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Suzuki type fixed point theorems for generalized multi-valued mappings in *b*-metric spaces

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Abstract
In this paper, we obtained a new condition for a multi-valued mapping in a *b*-metric space, which guarantees the existence of its fixed point.
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1 Introduction

Let (X,d) be a metric space, and let $\mathrm{CB}(X)$ be a collection of all non-empty closed and bounded subsets of X. For every $A,B\in \mathrm{CB}(X)$, a Hausdorff metric H induced by the metric d of X is given by

$$H(A,B)=\max\Big\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(b,A)\Big\},$$

where $d(a,B)=\inf\{d(a,x),x\in B\}$.

For a multi-valued mapping $T:X\to 2^X$, a point $x\in X$ is called a fixed point of T if $x\in Tx$. We denote the set of fixed points of T by $\mathrm{Fix}(T)$.

Banach's fixed point theorem is extended to the following result of Nadler [1] from the single-valued mappings to the multi-valued contractive mappings.

Theorem 1.1 [1] *Let (X,d) be a complete metric space, and let $T:X\to \mathrm{CB}(X)$ be a set-valued α-contraction, that is, a mapping, for which there exists a constant $\alpha\in(0,1)$ such that $H(Tx,Ty)\le \alpha d(x,y)$, $\forall x,y\in X$. Then T has at least one fixed point.*

The following remarkable generalization of the classical Banach contraction theorem due to Suzuki [2], states the following.

Theorem 1.2 [2] *For a metric space (X,d), define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r)=\begin{cases}1 & \text{if } 0\le r\le (\sqrt{5}-1)/2,\\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2\le r\le 2^{-1/2},\\ (1+r)^{-1} & \text{if } 2^{-1/2}\le r<1.\end{cases}$$

The following are equivalent:

(i) *X is complete.*

(ii) *Every mapping T on X such that there exists $r \in [0,1)$, $\theta(r)d(x,Tx) \le d(x,y)$ implies that $d(Tx,Ty) \le rd(x,y)$ for all $x,y \in X$ has a fixed point.*

Theorem 1.2 has been generalized to multi-valued mappings by Kikkawa and Suzuki [3], Mot and Petrusel [4], Dhompongsa and Yingtaweesittikul [5], Singh and Mishra [6], Shahzad and Bassindowa [7], and Aleomraninejad *et al.* [8].

The concept of a b-metric space was introduced by Czerwik (see [9] and [10]). We recall from [9] the following definition.

Definition 1.3 [9] Let X be a set, and let $s \ge 1$ be a given real number. A function $d: X \times X \to R^+$ is said to be a b-metric if and only if for all $x,y,z \in X$, the following conditions are satisfied:

1. $d(x,y) = 0$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$.
3. $d(x,z) \le s[d(x,y) + d(y,z)]$.

A pair (X,d) is called a b-metric space.

We remark that a metric space is evidently a b-metric space. However, Czerwik (see [9, 10]) has shown that a b-metric on X need not be a metric on X.

We cite the following lemmas from Czerwik [9–11] and Singh *et al.* [6]

Lemma 1.4 *Let (X,d) be a b-metric space. For any $A,B,C \in \mathrm{CB}(X)$ and any $x,y \in X$,*

1. *$d(x,B) \le d(x,b)$ for any $b \in B$,*
2. *$d(x,B) \le H(A,B)$ for any $x \in A$,*
3. *$d(x,A) \le s[d(x,y) + d(y,A)]$.*

Lemma 1.5 *Let (X,d) be a b-metric space, and let $A,B \in C(X)$. Then for each $\alpha > 0$ and for all $b \in B$, there exists $a \in A$ such that*

$$d(a,b) \le H(A,B) + \alpha.$$

Some examples of b-metric spaces and some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces can also be found in Czerwik [9], Boriceanu *et al.* [12], Boriceanu *et al.* [13], Aydi and Bota [14], Bota *et al.* [15], and Bota [16].

Theorem 1.6 [9] *Let (X,d) be a b-complete metric space, and let $T: X \to \mathrm{CB}(X)$ be a multi-valued mapping such that T satisfies the inequality*

$$H(Tx,Ty) \le rd(x,y) \quad \text{for all } x,y \in X,$$

where $0 < r < \frac{1}{s}$. Then T has a fixed point.

