction.

Hence, $\operatorname{Fix}(T) = S\operatorname{Fix}(T) = \{x^*\}$. □

Next we present some examples and counterexamples of multivalued operators which satisfy the hypothesis in Theorem 2.1.

Example 2.1 Let $X := \{(0,0), (0,1), (1,0), (1,1)\}$ and $T : X \to P_{\mathrm{cl}}(X)$ be given by

$$T(x) = \begin{cases} \{(0,0)\}, & x = (0,0), \\ \{(0,0)\}, & x = (0,1), \\ \{(0,0), (0,1)\}, & x = (1,0), \\ \{(0,0), (0,1)\}, & x = (1,1). \end{cases} \tag{2.7}$$

Let $E(G) := \{((0,1), (0,0)), ((1,0), (0,1)), ((1,1), (0,0))\} \cup \Delta$.

Notice that all the hypotheses in Theorem 2.1 are satisfied (the condition (i) is verified for $a = c = 0.01$, $b = 0.97$ and so $\operatorname{Fix}(T) = S\operatorname{Fix}(T) = \{(0,0)\}$.

The following remarks show that it is not possible to have elements in $F_T \setminus SF_T$.

Remark 2.1 If we suppose that there exists $x \in F_T \setminus SF_T$, then, since $(x, x) \in \Delta$, we get (using the condition (i) in the above theorem with $y = x$) that $\delta(T(x)) \leq (b+c)\delta(T(x))$, which is a contradiction with $a + b + c < 1$.

Remark 2.2 If, in the previous theorem, instead of the property (P), we suppose that T has a closed graph, then we obtain again the conclusion $\operatorname{Fix}(T) = S\operatorname{Fix}(T) \neq \emptyset$.

Remark 2.3 If, in the above remark, we additionally suppose that

$$x^*, y^* \in \operatorname{Fix}(T) \quad \Rightarrow \quad (x^*, y^*) \in E(G),$$

then $\operatorname{Fix}(T) = S\operatorname{Fix}(T) = \{x^*\}$.

The next result presents a strict fixed point theorem where the operator T satisfies a (δ, φ)-G-contractive condition on $E(G)$.

Theorem 2.2 *Let (X, d) be a complete metric space and let G be a directed graph such that the triple (X, d, G) satisfies the following property*:

$$(P) \quad \begin{array}{l} \text{for any sequence } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } x_n \to x \text{ as } n \to \infty, \\ \text{there exists a subsequence } (x_{k_n})_{n\in\mathbb{N}} \text{ of } (x_n)_{n\in\mathbb{N}} \text{ such that } (x_{k_n}, x) \in E(G). \end{array}$$

Let $T : X \to P_b(X)$ be a multivalued operator. Suppose that the following assertions hold:

(i) *T is a (δ, φ)-G-contraction.*

(ii) *For each $x \in X$, the set*

$$\tilde{X}_T := \left\{ y \in T(x) : (x, y) \in E(G) \text{ and } \delta\bigl(x, T(x)\bigr) \le q d(x, y) \text{ for some } q \in \left]1, \frac{1-a-c}{b}\right[\right\}$$

is nonempty.

Then we have:

(a) $\operatorname{Fix}(T) = S\operatorname{Fix}(T) \neq \emptyset$;

(b) *If, in addition, the following implication holds*:

$$x^*, y^* \in \operatorname{Fix}(T) \quad \Rightarrow \quad (x^*, y^*) \in E(G),$$

then $\operatorname{Fix}(T) = S\operatorname{Fix}(T) = \{x^*\}$.

Proof (a) Let $x_0 \in X$. Then, since $\tilde{X}_T(x_0)$ is nonempty, there exist $x_1 \in T(x_0)$ and $q \in]1, \frac{1-a-c}{b}[$ such that $(x_0, x_1) \in E(G)$ and

$$\delta\bigl(x_0, T(x_0)\bigr) \le q d(x_0, x_1).$$

By (i) we have that

$$\delta\bigl(x_1, T(x_1)\bigr) \le \delta\bigl(T(x_0), T(x_1)\bigr) \le \varphi\bigl(d(x_0, x_1)\bigr).$$

For $x_1 \in X$, by the same approach as before, there exists $x_2 \in T(x_1)$ such that $\delta(x_1, T(x_1)) \le q d(x_1, x_2)$ and $(x_1, x_2) \in E(G)$.

