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Some results on a modified Mann iterative scheme in a reflexive Banach space

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Abstract

The purpose of this paper is to study Mann iterative schemes. Strong convergence of a modified Mann iterative scheme is obtained in a reflexive Banach space.

Keywords: fixed point; nonexpansive mapping; asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense; relatively nonexpansive mapping; generalized projection

1 Introduction-Preliminaries

Normal Mann iterative scheme is an important iterative scheme to study the class of nonexpansive mappings [1]. However, the normal Mann iterative scheme is only weak convergence for nonexpansive mappings; see [2–4]. In many disciplines, including economics [5] and image recovery [6], problems arise in infinite dimensional spaces. In such problems, strong convergence is often much more desirable than weak convergence, for it translates the physically tangible property. Strong convergence of iterative sequences properties has a direct impact when the process is executed directly in the underlying infinite dimensional space. Recently, with the aid of projections, many authors studied Mann iterative schemes. Theoretically, the advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions. The purpose of this paper is to study a modified Mann iterative scheme. Strong convergence of the scheme is obtained in a reflexive Banach space.

Let E be a real Banach space, let C be a nonempty subset of E, and let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. Recall that T is said to be asymptotically regular on C iff for any bounded subset K of C,

$$\limsup_{n\to\infty}\left\{\left\|T^{n+1}x - T^n x\right\| : x \in K\right\} = 0.$$

Recall that T is said to be closed iff for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

T is said to be quasi-nonexpansive iff $F(T) \neq \emptyset$, and

$$\|p - Ty\| \leq \|p - y\|, \quad \forall p \in F(T), \forall y \in C.$$

T is said to be asymptotically nonexpansive iff there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \forall n \geq 1.$$

T is said to be asymptotically quasi-nonexpansive iff $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|p - T^n y\| \leq k_n \|p - y\|, \quad \forall p \in F(T), \forall y \in C, \forall n \geq 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [7] in 1972. In uniformly convex Banach spaces, they proved that if C is nonempty, bounded, closed, and convex, then every asymptotically nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence of iterative schemes for such a class of mappings.

Recall that T is said to be asymptotically nonexpansive in the intermediate sense iff it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\| \right) \leq 0.$$

T is said to be asymptotically quasi-nonexpansive in the intermediate sense iff $F(T) \neq \emptyset$ and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{p\in F(T), y\in C} \left(\|p - T^n y\| - \|p - y\| \right) \leq 0.$$

The class of the mappings that are asymptotically nonexpansive in the intermediate sense was considered by Bruck *et al.* [8] and Kirk [9]. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense may not be Lipschitz continuous. However, asymptotically nonexpansive mappings are Lipschitz continuous. For the existence of the mapping, we can find the details in [9].

In [10], Nakajo and Takahashi first investigated fixed point problems of nonexpansive mappings based on hybrid projection methods in the framework of Hilbert spaces. Subsequently, many authors investigated fixed point problems of nonlinear mappings based on the methods in the framework of Hilbert spaces. The advantage of the method is that strong convergence of iterative sequences can be guaranteed without any compact assumptions.

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on the Kadec-Klee property, the readers can refer to [11] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [12] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E.$$

Observe that, in a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x) = \min_{y\in C}\phi(y,x).$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J; see, for example, [11–13]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of a function ϕ that

$$\bigl(\|x\| - \|y\|\bigr)^2 \le \phi(x,y) \le \bigl(\|y\| + \|x\|\bigr)^2, \quad \forall x, y \in E \tag{1.1}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{1.2}$$

Remark 1.1 If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (2.1), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; see [11] for more details.

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [14] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive [15, 16] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be relatively asymptotically nonexpansive [17, 18] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, Tx) \leq k_n\phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$. The asymptotic behavior of relatively nonexpansive mappings was studied in [14–16].

The mapping T is said to be quasi-ϕ-nonexpansive [19] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be asymptotically quasi-ϕ-nonexpansive [20–22] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, Tx) \leq k_n\phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.2 The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings, which requires the restriction: $F(T) = \widetilde{F}(T)$.

T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense [23] if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T),x\in C} \bigl(\phi\bigl(p, T^n x\bigr) - \phi(p,x)\bigr) \leq 0. \tag{1.3}$$

Put

$$\xi_n = \max\Bigl\{0, \sup_{p\in F(T),x\in C} \bigl(\phi\bigl(p, T^n x\bigr) - \phi(p,x)\bigr)\Bigr\}.$$

It follows that $\xi_n \to 0$ as $n \to \infty$. Then (1.3) is reduced to the following:

$$\phi\bigl(p, T^n x\bigr) \leq \phi(p,x) + \xi_n, \quad \forall p \in F(T), \forall x \in C. \tag{1.4}$$

Remark 1.3 The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is a generalization of the class of asymptotically quasi-nonexpansive mappings in the intermediate sense in the framework of Banach spaces.