Theorem 1.7 [16] *Let (X,d) be a b-complete metric space, and let $T: X \to \mathrm{CB}(X)$ be a multi-valued mapping. Suppose that there exist $a,b,c > 0$ with $c < 1$ and $\frac{a+b}{1-c} < \frac{1}{s}$ such that*

T satisfies the inequality

$$H(Tx, Ty) \leq ad(x,y) + bd(x, Tx) + cd(y, Ty)$$

for all $x, y \in X$. Then T has a fixed point.

Theorem 1.8 [14] *Let (X,d) be a b-complete metric space, and let $T : X \to \mathrm{CB}(X)$ be a multi-valued mapping such that for all $x, y \in X$,*

$$H(Tx, Ty) \leq r \max\{d(x,y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

where $r < \frac{1}{s^2+s}$. Then T has a fixed point.

In 2011, Aleomraninejad *et al.* [17] gave a new condition for multi-valued mappings in a metric space, which guarantees the existence of its fixed point.

Consider a continuous function $g : [0,\infty)^5 \to [0,\infty)$ satisfying the following conditions:

(i) $g(1,1,1,2,0) = g(1,1,1,0,2) = h \in (0,1)$.

(ii) g is subhomogeneous, that is, for all $(x_1, x_2, x_3, x_4, x_5) \in [0,\infty)^5$, $\alpha > 0$

$$g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha g(x_1, x_2, x_3, x_4, x_5).$$

(iii) If $x_i, y_i \in [0,\infty)$, $x_i < y_i$ for $i = 1, \ldots, 4$, then

$$g(x_1, x_2, x_3, x_4, 0) < g(y_1, y_2, y_3, y_4, 0) \quad \text{and} \quad g(x_1, x_2, x_3, 0, x_4) < g(y_1, y_2, y_3, 0, y_4).$$

Theorem 1.9 [17] *Let (X,d) be a complete metric space, and let $F, G : X \to \mathrm{CB}(X)$ be two multi-valued mappings. Suppose that there exist $\alpha \in (0,1)$ and $g \in R$ such that $\alpha(h+1) \leq 1$ and $\alpha d(x, Fx) \leq d(x,y)$ or $\alpha d(y, Gy) \leq d(x,y)$ implies that*

$$H(Fx, Gy) \leq g\big(d(x,y), d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)\big)$$

for all $x, y \in X$. Then $\mathrm{Fix}(F) = \mathrm{Fix}(G)$ and $\mathrm{Fix}(F)$ is non-empty.

The aim of this paper is to apply the concept of this function g to b-metric spaces.

Let $s \geq 1$ be fixed, and let R_s be the set of all continuous functions $g : [0,\infty)^5 \to [0,\infty)$ satisfying the conditions (ii), (iii) and

(iv) $g(1,1,1,2s,0) = g(1,1,1,0,2s) = h_s \in (0, 1/s)$.

Following the proofs in [18] and [17] with minor modification, we get the following results, respectively.

Lemma 1.10 *If $g \in R_s$ and $u, v \in [0,\infty)$ are such that*

$$\begin{aligned} u \leq \max\{ & g\big(v, v, u, s(v+u), 0\big), g\big(v, v, u, 0, s(v+u)\big), \\ & g\big(v, u, v, s(v+u), 0\big), g\big(v, u, v, 0, s(v+u)\big)\}, \end{aligned}$$

then $u \leq h_s v$.

Proof Without loss of generality, we can suppose that $u \leq g(v, v, u, s(v+u), 0)$.

If $v < u$, then

$$u \leq g\bigl(v, v, u, s(v+u), 0\bigr) \leq g(u, u, u, 2us, 0) \leq ug(1, 1, 1, 2s, 0) = h_s u < u$$

which is a contradiction. Thus $u \leq v$. So,

$$u \leq g\bigl(v, v, u, s(u+v), 0\bigr) \leq g(v, v, v, 2vs, 0) \leq vg(1, 1, 1, 2s, 0) = h_s v. \qquad \square$$

Lemma 1.11 *Let (X, d) be a b-complete metric space, and let $F, G : X \to \mathrm{CB}(X)$ be two multi-valued mappings. Suppose that there exist $\alpha \in (0, \infty)$ and $g \in R_s$ such that $\alpha d(x, Fx) \leq d(x, y)$ or $\alpha d(y, Gy) \leq d(x, y)$ implies that*

$$H(Fx, Gy) \leq g\bigl(d(x, y), d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)\bigr)$$

for all $x, y \in X$. Then $\mathrm{Fix}(F) = \mathrm{Fix}(G)$.