We have

$$d(x_1, x_2) \le \delta\bigl(x_1, T(x_1)\bigr) \le \delta\bigl(T(x_0), T(x_1)\bigr) \le \varphi\bigl(d(x_0, x_1)\bigr). \tag{2.8}$$

On the other hand, by (i) we have that

$$\delta\big(x_2, T(x_2)\big) \le \delta\big(T(x_1), T(x_2)\big) \le \varphi\big(d(x_1,x_2)\big) \le \varphi^2\big(d(x_0,x_1)\big).$$

By the same procedure, for $x_2 \in X$ there exists $x_3 \in T(x_2)$ such that $\delta(x_2, T(x_2)) \le qd(x_2,x_3)$ and $(x_2,x_3) \in E(G)$. Thus

$$d(x_2,x_3) \le \delta\big(x_2, T(x_2)\big) \le \varphi^2\big(d(x_0,x_1)\big). \tag{2.9}$$

We have

$$\delta\big(x_3, T(x_3)\big) \le \delta\big(T(x_2), T(x_3)\big) \le \varphi\big(d(x_2,x_3)\big) \le \varphi^3\big(d(x_0,x_1)\big).$$

By these procedures, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

(1) $(x_n,x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$;
(2) $d(x_n,x_{n+1}) \le \varphi^n(d(x_0,x_1))$ for each $n \in \mathbb{N}$;
(3) $\delta(x_n, T(x_n)) \le \varphi^n(d(x_0,x_1))$ for each $n \in \mathbb{N}$.

By (2), using the properties of φ, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since the metric space is complete, we have that the sequence is convergent, *i.e.*, $x_n \to x^*$ as $n \to \infty$. By the property (P), we get that there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{k_n},x^*) \in E(G)$ for each $n \in \mathbb{N}$.

We shall prove now that $x^* \in S\operatorname{Fix}(T)$. We have

$$\begin{aligned}\delta\big(x^*, T\big(x^*\big)\big) &\le d\big(x^*,x_{k_{n+1}}\big) + \delta\big(x_{k_{n+1}}, T\big(x^*\big)\big) \le d\big(x^*,x_{k_{n+1}}\big) + \delta\big(T(x_{k_n}), T\big(x^*\big)\big)\\ &\le d\big(x^*,x_{k_{n+1}}\big) + \varphi\big(d\big(x_{k_n},x^*\big)\big).\end{aligned}$$

Since $d(x^*,x_{k_{n+1}}) \to 0$ as $n \to \infty$ and φ is continuous in 0 with $\varphi(0) = 0$, we get that $\delta(x^*, T(x^*)) = 0$.

Hence, $x^* \in S\operatorname{Fix}(T) \Rightarrow S\operatorname{Fix}(T) \ne \emptyset$.

We shall prove now that $\operatorname{Fix}(T) = S\operatorname{Fix}(T)$.

Because $S\operatorname{Fix}(T) \subset \operatorname{Fix}(T)$, we need to show that $\operatorname{Fix}(T) \subset S\operatorname{Fix}(T)$.

Let $x^* \in \operatorname{Fix}(T)$. Because $\Delta \subset E(G)$, we have that $(x^*,x^*) \in E(G)$. Using (i) with $x = y = x^*$, we obtain

$$\delta\big(T\big(x^*\big)\big) \le \varphi\big(d\big(x^*,x^*\big)\big) = 0.$$

Hence, $x^* \in S\operatorname{Fix}(T)$ and the proof of this conclusion is complete.

(b) Suppose that there exist $x^*, y^* \in \operatorname{Fix}(T) = S\operatorname{Fix}(T)$ with $x^* \ne y^*$. We have that

- $x^* \in S\operatorname{Fix}(T) \Rightarrow \delta(x^*, T(x^*)) = 0$;
- $y^* \in S\operatorname{Fix}(T) \Rightarrow \delta(y^*, T(y^*)) = 0$;
- $(x^*,y^*) \in E(G)$.

Using (i) we obtain

$$d\big(x^*,y^*\big) = \delta\big(T\big(x^*\big), T\big(y^*\big)\big) \le \varphi\big(d\big(x^*,y^*\big)\big) < d\big(x^*,y^*\big).$$

This is a contradiction. Hence, $\operatorname{Fix}(T) = S\operatorname{Fix}(T) = \{x^*\}$. □

In the next result, the operator T satisfies another contractive condition with respect to δ on $E(G) \cap \operatorname{Graph}(T)$.