Let $E = \mathbb{R}^1$ and $C = [0,1]$. Define the following mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{2}x, & x \in [0, \frac{1}{2}], \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then T is an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense with the fixed point set $\{0\}$. We also have the following

$$\phi\bigl(T^n x, T^n y\bigr) = \bigl|T^n x - T^n y\bigr|^2 = \frac{1}{2^{2n}}|x-y|^2 \leq |x-y|^2 = \phi(x,y), \quad \forall x,y \in \biggl[0, \frac{1}{2}\biggr],$$

$$\phi\bigl(T^n x, T^n y\bigr) = \bigl|T^n x - T^n y\bigr|^2 = 0 \leq |x-y|^2 = \phi(x,y), \quad \forall x,y \in \biggl(\frac{1}{2}, 1\biggr]$$

and

$$\begin{aligned}
\phi\left(T^n x, T^n y\right) &= \left|T^n x - T^n y\right|^2 \\
&= \left|\frac{1}{2^n}x - 0\right|^2 \\
&\leq \left(\frac{1}{2^n}|x-y| + \frac{1}{2^n}|y|\right)^2 \\
&\leq \left(|x-y| + \frac{1}{2^n}\right)^2 \\
&\leq |x-y|^2 + \xi_n \\
&= \phi(x,y) + \xi_n, \quad \forall x \in \left[0, \frac{1}{2}\right], \forall y \in \left(\frac{1}{2}, 1\right],
\end{aligned}$$

where $\xi_n = \frac{1}{2^{2n}} + \frac{1}{2^{n-1}}$. Hence, we have $\phi(T^n x, T^n y) \leq \phi(x,y) + \xi_n$, $\forall x, y \in [0,1]$.

Recently, Matsushita and Takahashi [24] first investigated fixed point problems of relatively nonexpansive mappings based on hybrid projection methods. A strong convergence theorem was established in a uniformly convex and uniformly smooth Banach space. To be more precise, they proved the following result.

Theorem MT *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases}
x_0 = x \in C, \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\
H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots,
\end{cases}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

Recently, Su and Qin [25] introduced a monotone projection method for computing fixed points of nonexpansive mappings. A strong convergence theorem was established in the framework of Hilbert spaces; for more details, see [25]. Takahashi *et al.* [26] further introduced the shrinking projection method for nonexpansive mappings; for more details, see [26]. Subsequently, a lot results were obtained on the two methods in Hilbert spaces or Banach spaces; see [27–38] and the references therein.

Motivated by the above results, some fixed point theorems of asymptotically quasi-ϕ-nonexpansive mappings based on hybrid projection methods were established in the framework of Banach spaces.

In this paper, motivated by Matsushita and Takahashi [24], Su and Qin [25] and Takahashi *et al.* [26], we investigate a fixed point problem of an asymptotically quasi-ϕ-

nonexpansive mapping in the intermediate sense. A strong convergence theorem is established in a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. The results improve and extend the corresponding results in the literature.

For our main results, we need the following lemmas.

Lemma 1.4 [12] *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty, closed, and convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 1.5 [12] *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

2 Main results

Theorem 2.1 *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that T is asymptotically regular on C and closed, and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where

$$\xi_n = \max\Big\{0, \sup_{p \in F(T), x \in C} \big(\phi\big(p, T^n x\big) - \phi(p, x)\big)\Big\}.$$

If the sequence $\{\alpha_n\}$ satisfies the restriction $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof The proof is split into six steps.

Step 1. Show that $F(T)$ is closed and convex so that $\Pi_{F(T)} x$ is well defined for any $x \in C$.