Proof Let $x \in \mathrm{Fix}(F)$, then $\alpha d(x, Fx) = 0 = d(x, x)$. Thus,

$$\begin{aligned} d(x, Gx) &\leq H(Fx, Gx) \\ &\leq g\bigl(d(x, x), d(x, Fx), d(x, Gx), d(x, Gx), d(x, Fx)\bigr) \\ &\leq g\bigl(0, 0, d(x, Gx), d(x, Gx), 0\bigr) \\ &\leq g\bigl(0, 0, d(x, Gx), sd(x, Gx), 0\bigr). \end{aligned}$$

Using Lemma 1.10, we have $d(x, Gx) \leq h_s 0 = 0$. So, $x \in \mathrm{Fix}(G)$.

Hence $\mathrm{Fix}(F) \subseteq \mathrm{Fix}(G)$. Similarly, we can obtain $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(F)$. $\square$

2 Main results

Theorem 2.1 *Let (X, d) be a b-complete metric space, and let $F, G : X \to \mathrm{CB}(X)$ be two multi-valued mappings. Suppose that there exist $\alpha \in (0, 1)$ and $g \in R_s$ such that $s\alpha(h_s + 1) \leq 1$ and $\alpha d(x, Fx) \leq d(x, y)$ or $\alpha d(y, Gy) \leq d(x, y)$ implies that*

$$H(Fx, Gy) \leq g\bigl(d(x, y), d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)\bigr)$$

for all $x, y \in X$. Then $\mathrm{Fix}(F) = \mathrm{Fix}(G)$ and $\mathrm{Fix}(F)$ is non-empty.

Proof The main idea of the proof follows from Theorem 1.9.

By Lemma 1.11, $\mathrm{Fix}(F) = \mathrm{Fix}(G)$. Let $r \in (h_s, \frac{1}{s})$ and $x_0 \in X$. If x_0 is not a fixed point, choose $x_1 \in Fx_0$ such that $\alpha d(x_0, Fx_0) < d(x_0, x_1)$. Thus,

$$\begin{aligned} d(x_1, Gx_1) &\leq H(Fx_0, Gx_1) \\ &\leq g\bigl(d(x_0, x_1), d(x_0, Fx_0), d(x_1, Gx_1), d(x_0, Gx_1), d(x_1, Fx_0)\bigr) \\ &\leq g\bigl(d(x_0, x_1), d(x_0, x_1), d(x_1, Gx_1), s\bigl[d(x_0, x_1) + d(x_1, Gx_1)\bigr], 0\bigr). \end{aligned}$$

By Lemma 1.10, we have $d(x_1, Gx_1) \leq h_s d(x_0, x_1) < rd(x_0, x_1)$. If x_1 is not a fixed point, there exists $x_2 \in Gx_1$ such that $d(x_1, x_2) < rd(x_0, x_1)$. Since $\alpha d(x_1, Gx_1) < d(x_1, x_2)$,

$$\begin{aligned} d(x_2, Fx_2) &\leq H(Fx_2, Gx_1) \\ &\leq g\big(d(x_1,x_2), d(x_2,Fx_2), d(x_1,Gx_1), d(x_2,Gx_1), d(x_1,Fx_2)\big) \\ &\leq g\big(d(x_1,x_2), d(x_2,Fx_2), d(x_1,x_2), 0, s\big[d(x_1,x_2) + d(x_2,Fx_2)\big]\big). \end{aligned}$$

By Lemma 1.10, we have $d(x_2, Fx_2) \leq h_s d(x_1, x_2) < rd(x_1, x_2)$.

