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Iterative scheme for a nonexpansive mapping, an η-strictly pseudo-contractive mapping and variational inequality problems in a uniformly convex and 2-uniformly smooth Banach space

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Abstract

In this paper, we introduce an iterative scheme by the modification of Mann's iteration process for finding a common element of the set of solutions of a finite family of variational inequality problems and the set of fixed points of an η-strictly pseudo-contractive mapping and a nonexpansive mapping. Moreover, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of η_i-strictly pseudo-contractive mappings for every $i = 1, 2, \dots, N$ in uniformly convex and 2-uniformly smooth Banach spaces.

Keywords: nonexpansive mapping; strictly pseudo-contractive mapping; variational inequality problem

1 Introduction

Let E be a Banach space with its dual space E^* and let C be a nonempty closed convex subset of E. Throughout this paper, we denote the norm of E and E^* by the same symbol $\|\cdot\|$. We use the symbol $\to$ to denote the strong convergence. Recall the following definition.

Definition 1.1 A Banach space E is said to be *uniformly convex* iff for any ϵ, $0 < \epsilon \le 2$, the inequalities $\|x\| \le 1$, $\|y\| \le 1$ and $\|x - y\| \ge \epsilon$ imply there exists a $\delta > 0$ such that $\|\frac{x+y}{2}\| \le 1 - \delta$.

Definition 1.2 Let E be a Banach space. Then a function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *the modulus of smoothness of E* if

$$\rho_E(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}.$$

A Banach space E is said to be *uniformly smooth* if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0.$$

Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

Definition 1.3 A mapping J from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$$

is called the normalized duality mapping of E. The duality pair $\langle x, f\rangle$ represents $f(x)$ for $f \in E^*$ and $x \in E$.

Definition 1.4 Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a self-mapping. T is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

T is called an η-strictly pseudo-contractive mapping if there exists a constant $\eta \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \eta \|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$. It is clear that (1.1) is equivalent to the following:

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \eta \|(I - T)x - (I - T)y\|^2 \tag{1.2}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$.

Let C and D be nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is sunny [1] provided $P(x + t(x - P(x))) = P(x)$ for all $x \in C$ and $t \geq 0$, whenever $x + t(x - P(x)) \in C$. The mapping $P : C \to D$ is called a retraction if $Px = x$ for all $x \in D$. Furthermore, P is a sunny nonexpansive retraction from C onto D if P is a retraction from C onto D which is also sunny and nonexpansive. The subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

An operator A of C into E is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to E$ is said to be α*-inverse strongly accretive* if there exists $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Remark 1.1 From (1.1) and (1.2), if T is an η-strictly pseudo-contractive mapping, then $I - T$ is η-inverse strongly accretive.

The variational inequality problem in a Banach space is to find a point $x^* \in C$ such that for some $j(x - x^*) \in J(x - x^*)$,

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This problem was considered by Aoyama *et al.* [2]. The set of solutions of the variational inequality in a Banach space is denoted by $S(C, A)$, that is,

$$S(C, A) = \{u \in C : \langle Au, J(v - u)\rangle \geq 0, \forall v \in C\}. \tag{1.4}$$

Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games reduce to find an element of (1.4); see [3, 4].

Recall that the normal Mann's iterative process was introduced by Mann [5] in 1953. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where the sequence $\{\alpha_n\} \subset (0, 1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.5) converges weakly to a fixed point of T.

In 2008, Cho *et al.* [6] modified the normal Mann's iterative process and proved strong convergence for a finite family of nonexpansive mappings in the framework of Banach spaces without any commutative assumption as follows.

Theorem 1.2 *Let C be a closed convex subset of a uniformly smooth and strictly convex Banach space E. Let $\{T_i\}$ be a nonexpansive mapping from C into itself for $i = 1, 2, \ldots, N$. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$, the following conditions are satisfied*:

$$\text{(i)} \quad \lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$\text{(ii)} \quad \lim_{n\to\infty} |\gamma_{ni} - \gamma_{n-1i}| = 0 \quad \textit{for all } i = 1, 2, \ldots, N,$$

$$\text{(iii)} \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Let $\{x_n\}$ be a sequence generated by $u, x_0 = x \in C$ and

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where W_n is the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\gamma_{n1}, \gamma_{n2}, \ldots, \gamma_{nN}$. Then $\{x_n\}$ converges strongly to $x^* \in F$, where $x^* = Q(u)$ and $Q : C \to F$ is the unique sunny nonexpansive retraction from C onto F.