Let $p_1, p_2 \in F(T)$, and $p = tp_1 + (1 - t)p_2$, where $t \in (0, 1)$. We see that $p = Tp$. Indeed, we see from (1.2) that

$$\phi\big(p_1, T^n p\big) = \phi(p_1, p) + \phi\big(p, T^n p\big) + 2\big\langle p_1 - p, Jp - JT^n p\big\rangle$$

and

$$\phi\big(p_1, T^n p\big) = \phi(p_1, p) + \phi\big(p, T^n p\big) + 2\big\langle p_1 - p, Jp - JT^n p\big\rangle.$$

It follows from the definition of T that

$$\phi(p, T^n p) \leq 2\langle p - p_1, Jp - JT^n p\rangle + \xi_n \tag{2.1}$$

and

$$\phi(p, T^n p) \leq 2\langle p - p_2, Jp - JT^n p\rangle + \xi_n. \tag{2.2}$$

Multiplying t and $(1 - t)$ on both sides of (2.1) and (2.2), respectively, yields that $\lim_{n\to\infty} \phi(p, T^n p) = 0$. In light of (1.1), we arrive at

$$\lim_{n\to\infty} \| T^n p\| = \|p\|. \tag{2.3}$$

It follows that

$$\lim_{n\to\infty} \|J(T^n p)\| = \|Jp\|. \tag{2.4}$$

Since E^* is reflexive, we may, without loss of generality, assume that $J(T^n p) \rightharpoonup q^* \in E^*$. In view of the reflexivity of E, we find that there exists an element $q \in E$ such that $Jq = q^*$. It follows that

$$\phi(p, T^n p) = \|p\|^2 - 2\langle p, J(T^n p)\rangle + \| T^n p\|^2 = \|p\|^2 - 2\langle p, J(T^n p)\rangle + \|J(T^n p)\|^2.$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above, we obtain that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, q^*\rangle + \|q^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jq\rangle + \|Jq\|^2 \\ &= \|p\|^2 - 2\langle p, Jq\rangle + \|q\|^2 \\ &= \phi(p, q). \end{aligned}$$

This implies that $p = q$, that is, $Jp = q^*$. It follows that $J(T^n p) \rightharpoonup Jp \in E^*$. In view of the Kadec-Klee property of E^*, we obtain from (2.4) that $\lim_{n\to\infty} \|J(T^n p) - Jp\| = 0$. Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T^n p \rightharpoonup p$. By virtue of the Kadec-Klee property of E, we see from (2.3) that $T^n p \to p$ as $n \to \infty$. Hence, $TT^n p = T^{n+1} p \to p$ as $n \to \infty$. In view of the closedness of T, we can obtain that $p \in F(T)$. This shows that $F(T)$ is convex. Since T is closed, we can easily conclude that $F(T)$ is also closed. This completes the proof that $F(T)$ is convex and closed.

Step 2. Show that C_n is closed and convex.

It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some $h \in \mathbb{N}$. We now show that C_{h+1} is also closed and convex.

For $z_1, z_2 \in C_{h+1}$, we see that $z_1, z_2 \in C_h$. It follows that $z = tz_1 + (1 - t)z_2 \in C_h$, where $t \in (0, 1)$. Notice that

$$\phi(z_1, y_h) \leq \phi(z_1, x_h) + \xi_h \tag{2.5}$$

and

$$\phi(z_1, y_h) \leq \phi(z_1, x_h) + \xi_h. \tag{2.6}$$

Notice that (2.5) and (2.6) are equivalent to

$$2\langle z_1, Jx_h - Jy_h\rangle \le \|x_h\|^2 - \|y_h\|^2 + \xi_h \tag{2.7}$$

and

$$2\langle z_2, Jx_h - Jy_h\rangle \le \|x_h\|^2 - \|y_h\|^2 + \xi_h. \tag{2.8}$$

Multiplying t and $(1-t)$ on both sides of (2.7) and (2.8), respectively, yields that

$$2\langle z, Jx_h - Jy_h\rangle \le \|x_h\|^2 - \|y_h\|^2 + \xi_h.$$

That is,

$$\phi(z, y_h) \le \phi(z, x_h) + \xi_h.$$

This implies that C_{h+1} is closed and convex. Then, for each $n \ge 1$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}}x_1$ is well defined.

Step 3. Show that $F(T) \subset C_n$.