Similarly, there exists $x_3 \in Fx_2$ such that $d(x_2, x_3) < rd(x_1, x_2) < r^2 d(x_0, x_1)$.

By continuing this process, we obtain a sequence $\{x_n\}$ in X such that

$$\begin{aligned} &x_{2n-1} \in Fx_{2n-2}, x_{2n} \in Gx_{2n-1}, \quad d(x_n, x_{n+1}) \leq r^n d(x_0, x_1) \\ &d(x_{2n}, Fx_{2n}) \leq hd(x_{2n-1}, x_{2n}) \quad \text{and} \quad d(x_{2n-1}, Gx_{2n-1}) \leq hd(x_{2n-2}, x_{2n-1}). \end{aligned}$$

We prove next that the sequence $\{x_n\}$ is Cauchy,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s\big[d(x_n,x_{n+1}) + d(x_{n+1},x_{n+p})\big] = sd(x_n,x_{n+1}) + sd(x_{n+1},x_{n+p}) \\ &\leq sd(x_n,x_{n+1}) + s^2\big[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_{n+p})\big] \\ &= sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + s^2 d(x_{n+2},x_{n+p}) \\ &\leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + s^3 d(x_{n+2},x_{n+3}) + \cdots \\ &\quad + s^{p-1} d(x_{n+p-2},x_{n+p-1}) + s^{p-1} d(x_{n+p-1},x_{n+p}) \\ &\leq sr^n d(x_0,x_1) + s^2 r^{n+1} d(x_0,x_1) + s^3 r^{n+2} d(x_0,x_1) + \cdots \\ &\quad + s^{p-1} r^{n+p-2} d(x_0,x_1) + s^{p-1} r^{n+p-1} d(x_0,x_1), \\ d(x_n, x_{n+p}) &\leq sr^n\big(1 + sr + s^2r^2 + \cdots + s^{p-2}r^{p-2} + s^{p-2}r^{p-1}\big) d(x_0,x_1) \\ &\leq sr^n\big(1 + sr + s^2r^2 + \cdots + s^{p-2}r^{p-2} + s^{p-1}r^{p-1}\big) d(x_0,x_1) \\ &= sr^n\left[\frac{1-(sr)^p}{1-sr}\right] d(x_0,x_1). \end{aligned}$$

Notice that

$$sr^n\left[\frac{1-(sr)^p}{1-sr}\right] d(x_0,x_1) \to 0 \quad \text{as } n \to \infty.$$

So $\{x_n\}$ is Cauchy, and $x_n \to x$ for some $x \in X$.

Now, we claim that for each $n \geq 1$,

$$\alpha d(x_{2n}, Fx_{2n}) \leq d(x_{2n}, x) \quad \text{or} \quad \alpha d(x_{2n+1}, Gx_{2n+1}) \leq d(x_{2n+1}, x).$$

If $\alpha d(x_{2n}, Fx_{2n}) > d(x_{2n}, x)$ and $\alpha d(x_{2n+1}, Gx_{2n+1}) > d(x_{2n+1}, x)$ for some $n \geq 1$, then

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq s\big[d(x_{2n},x) + d(x_{2n+1},x)\big] \\ &< s\big[\alpha d(x_{2n},Fx_{2n}) + \alpha d(x_{2n+1},Gx_{2n+1})\big] \end{aligned}$$

$$\begin{aligned}&\leq s\big[\alpha d(x_{2n},x_{2n+1})+\alpha h d(x_{2n},x_{2n+1})\big]\\&=s(\alpha+\alpha h)d(x_{2n},x_{2n+1})=s\alpha(h_s+1)d(x_{2n},x_{2n+1}).\end{aligned}$$

Thus, we get $s\alpha(h_s+1)>1$, which is a contradiction. By using the assumption, for each $n\geq 1$, either

$$H(Fx_{2n},Gx)\leq g\big(d(x_{2n},x),d(x_{2n},Fx_{2n}),d(x,Gx),d(x_{2n},Gx),d(x,Fx_{2n})\big)$$

or

$$H(Fx,Gx_{2n+1})\leq g\big(d(x,x_{2n+1}),d(x,Fx),d(x_{2n+1},Gx_{2n+1}),d(x,Gx_{2n+1}),d(x_{2n+1},Fx)\big).$$

Therefore, one of the following cases holds.