In 2008, Zhou [7] proved a strong convergence theorem for the modification of normal Mann's iteration algorithm generated by a strict pseudo-contraction in a real 2-uniformly smooth Banach space as follows.

Theorem 1.3 *Let C be a closed convex subset of a real 2-uniformly smooth Banach space E and let $T : C \to C$ be a λ-strict pseudo-contraction such that $F(T) \neq \emptyset$. Given $u, x_0 \in C$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$, the following control conditions are satisfied*:

(i) $a \leq \alpha_n \leq \frac{\lambda}{K^2}$ *for some $a > 0$ and for all $n \geq 0$*,

(ii) $\beta_n + \gamma_n + \delta_n = 1$ *for all $n \geq 0$*,

(iii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(iv) $\alpha_{n+1} - \alpha_n \to 0$ *as $n \to \infty$*,

(v) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let a sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \alpha_n T x_n + (1 - \alpha_n) x_n, \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \quad n \geq 0. \end{cases} \tag{1.7}$$

Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, where $x^* = Q_{F(T)}(u)$ and $Q_{F(T)} : C \to F(T)$ is the unique sunny nonexpansive retraction from C onto $F(T)$.

In 2005, Aoyama *et al.* [2] proved a weak convergence theorem for finding a solution of problem (1.3) as follows.

Theorem 1.4 *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$ and let A be an α-inverse strongly accretive operator of C into E with $S(C, A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E.

In this paper, motivated by Theorems 1.2, 1.3 and 1.4, we prove a strong convergence theorem for finding a common element of the set of solutions of a finite family of variational inequality problems and the set of fixed points of a nonexpansive mapping and an η-strictly pseudo-contractive mapping in uniformly convex and 2-uniformly smooth spaces. Moreover, by using our main result, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of η_i-strictly pseudo-contractive mappings for every $i = 1, 2, \dots, N$ in uniformly convex and 2-uniformly smooth Banach spaces.

2 Preliminaries

In this section, we collect and prove the following lemmas to use in our main result.

Lemma 2.1 (See [8]) *Let E be a real* 2*-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2$$

for any $x, y \in E$.

Definition 2.1 (See [9]) Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i = 1, \dots, N$. Define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_1 &= \lambda_1 T_1 + (1 - \lambda_1)I, \\
U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\
U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3)U_2, \\
&\vdots \\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\
K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}.
\end{aligned} \tag{2.1}$$

Such a mapping K is called the *K-mapping* generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$.

Lemma 2.2 (See [9]) *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpanxive mappings of C into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $\lambda_1, \dots, \lambda_N$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, \dots, N-1$ *and* $0 < \lambda_N \le 1$. *Let K be the K-mapping generated by* $T_1, \dots, T_N$ *and* $\lambda_1, \dots, \lambda_N$. *Then* $F(K) = \bigcap_{i=1}^N F(T_i)$.

Remark 2.3 From Lemma 2.2, it is easy to see that the K mapping is a nonexpansive mapping.

Lemma 2.4 (See [10]) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$$

for all integer $n \ge 0$ *and*

$$\limsup_{n\to\infty}\big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.5 (See [11]) *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0,\infty] \to [0,\infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g\big(\|x - y\|\big)$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.6 (See [2]) *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all $\lambda > 0$,*

$$S(C,A) = F\big(Q_C(I - \lambda A)\big).$$

Lemma 2.7 (See [12]) *Let C be a closed convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=1}^{\infty} \lambda_n T_n x$ for $x \in C$ is well defined, non-expansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.*

Lemma 2.8 (See [8]) *Let $r > 0$. If E is uniformly convex, then there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that for all $x, y \in B_r(0) = \{x \in E : \|x\| \leq r\}$ and for any $\alpha \in [0,1]$, we have $\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x - y\|)$.*

Lemma 2.9 (See [13]) *Let X be a uniformly smooth Banach space, C be a closed convex subset of X, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $f \in \prod_C$ where $\prod_C$ is to denote the collection of all contractions on C. Then the sequence $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converses strongly to a point in $F(T)$. If we define a mapping $Q : \prod_C \to F(T)$ by $Q(f) = \lim_{t\to 0} x_t$ for all $f \in \prod_C$, then $Q(f)$ solves the following variational inequality*:

$$\big\langle (I - f)Q(f), j\big(Q(f) - p\big)\big\rangle \leq 0$$

for all $f \in \prod_C$, $p \in F(T)$.

