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On generalized quasi-ϕ-nonexpansive mappings and their projection algorithms

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Abstract

A fixed point problem of a generalized asymptotically quasi-ϕ-nonexpansive mapping and an equilibrium problem are investigated. A strong convergence theorem for solutions of the fixed point problem and the equilibrium problem is established in a Banach space.

Keywords: asymptotically quasi-ϕ-nonexpansive mapping; equilibrium problem; fixed point; generalized asymptotically quasi-ϕ-nonexpansive mapping; generalized projection

1 Introduction and preliminaries

Let E be a real Banach space, and let E^* be the dual space of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on the Kadec-Klee property, the readers can refer to [1] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Let C be a nonempty subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. In this paper, we investigate the following equilibrium problem.

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Find $p \in C$ such that

$$f(p,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We use $\mathrm{EP}(f)$ to denote the solution set of the equilibrium problem (1.1). That is,

$$\mathrm{EP}(f) = \{p \in C : f(p,y) \geq 0, \forall y \in C\}.$$

Given a mapping $Q : C \to E^*$, let

$$f(x,y) = \langle Qx, y - x\rangle, \quad \forall x,y \in C.$$

Then $p \in \mathrm{EP}(f)$ iff p is a solution of the following variational inequality. Find p such that

$$\langle Qp, y - p\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

In order to study the solution problem of the equilibrium problem (1.1), we assume that f satisfies the following conditions:

(A1) $f(x,x) = 0$, $\forall x \in C$;

(A2) f is monotone, *i.e.*, $f(x,y) + f(y,x) \leq 0$, $\forall x,y \in C$;

(A3)

$$\limsup_{t \downarrow 0} f\big(tz + (1-t)x, y\big) \leq f(x,y), \quad \forall x,y,z \in C;$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and weakly lower semi-continuous.

As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E, which is an analogue of the metric projection P_C in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E.$$

Observe that in a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J; see, for example, [1] and [2]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x,y) \leq \big(\|y\| + \|x\|\big)^2, \quad \forall x,y \in E \tag{1.3}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x,y,z \in E. \tag{1.4}$$

Remark 1.1 If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$; for more details, see [1] and [3].

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. T is said to be asymptotically regular on C if, for any bounded subset K of C, $\lim_{n\to\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0$. T is said to be closed if, for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

A point p in C is said to be an asymptotic fixed point of T [3] iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. T is said to be relatively nonexpansive [4, 5] iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be relatively asymptotically nonexpansive [6, 7] iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{\mu_n\} \subset [1,\infty)$ with $\mu_n \to 1$ as $n \to \infty$ such that $\phi(p, Tx) \leq \mu_n \phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$. T is said to be quasi-ϕ-nonexpansive [8, 9] iff $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be asymptotically quasi-ϕ-nonexpansive [10–12] iff $F(T) \neq \emptyset$ and there exists a sequence $\{\mu_n\} \subset [1,\infty)$ with $\mu_n \to 1$ as $n \to \infty$ such that $\phi(p, Tx) \leq \mu_n \phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.2 The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings which requires the restriction $F(T) = \widetilde{F}(T)$.

Remark 1.3 The classes of asymptotically quasi-ϕ-nonexpansive mappings and quasi-ϕ-nonexpansive mappings are the generalizations of asymptotically quasi-nonexpansive mappings and quasi-nonexpansive mappings in Hilbert spaces.

Recently, Qin *et al.* [13] introduced a class of generalized asymptotically quasi-ϕ-nonexpansive mappings. Recall that a mapping T is said to be generalized asymptotically quasi-ϕ-nonexpansive iff $F(T) \neq \emptyset$ and there exist a sequence $\{\mu_n\} \subset [1,\infty)$ with $\mu_n \to 1$ as $n \to \infty$ and a sequence $\{\nu_n\} \subset [0,\infty)$ with $\nu_n \to 0$ as $n \to \infty$ such that $\phi(p, Tx) \leq \mu_n \phi(p,x) + \nu_n$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.4 The class of generalized asymptotically quasi-ϕ-nonexpansive mappings is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings which was studied in [14].

Recently, fixed point and equilibrium problems (1.1) have been intensively investigated based on iterative methods; see [15–28]. The projection method which grants strong convergence of the iterative sequences is one of efficient methods for the problems. In this paper, we investigate the equilibrium problem (1.1) and a fixed point problem of the generalized quasi-ϕ-nonexpansive mapping based on a projection method. A strong convergence theorem for solutions of the equilibrium and the fixed point problem is established in a Banach space.