$F(T) \subset C_1 = C$ is obvious. Suppose that $F(T) \subset C_h$ for some $h \in \mathbb{N}$. Then, $\forall w \in F(T) \subset C_h$, we have

$$\begin{aligned}
\phi(w, y_h) &= \phi\bigl(w, J^{-1}\bigl(\alpha_h Jx_n + (1-\alpha_h)JT^h x_h\bigr)\bigr) \\
&= \|w\|^2 - 2\bigl\langle w, \alpha_h Jx_h + (1-\alpha_h)JT^h x_h\bigr\rangle + \bigl\|\alpha_h Jx_n + (1-\alpha_h)JT^h x_h\bigr\|^2 \\
&\le \|w\|^2 - 2\alpha_h\langle w, Jx_n\rangle - 2(1-\alpha_h)\bigl\langle w, JT^h x_h\bigr\rangle + \alpha_h\|x_h\|^2 + (1-\alpha_h)\bigl\|T^h x_h\bigr\|^2 \\
&= \alpha_h\phi(w, x_h) + (1-\alpha_h)\phi\bigl(w, T^h x_h\bigr) \\
&= \phi(w, x_h) - (1-\alpha_h)\phi(w, x_h) + (1-\alpha_h)\bigl(\phi(w, x_h) + \xi_n\bigr) \\
&\le \phi(w, x_h) + \xi_h.
\end{aligned}$$

This shows that $w \in C_{h+1}$. This implies that $\mathcal{F} \subset C_n$.

Step 4. Show that $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$, where $\bar{x}$ is some point in C.

In view of $x_n = \Pi_{C_n}x_1$, we see from Lemma 1.5 that

$$\langle x_n - z, Jx_1 - Jx_n\rangle \ge 0, \quad \forall z \in C_n.$$

It follows from $F(T) \subset C_n$ that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \ge 0, \quad \forall w \in F(T). \tag{2.9}$$

It follows from Lemma 1.4 that

$$\begin{aligned}
\phi(x_n, x_1) &= \phi(\Pi_{C_n}x_1, x_1) \\
&\le \phi(\Pi_{F(T)}x_1, x_1) - \phi(\Pi_{F(T)}x_1, x_n) \\
&\le \phi(\Pi_{F(T)}x_1, x_1).
\end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_1)\}$ is bounded. It follows from (1.1) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Since C_n is closed and convex, we find that $\bar{x} \in C_n$. On the other hand, we see from the weak lower semicontinuity of the norm that

$$\begin{aligned}
\phi(\bar{x}, x_1) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_1\rangle + \|x_1\|^2 \\
&\le \liminf_{n\to\infty}\bigl(\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2\bigr) \\
&= \liminf_{n\to\infty} \phi(x_n, x_1) \\
&\le \limsup_{n\to\infty} \phi(x_n, x_1) \\
&\le \phi(\bar{x}, x_1),
\end{aligned}$$

which implies that $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. In view of the Kadec-Klee property of E, we find that $x_n \to \bar{x}$ as $n \to \infty$. This completes Step 4.

Step 5. Show that $\bar{x} \in F(T)$.

Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we find that $\phi(x_n, x_1) \le \phi(x_{n+1}, x_1)$. This shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. In view of construction of $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we arrive at

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\
&\le \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\
&= \phi(x_{n+1}, x_1) - \phi(x_n, x_1).
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.10}$$

In view of $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, we find that

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) + \xi_n.$$

This in turn implies from (2.10) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{2.11}$$

In view of (1.1), we see that

$$\lim_{n\to\infty} \bigl(\|x_{n+1}\| - \|y_n\|\bigr) = 0.$$

It follows that

$$\lim_{n\to\infty} \|y_n\| = \|\bar{x}\|.$$

This is equivalent to

$$\lim_{n\to\infty} \|Jy_n\| = \|J\bar{x}\|. \tag{2.12}$$

This implies that $\{Jy_n\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\phi(x_{n+1}, y_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2.$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above yields that

$$0 \geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2 = \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|Jy\|^2 = \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|y\|^2 = \phi(\bar{x}, y).$$

That is, $\bar{x} = y$, which in turn implies that $y^* = J\bar{x}$. It follows that $Jy_n \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (2.12) that

$$\lim_{n\to\infty} Jy_n = J\bar{x}.$$

On the other hand, we have

$$\|Jx_n - Jy_n\| \leq \|Jx_n - J\bar{x}\| + \|J\bar{x} - Jy_n\|.$$

It follows that

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{2.13}$$

In view of

$$y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n\big),$$

we find that

$$Jx_n - Jy_n = (1-\alpha_n)\big(Jx_n - JT^n x_n\big).$$

In view of the restriction $\limsup_{n\to\infty} \alpha_n < 1$, we find from (2.13) that

$$\lim_{n\to\infty} \big\|J\big(T^n x_n\big) - Jx_n\big\| = 0. \tag{2.14}$$

Notice that

$$\big\|J\big(T^n x_n\big) - J\bar{x}\big\| \leq \big\|J\big(T^n x_n\big) - Jx_n\big\| + \|Jx_n - J\bar{x}\|.$$