Lemma 2.10 (See [14]) *In a Banach space E, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\big\langle y, j(x + y)\big\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 2.11 (See [15]) *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy the conditions*

$$(1)\quad \{\alpha_n\} \subset [0,1], \qquad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2)\quad \limsup_{n\to\infty} \beta_n \leq 0 \quad \textit{or} \quad \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.12 *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudocontractive mapping with $F(S) \cap F(T) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $B_A x = T((1-\alpha)I + \alpha S)x$ for all $x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$, where K is the 2-uniformly smooth constant of E. Then $F(B_A) = F(S) \cap F(T)$.*

Proof It is easy to see that $F(T) \cap F(S) \subseteq F(B_A)$. Let $x_0 \in F(B_A)$ and $x^* \in F(T) \cap F(S)$, we have

$$\begin{aligned}
\|x_0 - x^*\|^2 &= \|T((1-\alpha)x_0 + \alpha S x_0) - x^*\|^2 \\
&\leq \|(1-\alpha)x_0 + \alpha S x_0 - x^*\|^2 \\
&= \|x_0 - x^* + \alpha(Sx_0 - x_0)\|^2 \\
&\leq \|x_0 - x^*\|^2 + 2\alpha\langle Sx_0 - x_0, j(x_0 - x^*)\rangle + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 + 2\alpha\langle Sx_0 - x^*, j(x_0 - x^*)\rangle + 2\alpha\langle x^* - x_0, j(x_0 - x^*)\rangle \\
&\quad + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 + 2\alpha\langle Sx_0 - x^*, j(x_0 - x^*)\rangle - 2\alpha\|x_0 - x^*\|^2 + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&\leq \|x_0 - x^*\|^2 + 2\alpha(\|x_0 - x^*\|^2 - \eta\|(I-S)x_0\|^2) - 2\alpha\|x_0 - x^*\|^2 \\
&\quad + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 - 2\alpha\eta\|x_0 - Sx_0\|^2 + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 - 2\alpha(\eta - K^2\alpha)\|x_0 - Sx_0\|^2.
\end{aligned} \tag{2.2}$$

(2.2) implies that

$$2\alpha(\eta - K^2\alpha)\|x_0 - Sx_0\|^2 \leq \|x_0 - x^*\|^2 - \|x_0 - x^*\|^2 = 0.$$

Then we have $Sx_0 = x_0$, that is, $x_0 \in F(S)$.

Since $x_0 \in F(B_A)$, from the definition of B_A, we have

$$x_0 = B_A x_0 = T((1-\alpha)x_0 + \alpha S x_0) = Tx_0.$$

Then we have $x_0 \in F(T)$. Therefore, $x_0 \in F(T) \cap F(S)$. It follows that $F(B_A) \subseteq F(T) \cap F(S)$. Hence, $F(B_A) = F(T) \cap F(S)$. □

Remark 2.13 Applying (2.2), we have that the mapping B_A is nonexpansive.

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1, 2, \dots, N$, let $A_i : C \to E$ be an α_i-inverse strongly accretive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 1, 2, \dots, N$, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1, 2, \dots, N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap F(T) \cap \bigcap_{i=1}^{N} S(C, A_i) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $T((1-\alpha)I + \alpha S)x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \geq 1, \tag{3.1}$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c, d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Proof First, we will show that G_i is a nonexpansive mapping for every $i = 1, 2, \dots, N$.

Let $x, y \in C$. From nonexpansiveness of Q_C, we have

$$\begin{aligned}
\|G_i x - G_i y\|^2 &= \|Q_C(I - \lambda_i A_i)x - Q_C(I - \lambda_i A_i)y\|^2 \\
&\leq \|(I - \lambda_i A_i)x - (I - \lambda_i A_i)y\|^2 \\
&= \|x - y - \lambda_i(A_i x - A_i y)\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_i \langle A_i x - A_i y, j(x-y) \rangle + 2K^2 \lambda_i^2 \|A_i x - A_i y\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_i \alpha_i \|A_i x - A_i y\|^2 + 2K^2 \lambda_i^2 \|A_i x - A_i y\|^2 \\
&= \|x-y\|^2 - 2\lambda_i (\alpha_i - K^2 \lambda_i) \|A_i x - A_i y\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Then we have G_i is a nonexpansive mapping for every $i = 1, 2, \dots, N$. Since $B : C \to C$ is the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$ and Lemma 2.2, we can conclude that $F(B) = \bigcap_{i=1}^N F(G_i)$. From Lemma 2.6 and the definition of G_i, we have $F(G_i) = S(C,A_i)$ for every $i = 1,2,\dots,N$. Hence, we have

$$F(B) = \bigcap_{i=1}^{N} F(G_i) = \bigcap_{i=1}^{N} S(C,A_i). \tag{3.2}$$

Next, we will show that the sequence $\{x_n\}$ is bounded.