In order to state our main results, we need the following lemmas.

Lemma 1.5 [2] *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty, closed, and convex subset of E, and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \forall y \in C.$$

Lemma 1.6 [2] *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \ge 0, \quad \forall y \in C.$$

Lemma 1.7 [11] *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Lemma 1.8 [29, 30] *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that $f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0$, $\forall y \in C$. Define a mapping $S_r : E \to C$ by $S_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in C\}$. Then the following conclusions hold*:

(1) *S_r is a single-valued and firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \le \langle S_r x - S_r y, Jx - Jy\rangle;$$

(2) *$F(S_r) = \mathrm{EP}(f)$ is closed and convex*;

(3) *S_r is quasi-ϕ-nonexpansive*;

(4) *$\phi(q, S_r x) + \phi(S_r x, x) \le \phi(q, x)$, $\forall q \in F(S_r)$.*

Lemma 1.9 [31] *Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r = \{x \in E : \|x\| \le r\}$ and $t \in [0, 1]$.

2 Main results

Theorem 2.1 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $T : C \to C$ be a closed generalized asymptotically quasi-ϕ-nonexpansive mapping. Assume that T is asymptotically regular* *on C and that $\mathcal{F} = F(T) \cap \mathrm{EP}(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (\mu_n - 1)M_n + \nu_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \mathcal{F}\}$, $\{\alpha_n\}$ is a real sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, and $\{r_n\}$ is a real sequence in $[a, \infty)$, where a is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof In view of Lemma 1.7 and Lemma 1.8, we find that $\mathcal{F}$ is closed and convex, so that $\Pi_{\mathcal{F}} x$ is well defined for any $x \in C$. Next, we show that C_n is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \in \mathbb{N}$. We now show that C_{m+1} is also closed and convex. For $z_1, z_2 \in C_{m+1}$, we see that $z_1, z_2 \in C_m$. It follows that $z = tz_1 + (1-t)z_2 \in C_m$, where $t \in (0,1)$. Notice that

$$\phi(z_1, u_m) \leq \phi(z_1, x_m) + (\mu_m - 1)M_m + \nu_m$$

and

$$\phi(z_1, u_h) \leq \phi(z_1, x_m) + (\mu_m - 1)M_m + \nu_m.$$

The above inequalities are equivalent to

$$2\langle z_1, Jx_m - Ju_m\rangle \leq \|x_m\|^2 - \|u_m\|^2 + (\mu_m - 1)M_m + \nu_m \tag{2.1}$$

and

$$2\langle z_2, Jx_m - Ju_m\rangle \leq \|x_m\|^2 - \|u_m\|^2 + (\mu_m - 1)M_m + \nu_m. \tag{2.2}$$

Multiplying t and $(1-t)$ on both sides of (2.1) and (2.2), respectively, yields that

$$2\langle z, Jx_m - Ju_m\rangle \leq \|x_m\|^2 - \|y_m\|^2 + (\mu_m - 1)M_m + \nu_m.$$

That is,

$$\phi(z, u_m) \leq \phi(z, x_m) + (\mu_m - 1)M_m + \nu_m, \tag{2.3}$$

where $z \in C_m$. This gives that C_{m+1} is closed and convex. Then C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_1$ is well defined.

Next, we show that $\mathcal{F} \subset C_n$. $\mathcal{F} \subset C_1 = C$ is obvious. Suppose that $\mathcal{F} \subset C_m$ for some $m \in \mathbb{N}$. Fix $w \in \mathcal{F} \subset C_m$. It follows that

$$\begin{aligned}
\phi(w,u_m) &= \phi(w,S_{r_m}y_m) \\
&\leq \phi(w,y_m) \\
&= \phi\big(w,J^{-1}\big(\alpha_m Jx_m + (1-\alpha_m)JT^m x_m\big)\big) \\
&= \|w\|^2 - 2\big\langle w,\alpha_m Jx_m + (1-\alpha_m)JT^m x_m\big\rangle + \big\|\alpha_m Jx_m + (1-\alpha_m)JT^m x_m\big\|^2 \\
&\leq \|w\|^2 - 2\alpha_m\langle w,Jx_m\rangle - 2(1-\alpha_m)\big\langle w,JT^m x_m\big\rangle + \alpha_m\|x_m\|^2 + (1-\alpha_m)\big\|T^m x_m\big\|^2 \\
&= \alpha_m\phi(w,x_m) + (1-\alpha_m)\phi\big(w,T^m x_m\big) \\
&\leq \alpha_m\phi(w,x_m) + (1-\alpha_m)\mu_m\phi(w,x_m) + (1-\alpha_m)\nu_m \\
&= \phi(w,x_m) - (1-\alpha_m)\phi(w,x_m) + (1-\alpha_m)\mu_m\phi(w,x_m) + (1-\alpha_m)\nu_m \\
&\leq \phi(w,x_m) + (1-\alpha_m)(\mu_m-1)\phi(w,x_m) + \nu_m \\
&\leq \phi(w,x_m) + (\mu_m-1)M_m + \nu_m,
\end{aligned} \tag{2.4}$$