Theorem 2.3 *Let (X,d) be a complete metric space, let G be a directed graph, and let $T : X \to P_b(X)$ be a multivalued operator. Suppose that $f : X \to \mathbb{R}_+$ defined $f(x) := \delta(x, T(x))$ is a lower semicontinuous mapping. Suppose that the following assertions hold:*

(i) *There exist $a, b \in \mathbb{R}_+$, with $b \neq 0$ and $a + b < 1$, such that*

$$\delta(y, T(y)) \leq ad(x,y) + b\delta(x, T(x)) \quad \textit{for all } (x,y) \in E(G) \cap \operatorname{Graph}(T).$$

(ii) *For each $x \in X$, the set*

$$\tilde{X}_T(x) := \left\{ y \in T(x) : (x,y) \in E(G) \textit{ and } \delta(x, T(x)) \leq qd(x,y) \textit{ for some } q \in \left]1, \frac{1-a}{b}\right[\right\}$$

is nonempty.

Then $\operatorname{Fix}(T) = S\operatorname{Fix}(T) \neq \emptyset$.

Proof Let $x_0 \in X$. Then, since $\tilde{X}_T(x_0)$ is nonempty, there exist $x_1 \in T(x_0)$ and $1 < q < \frac{1-a}{b}$ such that

$$\delta(x_0, T(x_0)) \leq qd(x_0, x_1)$$

and $(x_0, x_1) \in E(G)$. Since $x_1 \in T(x_0)$, we get that $(x_0, x_1) \in E(G) \cap \operatorname{Graph}(T)$.

By (i), taking $y = x_1$ and $x = x_0$, we have that

$$\begin{aligned}\delta(x_1, T(x_1)) &\leq ad(x_0,x_1) + b\delta(x_0, T(x_0)) \\ &\leq ad(x_0,x_1) + bqd(x_0,x_1).\end{aligned}$$

Hence,

$$\delta(x_1, T(x_1)) \leq (a + bq)d(x_0, x_1). \tag{2.10}$$

For $x_1 \in X$ (since $\tilde{X}_T(x_1) \neq \emptyset$), there exists $x_2 \in T(x_1)$ such that $\delta(x_1, T(x_1)) \leq qd(x_1, x_2)$ and $(x_1, x_2) \in E(G)$. But $x_2 \in T(x_1)$ and so $(x_1, x_2) \in E(G) \cap \operatorname{Graph}(T)$.

Then

$$d(x_1, x_2) \leq \delta(x_1, T(x_1)) \leq (a + bq)d(x_0, x_1). \tag{2.11}$$

By (i), taking $y = x_2$ and $x = x_1$, we have that

$$\begin{aligned}\delta(x_2, T(x_2)) &\leq ad(x_1,x_2) + b\delta(x_1, T(x_1)) \\ &\leq ad(x_1,x_2) + bqd(x_1,x_2) = (a+bq)d(x_1,x_2) \leq (a+bq)^2 d(x_0,x_1).\end{aligned}$$

By these procedures, we obtain a sequence $(x_n)_{n \in \mathbb{N}}$ with the following properties:

(1) $(x_n, x_{n+1}) \in E(G) \cap \operatorname{Graph}(T)$ for each $n \in \mathbb{N}$;

(2) $d(x_n, x_{n+1}) \leq (a + bq)^n d(x_0, x_1)$ for each $n \in \mathbb{N}$;
(3) $\delta(x_n, T(x_n)) \leq (a + bq)^n d(x_0, x_1)$ for each $n \in \mathbb{N}$.

From (2) we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since the metric space X is complete, we have that the sequence is convergent, *i.e.*, $x_n \to x^*$ as $n \to \infty$. Now, by the lower semicontinuity of the function f, we have

$$0 \leq f(x^*) \leq \liminf_{n\to\infty} f(x_n) = 0.$$

Thus $f(x^*) = 0$, which means that $\delta(x^*, T(x^*)) = 0$. Thus $x^* \in S\mathrm{Fix}(T)$.

Let $x^* \in \mathrm{Fix}(T)$. Then $(x^*, x^*) \in \mathrm{Graph}(T)$ and hence $(x^*, x^*) \in E(G) \cap \mathrm{Graph}(T)$.

Using (i) with $x = y = x^*$, we obtain

$$\delta(T(x^*)) = \delta(x^*, T(x^*)) \leq ad(x^*, x^*) + b\delta(x^*, T(x^*)).$$

So, $\delta(T(x^*)) \leq b\delta(T(x^*))$. If we suppose that card $T(x^*) > 1$, then $\delta(T(x^*)) > 0$. Thus, $b \geq 1$, which contradicts the hypothesis.

Thus $\delta(T(x^*)) = 0$ and so $T(x^*) = \{x^*\}$. The proof is now complete. □

Remark 2.4 Example 2.1 satisfies the conditions from Theorem 2.3 for $a = 0.01$ and $b = 0.97$.