Let $z \in \mathcal{F}$; from the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\| &\le \alpha_n \|f(x_n) - z\| + \beta_n\|x_n - z\| + \gamma_n\|Bx_n - z\| + \delta_n\|B_Ax_n - z\| \\
&\le \alpha_n \|f(x_n) - z\| + (1-\alpha_n)\|x_n - z\| \\
&\le \alpha_n \|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&\le \alpha_n a\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&= \bigl(1 - \alpha_n(1-a)\bigr)\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\le \max\left\{\|x_1 - z\|, \frac{\|f(z) - z\|}{1-a}\right\}.
\end{aligned}$$

By induction, we can conclude that the sequence $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{Bx_n\}$, $\{B_Ax_n\}$.

Next, we will show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

From the definition of x_n, we can rewrite x_n by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \tag{3.4}$$

where $z_n = \frac{\alpha_n f(x_n)+\gamma_n Bx_n+\delta_n B_Ax_n}{1-\beta_n}$.

Since

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}Bx_{n+1} + \delta_{n+1}B_Ax_{n+1}}{1-\beta_{n+1}} \right. \\
&\qquad \left. - \left(\frac{\alpha_n f(x_n) + \gamma_n Bx_n + \delta_n B_Ax_n}{1-\beta_n}\right)\right\| \\
&= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \right\| \\
&= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_{n+1}} + \frac{x_{n+1} - \beta_n x_n}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \right\| \\
&\le \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_{n+1}} \right\| + \left\| \frac{x_{n+1} - \beta_n x_n}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \right\| \\
&= \frac{1}{1-\beta_{n+1}} \|x_{n+2} - \beta_{n+1}x_{n+1} - (x_{n+1} - \beta_n x_n)\| \\
&\qquad + \left| \frac{1}{1-\beta_{n+1}} - \frac{1}{1-\beta_n} \right| \|x_{n+1} - \beta_n x_n\|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-\beta_{n+1}}\left\|x_{n+2}-\beta_{n+1}x_{n+1}-(x_{n+1}-\beta_n x_n)\right\| \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&= \frac{1}{1-\beta_{n+1}}\left\|\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Bx_{n+1}+\delta_{n+1}B_A x_{n+1}\right. \\
&\quad \left.-\left(\alpha_n f(x_n)+\gamma_n Bx_n+\delta_n B_A x_n\right)\right\| + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&= \frac{1}{1-\beta_{n+1}}\left(\left\|\alpha_{n+1}f(x_{n+1})-\alpha_n f(x_n)\right\|+\gamma_{n+1}\|Bx_{n+1}-Bx_n\|\right. \\
&\quad \left.+\delta_{n+1}\|B_A x_{n+1}-B_A x_n\|+|\gamma_{n+1}-\gamma_n|\|Bx_n\|+|\delta_{n+1}-\delta_n|\|B_A x_n\|\right) \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&\leq \frac{1}{1-\beta_{n+1}}\left(\alpha_{n+1}\left\|f(x_{n+1})\right\|+\alpha_n\left\|f(x_n)\right\|+(\gamma_{n+1}+\delta_{n+1})\|x_{n+1}-x_n\|\right. \\
&\quad \left.+|\gamma_{n+1}-\gamma_n|\|Bx_n\|+|\delta_{n+1}-\delta_n|\|B_A x_n\|\right) \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|f(x_{n+1})\right\|+\frac{\alpha_n}{1-\beta_{n+1}}\left\|f(x_n)\right\|+\frac{\gamma_{n+1}+\delta_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\
&\quad + \frac{|\gamma_{n+1}-\gamma_n|}{1-\beta_{n+1}}\|Bx_n\|+\frac{|\delta_{n+1}-\delta_n|}{1-\beta_{n+1}}\|B_A x_n\| \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|f(x_{n+1})\right\|+\frac{\alpha_n}{1-\beta_{n+1}}\left\|f(x_n)\right\|+\|x_{n+1}-x_n\|+\frac{|\gamma_{n+1}-\gamma_n|}{1-\beta_{n+1}}\|Bx_n\| \\
&\quad + \frac{|\delta_{n+1}-\delta_n|}{1-\beta_{n+1}}\|B_A x_n\|+\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\|. \qquad (3.5)
\end{aligned}$$