which shows that $w \in C_{m+1}$. This implies that $\mathcal{F} \subset C_n$ for each $n \geq 1$. In view of $x_n = \Pi_{C_n}x_1$, from Lemma 1.6 we find that $\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0$ for any $z \in C_n$. It follows from $\mathcal{F} \subset C_n$ that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \geq 0, \quad \forall w \in \mathcal{F}. \tag{2.5}$$

It follows from Lemma 1.5 that

$$\begin{aligned}
\phi(x_n,x_1) &= \phi(\Pi_{C_n}x_1,x_1) \\
&\leq \phi(\Pi_{\mathcal{F}}x_1,x_1) - \phi(\Pi_{\mathcal{F}}x_1,x_n) \\
&\leq \phi(\Pi_{\mathcal{F}}x_1,x_1).
\end{aligned}$$

This implies that the sequence $\{\phi(x_n,x_1)\}$ is bounded. It follows from (1.3) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Since C_n is closed and convex, we find that $\bar{x} \in C_n$. On the other hand, we see from the weak lower semicontinuity of the norm that

$$\begin{aligned}
\phi(\bar{x},x_1) &= \|\bar{x}\|^2 - 2\langle\bar{x},Jx_1\rangle + \|x_1\|^2 \\
&\leq \liminf_{n\to\infty}\big(\|x_n\|^2 - 2\langle x_n,Jx_1\rangle + \|x_1\|^2\big) \\
&= \liminf_{n\to\infty}\phi(x_n,x_1) \\
&\leq \limsup_{n\to\infty}\phi(x_n,x_1) \\
&\leq \phi(\bar{x},x_1),
\end{aligned}$$

which implies that $\lim_{n\to\infty}\phi(x_n,x_1) = \phi(\bar{x},x_1)$. Hence, we have $\lim_{n\to\infty}\|x_n\| = \|\bar{x}\|$. In view of the Kadec-Klee property of E, we find that $x_n \to \bar{x}$ as $n \to \infty$.

Now, we are in a position to prove that $\bar{x} \in F(T)$. Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we find that $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. In view of the construction of $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we arrive at

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.6}$$

In light of $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, we find that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + (\mu_n - 1)M_n + \nu_n.$$

Thanks to (2.6), we find that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{2.7}$$

In view of (1.3), we see that $\lim_{n\to\infty}(\|x_{n+1}\| - \|u_n\|) = 0$. It follows that $\lim_{n\to\infty} \|u_n\| = \|\bar{x}\|$. This is equivalent to

$$\lim_{n\to\infty} \|Ju_n\| = \|J\bar{x}\|. \tag{2.8}$$

This implies that $\{Ju_n\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $Ju_n \rightharpoonup u^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $u \in E$ such that $Ju = u^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above yields that

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, u^*\rangle + \|u^*\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju\rangle + \|Ju\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju\rangle + \|u\|^2 \\ &= \phi(\bar{x}, u).\end{aligned}$$

That is, $\bar{x} = u$, which in turn implies that $u^* = J\bar{x}$. It follows that $Ju_n \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (2.8) that

$$\lim_{n\to\infty} Ju_n = J\bar{x}.$$

Since E enjoys the Kadec-Klee property, we obtain that $u_n \to \bar{x}$ as $n \to \infty$. Note that $\|x_n - u_n\| \le \|x_n - \bar{x}\| + \|\bar{x} - u_n\|$. It follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.9}$$

This gives that

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{2.10}$$

Notice that

$$\begin{aligned}\phi(w,x_n) - \phi(w,u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|\bigl(\|x_n\| + \|u_n\|\bigr) + 2\|w\|\|Jx_n - Ju_n\|.\end{aligned}$$