3 Well-posedness of the fixed point problem

In this section we present some well-posedness results for the fixed point problem. We consider both the well-posedness and the well-posedness in the generalized sense for a multivalued operator T.

We begin by recalling the definition of these notions from [14] and [15].

Definition 3.1 Let (X, d) be a metric space and let $T : X \to P(X)$ be a multivalued operator. By definition, the fixed point problem is well posed for T with respect to H if:

(i) $S\mathrm{Fix}(T) = \{x^*\}$;
(ii) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H(x_n, T(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x^*$ as $n \to \infty$.

Definition 3.2 Let (X, d) be a metric space and let $T : X \to P(X)$ be a multivalued operator. By definition, the fixed point problem is well posed in the generalized sense for T with respect to H if:

(i) $S\mathrm{Fix}\, T \neq \emptyset$;
(ii) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H(x_n, T(x_n)) \to 0$ as $n \to \infty$, then there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{k_n} \xrightarrow{d} x^*$ as $n \to \infty$.

In our first result we will establish the well-posedness of the fixed point problem for the operator T, where T is a Reich-type δ-contraction.

Theorem 3.1 *Let (X, d) be a complete metric space and let G be a directed graph such that the triple (X, d, G) satisfies the property (P).*

Let $T : X \to P_{\mathrm{b}}(X)$ be a multivalued operator. Suppose that

(i) *conditions* (i) *and* (ii) *in Theorem* 2.1 *hold*;

(ii) *if* $x^*, y^* \in \mathrm{Fix}(T)$, *then* $(x^*, y^*) \in E(G)$;
(iii) *for any sequence* $(x_n)_{n\in\mathbb{N}}$, $x_n \in X$ *with* $H(x_n, T(x_n)) \to 0$ *as* $n \to \infty$, *we have* $(x_n, x^*) \in E(G)$.

In these conditions the fixed point problem is well posed for T *with respect to* H.

Proof From (i) and (ii) we obtain that $S\mathrm{Fix}(T) = \{x^*\}$. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence which satisfies (iii). It is obvious that $H(x_n, T(x_n)) = \delta(x_n, T(x_n))$,

$$\begin{aligned} d(x_n, x^*) &\le \delta(x_n, T(x^*)) \le \delta(x_n, T(x_n)) + \delta(T(x_n), T(x^*)) \\ &\le \delta(x_n, T(x_n)) + ad(x_n, x^*) + b\delta(x_n, T(x_n)) + c\delta(x^*, T(x^*)). \end{aligned}$$

Thus

$$d(x_n, x^*) \le \frac{1+b}{1-a}\delta(x_n, T(x_n)) \to 0 \quad \text{as } n \to \infty.$$

Hence, $x_n \to x^*$ as $n \to \infty$. □

Remark 3.1 If we replace the property (P) with the condition that T has a closed graph, we reach the same conclusion.

The next result deals with the well-posedness of the fixed point problem in the generalized sense.

Theorem 3.2 *Let* (X, d) *be a complete metric space and let G be a directed graph such that the triple* (X, d, G) *satisfies the property* (P).

Let $T : X \to P_b(X)$ *be a multivalued operator. Suppose that*

(i) *conditions* (i) *and* (ii) *in Theorem* 2.1 *hold*;
(ii) *for any sequence* $(x_n)_{n\in\mathbb{N}}$, $x_n \in X$ *with* $H(x_n, T(x_n)) \to 0$ *as* $n \to \infty$, *there exists a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *such that* $(x_{k_n}, x^*) \in E(G)$ *and* $H(x_{k_n}, T(x_{k_n})) \to 0$.

In these conditions the fixed point problem is well posed in the generalized sense for T *with respect to* H.

Proof From (i) we have that $S\mathrm{Fix}(T) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence which satisfies (ii). Then there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such that $(x_{k_n}, x^*) \in E(G)$.

We have $H(x_{k_n}, T(x_{k_n})) = \delta(x_{k_n}, T(x_{k_n}))$,

$$\begin{aligned} d(x_{k_n}, x^*) &\le \delta(x_{k_n}, T(x^*)) \le \delta(x_{k_n}, T(x_{k_n})) + \delta(T(x_{k_n}), T(x^*)) \\ &\le \delta(x_{k_n}, T(x_{k_n})) + ad(x_{k_n}, x^*) + b\delta(x_{k_n}, T(x_{k_n})) + c\delta(x^*, T(x^*)). \end{aligned}$$

Thus

$$d(x_{k_n}, x^*) \le \frac{1+b}{1-a}\delta(x_{k_n}, T(x_{k_n})) \to 0 \quad \text{as } n \to \infty.$$

Hence, $x_{k_n} \to x^*$. □

Remark 3.2 If we replace the property (P) with the condition that T has a closed graph, we reach the same conclusion.