From (3.5) and the conditions (i)-(iv), we have

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\right) \leq 0. \qquad (3.6)$$

From Lemma 2.4 and (3.4), we have

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \qquad (3.7)$$

From (3.4), we have

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|z_n-x_n\|,$$

and from the condition (iv) and (3.7), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Next, we will show that

$$\lim_{n\to\infty} \|Bx_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|B_A x_n - x_n\| = 0.$$

From the definition of x_n, we can rewrite x_{n+1} by

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n \\ &= \alpha_n f(x_n) + \beta_n x_n + (\gamma_n + \delta_n) \frac{(\gamma_n B x_n + \delta_n B_A x_n)}{\gamma_n + \delta_n} \\ &= \alpha_n f(x_n) + \beta_n x_n + e_n z_n', \end{aligned} \tag{3.8}$$

where $e_n = \gamma_n + \delta_n$ and $z_n' = \frac{(\gamma_n B x_n + \delta_n B_A x_n)}{\gamma_n + \delta_n}$.

From Lemma 2.5 and (3.8), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\|\alpha_n \big(f(x_n) - z\big) + \beta_n (x_n - z) + e_n \big(z_n' - z\big)\right\|^2 \\ &\le \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n \|x_n - z\|^2 + e_n \left\|z_n' - z\right\|^2 - \beta_n e_n g_1\left(\left\|z_n' - x_n\right\|\right) \\ &= \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n \|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z_n' - x_n\right\|\right) \\ &\quad + e_n \left\| \frac{(\gamma_n B x_n + \delta_n B_A x_n)}{\gamma_n + \delta_n} - z \right\|^2 \\ &= \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n \|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z_n' - x_n\right\|\right) \\ &\quad + e_n \left\| \left(1 - \frac{\delta_n}{\gamma_n + \delta_n}\right)(B x_n - z) + \frac{\delta_n}{\gamma_n + \delta_n}(B_A x_n - z) \right\|^2 \\ &\le \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n \|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z_n' - x_n\right\|\right) \\ &\quad + e_n \left(\left(1 - \frac{\delta_n}{\gamma_n + \delta_n}\right) \|B x_n - z\| + \frac{\delta_n}{\gamma_n + \delta_n} \|B_A x_n - z\| \right)^2 \\ &\le \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n \|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z_n' - x_n\right\|\right) + e_n \|x_n - z\|^2 \\ &\le \alpha_n \left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z_n' - x_n\right\|\right), \end{aligned}$$

which implies that

$$\begin{aligned} \beta_n e_n g_1\left(\left\|z_n' - x_n\right\|\right) &\le \alpha_n \left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\le \alpha_n \left\|f(x_n) - z\right\|^2 + \big(\|x_n - z\| + \|x_{n+1} - z\|\big) \|x_{n+1} - x_n\|. \end{aligned} \tag{3.9}$$

From the conditions (i), (ii), (iv) and (3.3), we have

$$\lim_{n\to\infty} g_1\left(\left\|z_n' - x_n\right\|\right) = 0.$$

From the properties of g_1, we have

$$\lim_{n\to\infty} \left\|z_n' - x_n\right\| = 0. \tag{3.10}$$

From Lemma 2.8 and the definition of z_n', we have

$$\begin{aligned}\|z_n' - z\|^2 &= \left\| \frac{(\gamma_n B x_n + \delta_n B_A x_n)}{\gamma_n + \delta_n} - z \right\|^2 \\ &= \left\| \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)(Bx_n - z) + \frac{\delta_n}{\delta_n + \gamma_n}(B_A x_n - z) \right\|^2 \\ &\leq \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\|Bx_n - z\|^2 + \frac{\delta_n}{\delta_n + \gamma_n}\|B_A x_n - z\|^2 \\ &\quad - \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\frac{\delta_n}{\delta_n + \gamma_n} g_2\big(\|Bx_n - B_A x_n\|\big) \\ &\leq \|x_n - z\|^2 - \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\frac{\delta_n}{\delta_n + \gamma_n} g_2\big(\|Bx_n - B_A x_n\|\big),\end{aligned}$$

which implies that

$$\begin{aligned}\left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\frac{\delta_n}{\delta_n + \gamma_n} g_2\big(\|Bx_n - B_A x_n\|\big) &\leq \|x_n - z\|^2 - \|z_n' - z\|^2 \\ &\leq \big(\|x_n - z\| + \|z_n' - z\|\big)\|z_n' - x_n\|.\end{aligned}$$