It follows from (2.9) and (2.10) that

$$\lim_{n\to\infty} \bigl(\phi(w,x_n) - \phi(w,u_n)\bigr) = 0. \tag{2.11}$$

Since E is uniformly smooth, we know that E^* is uniformly convex. In view of Lemma 1.9, we see that

$$\begin{aligned}&\phi(w,u_n) \\ &\quad= \phi(w, S_{r_n} y_n) \\ &\quad\le \phi(w, y_n) \\ &\quad= \phi\bigl(w, J^{-1}\bigl[\alpha_n Jx_n + (1-\alpha_n)JT^n x_n\bigr]\bigr) \\ &\quad= \|w\|^2 - 2\bigl\langle w, \alpha_n Jx_n + (1-\alpha_n)JT^n x_n\bigr\rangle + \bigl\|\alpha_n Jx_n + (1-\alpha_n)JT^n x_n\bigr\|^2 \\ &\quad\le \|w\|^2 - 2\alpha_n\langle w, Jx_n\rangle - 2(1-\alpha_n)\bigl\langle w, JT^n x_n\bigr\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\bigl\|T^n x_n\bigr\|^2 \\ &\qquad - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - J\bigl(T^n x_n\bigr)\bigr\|\bigr) \\ &\quad= \alpha_n\phi(w,x_n) + (1-\alpha_n)\phi\bigl(w, T^n x_n\bigr) - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - J\bigl(T^n x_n\bigr)\bigr\|\bigr) \\ &\quad\le \alpha_n\phi(w,x_n) + (1-\alpha_n)\mu_n\phi(w,x_n) + \nu_n - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - J\bigl(T^n x_n\bigr)\bigr\|\bigr) \\ &\quad\le \phi(w,x_n) + (1-\alpha_n)(\mu_n - 1)\phi(w,x_n) + \nu_n - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - J\bigl(T^n x_n\bigr)\bigr\|\bigr).\end{aligned}$$

This implies that

$$\alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - J\bigl(T^n x_n\bigr)\bigr\|\bigr) \le \phi(w,x_n) - \phi(w,u_n) + (1-\alpha_n)(\mu_n - 1)\phi(w,x_n) + \nu_n.$$

In view of the restrictions on the sequence $\{\alpha_n\}$, we find from (2.11) that

$$\lim_{n\to\infty} \bigl\|J\bigl(T^n x_n\bigr) - Jx_n\bigr\| = 0. \tag{2.12}$$

Notice that

$$\bigl\|J\bigl(T^n x_n\bigr) - J\bar{x}\bigr\| \le \bigl\|J\bigl(T^n x_n\bigr) - Jx_n\bigr\| + \|Jx_n - J\bar{x}\|.$$

It follows from (2.12) that

$$\lim_{n\to\infty}\left\|J\left(T^n x_n\right)-J\bar{x}\right\|=0. \tag{2.13}$$

The demicontinuity of $J^{-1}:E^*\to E$ implies that $T^n x_n \rightharpoonup \bar{x}$. Note that

$$\left|\left\|T^n x_n\right\|-\|\bar{x}\|\right|=\left|\left\|J\left(T^n x_n\right)\right\|-\|J\bar{x}\|\right|\le\left\|J\left(T^n x_n\right)-J\bar{x}\right\|.$$

This implies from (2.13) that $\lim_{n\to\infty}\|T^n x_n\|=\|\bar{x}\|$. Since E has the Kadec-Klee property, we obtain that $\lim_{n\to\infty}\|T^n x_n-\bar{x}\|=0$. Since

$$\left\|T^{n+1}x_n-\bar{x}\right\|\le\left\|T^{n+1}x_n-T^n x_n\right\|+\left\|T^n x_n-\bar{x}\right\|,$$

we find from the asymptotic regularity of T that $\lim_{n\to\infty}\|T^{n+1}x_n-\bar{x}\|=0$, that is, $TT^n x_n-\bar{x}\to 0$ as $n\to\infty$. It follows from the closedness of T that $T\bar{x}=\bar{x}$.