(a) There exists an infinite subset $I\subseteq N$ such that

$$\begin{aligned}d(x_{2n+1},Gx)&\leq H(Fx_{2n},Gx)\\&\leq g\big(d(x_{2n},x),d(x_{2n},Fx_{2n}),d(x,Gx),d(x_{2n},Gx),d(x,Fx_{2n})\big)\end{aligned}$$

for all $n\in I$.

(b) There exists an infinite subset $J\subseteq N$ such that

$$\begin{aligned}d(Fx,x_{2n+2})&\leq H(Fx,Gx_{2n+1})\\&\leq g\big(d(x,x_{2n+1}),d(x,Fx),d(x_{2n+1},Gx_{2n+1}),d(x,Gx_{2n+1}),d(x_{2n+1},Fx)\big)\end{aligned}$$

for all $n\in J$.

In case (a), we obtain

$$\begin{aligned}d(x,Gx)&\leq s\big[d(x,x_{2n+1})+d(x_{2n+1},Gx)\big]\\&\leq s\big[d(x,x_{2n+1})+g\big(d(x_{2n},x),d(x_{2n},Fx_{2n}),d(x,Gx),d(x_{2n},Gx),d(x,Fx_{2n})\big)\big]\\&\leq s\big[d(x,x_{2n+1})\\&\quad+g\big(d(x_{2n},x),d(x_{2n},x_{2n+1}),d(x,Gx),s\big[d(x_{2n},x)+d(x,Gx)\big],d(x,x_{2n+1})\big)\big]\end{aligned}$$

for all $n\in I$. Since g is continuous, $d(x,Gx)\leq s(g(0,0,d(x,Gx),sd(x,Gx),0))$. Using Lemma 1.10, $d(x,Gx)=0$. We have $x\in Gx$.

In case (b), we obtain

$$\begin{aligned}d(x,Fx)&\leq s\big[d(x,x_{2n+2})+d(x_{2n+2},Fx)\big]\\&\leq s\big[d(x,x_{2n+2})\\&\quad+g\big(d(x,x_{2n+1}),d(x,Fx),d(x_{2n+1},Gx_{2n+1}),d(x,Gx_{2n+1}),d(x_{2n+1},Fx)\big)\big]\\&\leq s\big[d(x,x_{2n+2})\\&\quad+g\big(d(x,x_{2n+1}),d(x,Fx),d(x_{2n+1},x_{2n+2}),d(x,x_{2n+2}),s\big[d(x_{2n+1},x)+d(x,Fx)\big]\big)\big]\end{aligned}$$

for all $n \in J$. Since g is continuous, $d(x, Fx) \leq s(g(0, d(x, Fx), 0, 0, sd(x, Fx)))$. Using Lemma 1.10, $d(x, Fx) = 0$. We have $x \in Fx$. This completes the proof. □

Remark 2.2 Taking $s = 1$ in Theorem 2.1 (case of metric spaces), we recover Theorem 1.9.

The following result is a consequence of Theorem 2.1.

Corollary 2.3 *Let (X, d) be a b-complete metric space, and let $T : X \to \mathrm{CB}(X)$ be a multi-valued mapping. Suppose that there exist $\alpha \in (0, 1)$ and $g \in R_s$ such that $s\alpha(h_s + 1) \leq 1$ and $\alpha d(x, Tx) \leq d(x, y)$ implies that*

$$H(Tx, Ty) \leq g\bigl(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\bigr)$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 2.4 *Let (X, d) be a b-complete metric space, and let $T : X \to \mathrm{CB}(X)$ be a multi-valued mapping. Suppose that there exists $r \in (0, \frac{1}{s})$ such that $\frac{1}{s(r+1)} d(x, Tx) \leq d(x, y)$ implies*

$$H(Tx, Ty) \leq r \max\bigl\{d(x, y), d(x, Tx), d(y, Ty)\bigr\}$$

for all $x, y \in X$. Then T has a fixed point.