From the condition (iii) and (3.10), we have

$$\lim_{n\to\infty} g_2\big(\|Bx_n - B_A x_n\|\big) = 0.$$

From the properties of g_2, we have

$$\lim_{n\to\infty} \|Bx_n - B_A x_n\| = 0. \tag{3.11}$$

From the definition of x_n, we can rewrite x_{n+1} by

$$\begin{aligned}x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n \\ &= \beta_n x_n + \gamma_n B x_n + (\alpha_n + \delta_n)\frac{\alpha_n f(x_n) + \delta_n B_A x_n}{\alpha_n + \delta_n} \\ &= \beta_n x_n + \gamma_n B x_n + d_n z_n'', \end{aligned} \tag{3.12}$$

where $d_n = \alpha_n + \delta_n$ and $z_n'' = \frac{\alpha_n f(x_n) + \delta_n B_A x_n}{\alpha_n + \delta_n}$.

From Lemma 2.5 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \big\|\beta_n(x_n - z) + \gamma_n(Bx_n - z) + d_n\big(z_n'' - z\big)\big\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\|z_n'' - z\|^2 - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big) \\ &= \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\left\|\frac{\alpha_n f(x_n) + \delta_n B_A x_n}{\alpha_n + \delta_n} - z\right\|^2 \\ &\quad - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big) \\ &= \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\left\|\frac{\alpha_n}{\alpha_n + \delta_n}\big(f(x_n) - z\big)\right. \\ &\quad \left. + \left(1 - \frac{\alpha_n}{\alpha_n + \delta_n}\right)(B_A x_n - z)\right\|^2 - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big)\end{aligned}$$

$$\begin{aligned}
&\le \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\left(\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2\right.\\
&\quad \left. + \left(1 - \frac{\alpha_n}{\alpha_n+\delta_n}\right)\|B_Ax_n - z\|^2\right) - \beta_n\gamma_n g_3\left(\|x_n - Bx_n\|\right)\\
&= \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2\\
&\quad + d_n\left(1 - \frac{\alpha_n}{\alpha_n+\delta_n}\right)\|B_Ax_n - z\|^2 - \beta_n\gamma_n g_3\left(\|x_n - Bx_n\|\right)\\
&\le \beta_n\|x_n - z\|^2 + \gamma_n\|x_n - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2\\
&\quad + d_n\|x_n - z\|^2 - \beta_n\gamma_n g_3\left(\|x_n - Bx_n\|\right)\\
&\le \|x_n - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2 - \beta_n\gamma_n g_3\left(\|x_n - Bx_n\|\right),
\end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}
\beta_n\gamma_n g_3\left(\|x_n - Bx_n\|\right) &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2\\
&\le \left(\|x_n - z\| + \|x_{n+1} - z\|\right)\|x_{n+1} - x_n\|\\
&\quad + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2.
\end{aligned} \tag{3.14}$$

From the conditions (i), (ii), (iv) (3.14) and (3.3), we have

$$\lim_{n\to\infty} g_3\left(\|x_n - Bx_n\|\right) = 0.$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|x_n - Bx_n\| = 0. \tag{3.15}$$

From (3.11), (3.15) and

$$\|x_n - B_Ax_n\| \le \|x_n - Bx_n\| + \|Bx_n - B_Ax_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - B_Ax_n\| = 0. \tag{3.16}$$

Define a mapping $L : C \to C$ by $Lx = (1-\epsilon)Bx + \epsilon B_Ax$ for all $x \in C$ and $\epsilon \in (0,1)$. From Lemma 2.7, 2.12 and (3.2), we have $F(L) = F(B) \cap F(B_A) = \bigcap_{i=1}^{N} S(C, A_i) \cap F(S) \cap F(T) = \mathcal{F}$.

From (3.15) and (3.16) and

$$\begin{aligned}
\|x_n - Lx_n\| &= \left\|(1-\epsilon)(x_n - Bx_n) + \epsilon(x_n - B_Ax_n)\right\|\\
&\le (1-\epsilon)\|x_n - Bx_n\| + \epsilon\|x_n - B_Ax_n\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - Lx_n\| = 0. \tag{3.17}$$

Next, we will show that

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle \le 0, \tag{3.18}$$

where $\lim_{t\to 0} x_t = q \in \mathcal{F}$ and x_t begins the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Lx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Lx_t$.