This implies from (2.14) that

$$\lim_{n\to\infty} \big\|J\big(T^n x_n\big) - J\bar{x}\big\| = 0. \tag{2.15}$$

The demicontinuity of $J^{-1} : E^* \to E$ implies that $T^n x_n \rightharpoonup \bar{x}$. Note that

$$\big|\big\|T^n x_n\big\| - \|\bar{x}\|\big| = \big|\big\|J\big(T^n x_n\big)\big\| - \|J\bar{x}\|\big| \leq \big\|J\big(T^n x_n\big) - J\bar{x}\big\|.$$

With the aid of (2.15), we see that $\lim_{n\to\infty} \|T^n x_n\| = \|\bar{x}\|$. Since E has the Kadec-Klee property, we find that

$$\lim_{n\to\infty} \|T^n x_n - \bar{x}\| = 0. \tag{2.16}$$

Notice that

$$\|T^{n+1}x_n - \bar{x}\| \le \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - \bar{x}\|.$$

In view of the asymptotic regularity of T, we find from (2.16) that

$$\lim_{n\to\infty} \|T^{n+1}x_n - \bar{x}\| = 0,$$

that is, $TT^n x_n - \bar{x} \to 0$ as $n \to \infty$. It follows from the closedness of T that $T\bar{x} = \bar{x}$. This completes Step 5.

Step 6. Show that $\bar{x} = \Pi_{F(T)}x_1$.

Letting $n \to \infty$ in (2.9), we arrive at

$$\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \ge 0, \quad \forall w \in F(T).$$

It follows from Lemma 1.5 that $\bar{x} = \Pi_{F(T)}x_1$. This completes the proof of Theorem 2.1. □

Remark 2.2 The sets C_{n+1} become increasingly complicated, which may render the algorithm unimplementable. One may use an inner-loop for calculating an approximation of $\Pi_{C_{n+1}}$ at each iterative step. The advantage of the algorithm is that strong convergence of iterative sequences can be guaranteed in a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property without any compact assumptions.

If T is quasi-ϕ-nonexpansive, then Theorem 2.1 is reduced to the following.

Corollary 2.3 *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let $T: C \to C$ be a quasi-ϕ-nonexpansive mapping with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1. \end{cases}$$

If the sequence $\{\alpha_n\}$ satisfies the restriction $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Remark 2.4 Corollary 2.3 mainly improves the corresponding results [24] in the following aspects: (1) from the relatively nonexpansive mapping to the quasi-ϕ-nonexpansive mapping; (2) from a uniformly convex and uniformly smooth Banach space to a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. The algorithm is also different from the one in [24].

If E is a Hilbert space, then we have the following result.

Corollary 2.5 *Let E be a Hilbert space. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense. Assume that T is asymptotically regular on C and closed and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \le \|z - x_n\|^2 + \xi_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where

$$\xi_n = \max\Big\{0, \sup_{p \in F(T), x \in C} \big(\big\|p - T^n x\big\|^2 - \|p - x\|^2\big)\Big\}.$$

If the sequence $\{\alpha_n\}$ satisfies the restriction $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_1$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.

Proof In Hilbert spaces, we find that J is the identity and $\phi(x, y) = \|x - y\|^2$. We can immediately derive from Theorem 2.1 the desired conclusion. □

Remark 2.6 Corollary 2.5 can be viewed as an improvement of the corresponding result in Su and Qin [38]. The mapping is extended from asymptotically nonexpansive mappings to asymptotically quasi-nonexpansive mappings in the intermediate sense.

If T is quasi-nonexpansive, then we have the following result.

Corollary 2.7 *Let E be a Hilbert space. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a quasi-nonexpansive mapping with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \le \|z - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1. \end{cases}$$

If the sequence $\{\alpha_n\}$ *satisfies the restriction* $\limsup_{n\to\infty} \alpha_n < 1$, *then the sequence* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_1$, *where* $P_{F(T)}$ *is the metric projection from* C *onto* $F(T)$.

Proof In Hilbert spaces, we find that J is the identity and $\phi(x, y) = \|x - y\|^2$. We can immediately derive from Theorem 2.1 the desired conclusion. □

Remark 2.8 Corollary 2.7 can be viewed as an improvement of the corresponding result in Nakajo and Takahashi [10]. The mapping has been extended from a nonexpansive mapping to a quasi-nonexpansive mapping. The sets Q_n have also been relaxed.

Competing interests

The author declares that she has no competing interests.

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