Next we consider the case where the operator T satisfies a φ-contraction condition.

Theorem 3.3 *Let (X,d) be a complete metric space and let G be a directed graph such that the triple (X,d,G) satisfies the property (P).*

Let $T : X \to P_b(X)$ be a multivalued operator. Suppose that

(i) *conditions* (i) *and* (ii) *in Theorem* 2.2 *hold*;
(ii) *the following implication holds*: $x^*, y^* \in \mathrm{Fix}(T)$ *implies* $(x^*, y^*) \in E(G)$;
(iii) *the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, *given by* $\psi(t) = t - \varphi(t)$, *has the following property*: *if* $\psi(t_n) \to 0$ *as* $n \to \infty$, *then* $t_n \to 0$ *as* $n \to \infty$;
(iv) *for any sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *with* $H(x_n, T(x_n)) \to 0$ *as* $n \to \infty$, *we have* $(x_n, x^*) \in E(G)$ *for all* $n \in \mathbb{N}$.

In these conditions the fixed point problem is well posed for T with respect to H.

Proof From (i) and (ii) we obtain that $S\mathrm{Fix}(T) = \{x^*\}$. Let $(x_n)_{n\in\mathbb{N}}$, $x_n \in X$ be a sequence which satisfies (iv). It is obvious that $H(x_n, T(x_n)) = \delta(x_n, T(x_n))$,

$$d(x_n, x^*) \le \delta(x_n, T(x^*)) \le \delta(x_n, T(x_n)) + \delta(T(x_n), T(x^*)) \\ \le \delta(x_n, T(x_n)) + \varphi(d(x_n, x^*)).$$

Thus

$$d(x_n, x^*) - \varphi(d(x_n, x^*)) \le \delta(x_n, T(x_n)) \to 0 \quad \text{as } n \to \infty.$$

Using condition (iii), we get that $d(x_n, x^*) \to 0$ as $n \to \infty$. Hence, $x_n \to x^*$. □

Remark 3.3 If we replace the property (P) with the condition that T has a closed graph, we reach the same conclusion.

The next result gives a well-posedness (in the generalized sense) criterion for the fixed point problem.

Theorem 3.4 *Let (X,d) be a complete metric space and let G be a directed graph such that the triple (X,d,G) satisfies the property (P).*

Let $T : X \to P_b(X)$ be a multivalued operator. Suppose that

(i) *the conditions* (i) *and* (ii) *in Theorem* 2.2 *hold*;
(ii) *the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, *given* $\psi(t) = t - \varphi(t)$, *has the following property*: *for any sequence* $(t_n)_{n\in\mathbb{N}}$, *there exists a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *such that if* $\psi(t_{k_n}) \to 0$ *as* $n \to \infty$, *then* $t_{k_n} \to 0$ *as* $n \to \infty$;
(iii) *for any sequence* $(x_n)_{n\in\mathbb{N}}$, $x_n \in X$ *with* $H(x_n, T(x_n)) \to 0$ *as* $n \to \infty$, *there exists a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *such that* $(x_{k_n}, x^*) \in E(G)$ *and* $H(x_{k_n}, T(x_{k_n})) \to 0$.

In these conditions the fixed point problem is well posed in the generalized sense for T with respect to H.

Proof From (i) we have that $S\mathrm{Fix}(T) \ne \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$, $x_n \in X$ be a sequence which satisfies (iii). Then there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such that $(x_{k_n}, x^*) \in E(G)$.

We have $H(x_{k_n}, T(x_{k_n})) = \delta(x_{k_n}, T(x_{k_n}))$,

$$\begin{aligned} d(x_{k_n}, x^*) &\leq \delta(x_{k_n}, T(x^*)) \leq \delta(x_{k_n}, T(x_{k_n})) + \delta(T(x_{k_n}), T(x^*)) \\ &\leq \delta(x_{k_n}, T(x_{k_n})) + \varphi(d(x_{k_n}, x^*)). \end{aligned}$$

Thus

$$d(x_{k_n}, x^*) - \varphi(d(x_{k_n}, x^*)) \leq \delta(x_{k_n}, T(x_{k_n})) \to 0 \quad \text{as } n \to \infty.$$

Using condition (ii), we get that $d(x_{k_n}, x^*) \to 0$ as $n \to \infty$. Hence, $x_n \to x^*$. □

Remark 3.4 If we replace the property (P) with the condition that T has a closed graph, we reach the same conclusion.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript. The authors have equal contributions to this paper.

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