From the definition of x_t, we have

$$\begin{aligned}
\|x_t - x_n\|^2 &= \big\| t\big(f(x_t)-x_n\big) + (1-t)(Lx_t - x_n)\big\|^2 \\
&\le (1-t)^2\|Lx_t - x_n\|^2 + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&\le (1-t)^2\big(\|Lx_t - Lx_n\| + \|Lx_n - x_n\|\big)^2 + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&\le (1-t)^2\big(\|x_t - x_n\| + \|Lx_n - x_n\|\big)^2 + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&= (1-t)^2\big(\|x_t-x_n\|^2 + 2\|x_t-x_n\|\|Lx_n-x_n\| + \|Lx_n-x_n\|^2\big) \\
&\quad + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&= (1-t)^2\big(\|x_t-x_n\|^2 + 2\|x_t-x_n\|\|Lx_n-x_n\| + \|Lx_n-x_n\|^2\big) \\
&\quad + 2t\langle f(x_t)-x_t, j(x_t-x_n)\rangle + 2t\langle x_t-x_n, j(x_t-x_n)\rangle \\
&= \big(1-2t+t^2\big)\|x_t-x_n\|^2 + (1-t)^2\big(2\|x_t-x_n\|\|Lx_n-x_n\| + \|Lx_n-x_n\|^2\big) \\
&\quad + 2t\langle f(x_t)-x_t, j(x_t-x_n)\rangle + 2t\|x_t-x_n\|^2 \\
&= \big(1+t^2\big)\|x_t-x_n\|^2 + f_n(t) + 2t\langle f(x_t)-x_t, j(x_t-x_n)\rangle,
\end{aligned} \tag{3.19}$$

where $f_n(t) = (1-t)^2(2\|x_t-x_n\|\|Lx_n-x_n\| + \|Lx_n-x_n\|^2)$. From (3.17), we have

$$\lim_{n\to\infty} f_n(t) = 0. \tag{3.20}$$

(3.19) implies that

$$\begin{aligned}
\langle x_t - f(x_t), j(x_t-x_n)\rangle &\le \frac{t}{2}\|x_t-x_n\|^2 + \frac{1}{2t}f_n(t) \\
&\le \frac{t}{2}D + \frac{1}{2t}f_n(t),
\end{aligned} \tag{3.21}$$

where $D > 0$ such that $\|x_t - x_n\|^2 \le D$ for all $t \in (0,1)$ and $n \ge 1$. From (3.20) and (3.21), we have

$$\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t-x_n)\rangle \le \frac{t}{2}D. \tag{3.22}$$

From (3.22) taking $t \to 0$, we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t-x_n)\rangle \le 0. \tag{3.23}$$

Since

$$\begin{aligned}\langle f(q)-q,j(x_n-q)\rangle &= \langle f(q)-q,j(x_n-q)\rangle - \langle f(q)-q,j(x_n-x_t)\rangle + \langle f(q)-q,j(x_n-x_t)\rangle \\ &\quad - \langle f(q)-x_t,j(x_n-x_t)\rangle + \langle f(q)-x_t,j(x_n-x_t)\rangle \\ &\quad - \langle f(x_t)-x_t,j(x_n-x_t)\rangle + \langle f(x_t)-x_t,j(x_n-x_t)\rangle \\ &= \langle f(q)-q,j(x_n-q)-j(x_n-x_t)\rangle + \langle x_t-q,j(x_n-x_t)\rangle \\ &\quad + \langle f(q)-f(x_t),j(x_n-x_t)\rangle + \langle f(x_t)-x_t,j(x_n-x_t)\rangle \\ &\leq \langle f(q)-q,j(x_n-q)-j(x_n-x_t)\rangle + \|x_t-q\|\|x_n-x_t\| \\ &\quad + a\|q-x_t\|\|x_n-x_t\| + \langle f(x_t)-x_t,j(x_n-x_t)\rangle,\end{aligned}$$

it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)\rangle &\leq \limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)-j(x_n-x_t)\rangle \\ &\quad + \|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\| + a\|q-x_t\|\limsup_{n\to\infty}\|x_n-x_t\| \\ &\quad + \limsup_{n\to\infty}\langle f(x_t)-x_t,j(x_n-x_t)\rangle. \end{aligned} \tag{3.24}$$

Since j is norm-to-norm uniformly continuous on a bounded subset of C and (3.24), then we have

$$\limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)\rangle = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)\rangle \leq 0.$$