Next, we show that $\bar{x}\in \mathrm{EP}(f)$. In view of Lemma 1.8, we find from (2.4) that

$$\begin{aligned}\phi(u_n,y_n)&\le\phi(w,y_n)-\phi(w,u_n)\\&\le\phi(w,x_n)+(\mu_n-1)M_n+\nu_n-\phi(w,u_n)\\&=\phi(w,x_n)-\phi(w,u_n)+(k_n-1)M_n.\end{aligned} \tag{2.14}$$

It follows from (2.11) that $\lim_{n\to\infty}\phi(u_n,y_n)=0$. This implies that $\lim_{n\to\infty}(\|u_n\|-\|y_n\|)=0$. In view of (2.9), we see that $u_n\to\bar{x}$ as $n\to\infty$. This implies that $\|y_n\|-\|\bar{x}\|\to 0$ as $n\to\infty$. It follows that $\lim_{n\to\infty}\|Jy_n\|=\|J\bar{x}\|$. Since E^* is reflexive, we may assume that $Jy_n\rightharpoonup r^*\in E^*$. In view of $J(E)=E^*$, we see that there exists $r\in E$ such that $Jr=r^*$. It follows that

$$\begin{aligned}\phi(u_n,y_n)&=\|u_n\|^2-2\langle u_n,Jy_n\rangle+\|y_n\|^2\\&=\|u_n\|^2-2\langle u_n,Jy_n\rangle+\|Jy_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above yields that

$$\begin{aligned}0&\ge\|\bar{x}\|^2-2\langle\bar{x},r^*\rangle+\left\|r^*\right\|^2\\&=\|\bar{x}\|^2-2\langle\bar{x},Jr\rangle+\|Jr\|^2\\&=\|\bar{x}\|^2-2\langle\bar{x},Jr\rangle+\|r\|^2\\&=\phi(\bar{x},r).\end{aligned}$$

That is, $\bar{x}=r$, which in turn implies that $r^*=J\bar{x}$. It follows that $Jy_n\rightharpoonup J\bar{x}\in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Jy_n-J\bar{x}\to 0$ as $n\to\infty$. Note that $J^{-1}:E^*\to E$ is demicontinuous. It follows that $y_n\rightharpoonup\bar{x}$. Since E enjoys the Kadec-Klee property, we obtain that $y_n\to\bar{x}$ as $n\to\infty$. Note that

$$\|u_n-y_n\|\le\|u_n-\bar{x}\|+\|\bar{x}-y_n\|.$$

This implies that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{2.15}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the assumption $r_n \geq a$, we see that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{2.16}$$

In view of $u_n = S_{r_n} y_n$, we see that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in the above inequality, from (A4) and (2.16) we obtain that

$$f(y, \bar{x}) \leq 0, \quad \forall y \in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\bar{x}$. It follows that $y_t \in C$, which yields that $f(y_t, \bar{x}) \leq 0$. It follows from (A1) and (A4) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, \bar{x}) \leq tf(y_t, y).$$

That is,

$$f(y_t, y) \geq 0.$$

Letting $t \downarrow 0$, we obtain from (A3) that $f(\bar{x}, y) \geq 0$, $\forall y \in C$. This implies that $\bar{x} \in \mathrm{EP}(f)$. This shows that $\bar{x} \in \mathcal{F} = \mathrm{EP}(f) \cap F(T)$.

Finally, we prove that $\bar{x} = \Pi_{\mathcal{F}} x_1$. Letting $n \to \infty$ in (2.5), we see that

$$\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \geq 0, \quad \forall w \in \mathcal{F}.$$

In view of Lemma 1.6, we find that $\bar{x} = \Pi_{\mathcal{F}} x_1$. This completes the proof. □

If T is asymptotically quasi-ϕ-nonexpansive, then Theorem 2.1 is reduced to the following.

Corollary 2.2 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $T: C \to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping. Assume that T is asymptotically regular on C and that $\mathcal{F} = F(T) \cap \mathrm{EP}(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (\mu_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \mathcal{F}\}$, $\{\alpha_n\}$ is a real sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and $\{r_n\}$ is a real sequence in $[a, \infty)$, where a is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

If T is quasi-ϕ-nonexpansive, then Theorem 2.1 is reduced to the following.

Corollary 2.3 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $T: C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Assume that $\mathcal{F} = F(T) \cap \mathrm{EP}(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and $\{r_n\}$ is a real sequence in $[a, \infty)$, where a is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

If T is the identity, then Theorem 2.1 is reduced to the following.

Corollary 2.4 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Assume that* $\mathrm{EP}(f)$ *is nonempty. Let*

$\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ u_n \in C \quad \textit{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (\mu_n - 1)M_n + \nu_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \mathcal{F}\}$, $\{\alpha_n\}$ *is a real sequence in* $[0,1]$ *such that* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, *and* $\{r_n\}$ *is a real sequence in* $[a, \infty)$, *where a is some positive real number. Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathrm{EP}(f)} x_1$.

Competing interests
The author declares that she has no competing interests.

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