Proof Let $g \in R_s$ by $g(x_1, x_2, x_3, x_4, x_5) = r \max\{x_1, x_2, x_3\}$, where $r < \frac{1}{s}$. Put $\alpha = \frac{1}{s(r+1)}$. Since $h_s = r < \frac{1}{s}$ and $s\alpha(h_s + 1) \leq 1$, by using Corollary 2.3, T has a fixed point. □

Remark 2.5 Corollary 2.4 is an extension of Theorem 1.6.

Corollary 2.6 *Let (X, d) be a b-complete metric space, and let $T : X \to \mathrm{CB}(X)$ be a multi-valued mapping. Suppose that there exists $a, b \in [0, 1)$ and $a + 2b < \frac{1}{s}$ such that $\frac{1}{s(1+a+2b)} d(x, Tx) \leq d(x, y)$ implies that*

$$H(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + bd(y, Ty)$$

for all $x, y \in X$. Then T has a fixed point.

Proof Let $g \in R_s$ be $g(x_1, x_2, x_3, x_4, x_5) = ax_1 + b(x_2 + x_3)$, where $a + 2b < \frac{1}{s}$. Put $\alpha = \frac{1}{s(1+a+2b)}$. Since $h_s = a + 2b < \frac{1}{s}$ and $s\alpha(h_s + 1) \leq 1$, by using Corollary 2.3, T has a fixed point. □

The following examples show that we can apply Corollary 2.3 but cannot apply Theorem 1.8.

Example 2.7 Let $X = [0, 1]$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$. It is obvious that d is a b-metric on X with $s = 2$ and (X, d) is complete. Also, d is not a metric on X. Define $T : X \to \mathrm{CB}(X)$ by

$$Tx = \begin{cases} \{\frac{1}{3}, \frac{2}{3}\} & \text{if } 0 \leq x < 1, \\ \{\frac{1}{3}\} & \text{if } x = 1. \end{cases}$$

Let $x, y \in X$. Without loss of generality, take $x \leq y$.

If $x = y$ or $x, y < 1$, then $Tx = Ty$. Hence $H(Tx, Ty) = 0$.

If $x < 1$ and $y = 1$, then

$$H(Tx, Ty) = \frac{1}{9} \leq \frac{4}{27} = \frac{1}{3} \cdot \frac{4}{9} = \frac{1}{3} d(y, Ty) \leq r \max\bigl\{d(x, y), d(x, Tx), d(y, Ty)\bigr\},$$

where $r = \frac{1}{3} < \frac{1}{2} = \frac{1}{s}$. So all the conditions of Corollary 2.4 are satisfied. Moreover, $\frac{1}{3}$ and $\frac{2}{3}$ are the two fixed points of T.

On the other hand, if we choose $x = \frac{1}{3}$ and $y = 1$, then

$$\begin{aligned} H(Tx, Ty) = \frac{1}{9} > \frac{1}{6} \cdot \frac{4}{9} &= \frac{1}{s^2 + s} \max\left\{\frac{4}{9}, 0, \frac{4}{9}, 0, \frac{1}{9}\right\} \\ &= \frac{1}{s^2 + s} \max\bigl\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\bigr\}. \end{aligned}$$

So we could not apply Theorem 1.8.

Example 2.8 Let $X = [1, \infty)$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with $s = 2$. Define $T : X \to \mathrm{CB}(X)$ by

$$Tx = \left[1, 1 + \frac{x}{2}\right] \quad \text{for all } x \in X.$$

Consider $H(Tx, Ty) = \frac{1}{4}(x - y)^2 = \frac{1}{4} d(x, y) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\}$, where $r = \frac{1}{4} < \frac{1}{2} = \frac{1}{s}$ for all $x, y \in X$. So all the conditions of Corollary 2.4 are satisfied. Moreover, 1 and 2 are the two fixed points of T.

On the other hand, if we choose $x = 1$ and $y = 2$, then

$$\begin{aligned} H(Tx, Ty) = \frac{1}{4} > \frac{1}{6} &= \frac{1}{s^2 + s} \max\left\{1, 0, 0, 0, \frac{1}{4}\right\} \\ &= \frac{1}{s^2 + s} \max\bigl\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\bigr\}. \end{aligned}$$

So we could not apply Theorem 1.8.

Competing interests

The author declares that she has no competing interests.

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