Finally, we will show the sequence $\{x_n\}$ converses strongly to $q\in\mathcal{F}$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\alpha_n(f(x_n)-q)+\beta_n(x_n-q)+\gamma_n(Bx_n-q)+\delta_n(B_Ax_n-q)\|^2 \\ &\leq \|\beta_n(x_n-q)+\gamma_n(Bx_n-q)+\delta_n(B_Ax_n-q)\|^2 \\ &\quad + 2\alpha_n\langle f(x_n)-q,j(x_{n+1}-q)\rangle \\ &\leq (\beta_n\|x_n-q\|+\gamma_n\|Bx_n-q\|+\delta_n\|B_Ax_n-q\|)^2 \\ &\quad + 2\alpha_n\langle f(x_n)-f(q),j(x_{n+1}-q)\rangle + 2\alpha_n\langle f(q)-q,j(x_{n+1}-q)\rangle \\ &\leq (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n\langle f(x_n)-f(q),j(x_{n+1}-q)\rangle \\ &\quad + 2\alpha_n\langle f(q)-q,j(x_{n+1}-q)\rangle \\ &\leq (1-\alpha_n)^2\|x_n-q\|^2 + 2a\alpha_n\|x_n-q\|\|x_{n+1}-q\| \\ &\quad + 2\alpha_n\langle f(q)-q,j(x_{n+1}-q)\rangle \\ &\leq (1-\alpha_n)^2\|x_n-q\|^2 + a\alpha_n\|x_n-q\|^2 + a\alpha_n\|x_{n+1}-q\|^2 \\ &\quad + 2\alpha_n\langle f(q)-q,j(x_{n+1}-q)\rangle \\ &= (1-2\alpha_n+\alpha_n^2)\|x_n-q\|^2 + a\alpha_n\|x_n-q\|^2 + a\alpha_n\|x_{n+1}-q\|^2 \\ &\quad + 2\alpha_n\langle f(q)-q,j(x_{n+1}-q)\rangle\end{aligned}$$

$$
\begin{aligned}
&= (1-2\alpha_n + a\alpha_n)\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + a\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&= (1 - a\alpha_n - 2\alpha_n + 2a\alpha_n)\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + a\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&= \bigl(1 - a\alpha_n - 2\alpha_n(1-a)\bigr)\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + a\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\le \left(1 - \frac{2\alpha_n(1-a)}{1 - a\alpha_n}\right)\|x_n - q\|^2 \\
&\quad + \frac{\alpha_n}{1 - a\alpha_n}\bigl(\alpha_n\|x_n - q\|^2 + 2\langle f(q) - q, j(x_{n+1} - q)\rangle\bigr) \\
&\le \left(1 - \frac{2\alpha_n(1-a)}{1 - a\alpha_n}\right)\|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n(1-a)}{1 - a\alpha_n} \cdot \frac{1}{2(1-a)}\bigl(\alpha_n\|x_n - q\|^2 + 2\langle f(q) - q, j(x_{n+1} - q)\rangle\bigr).
\end{aligned}
$$

From the condition (i) and Lemma 2.11, we can imply that $\{x_n\}$ converses strongly to $q \in \mathcal{F}$. This completes the proof. □

The following results can be obtained from Theorem 3.1. We, therefore, omit the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1, 2, \dots, N$, let $A : C \to E$ be a ν-inverse strongly accretive mapping. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap F(T) \cap S(C, A) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $T((1-\alpha)I + \alpha S)x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$, where K is the 2-uniformly smooth constant of E. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Q_C(I - \lambda A)x_n + \delta_n B_A x_n, \quad \forall n \ge 1,
$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, $\lambda \in (0, \frac{\nu}{K^2})$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c, d] \subset (0, 1)$ *for some* $c, d > 0$ *and* $\forall n \ge 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p)\rangle \le 0, \quad \forall p \in \mathcal{F}.$$

Corollary 3.3 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $A_i : C \to E$ be an α_i-inverse strongly accretive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F} = F(T) \cap \bigcap_{i=1}^N S(C, A_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n T x_n, \quad \forall n \ge 1,$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some $c, d > 0$ and $\forall n \ge 1$*,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p)\rangle \le 0, \quad \forall p \in \mathcal{F}.$$

Corollary 3.4 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $A_i : C \to E$ be an α_i-inverse strongly accretive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N S(C, A_i) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $(1-\alpha)x + \alpha S x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \ge 1,$$

where $f : C \to C$ *is a contractive mapping and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c,d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q-p)\rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

4 Applications

To prove the next theorem, we needed the following lemma.

Lemma 4.1 *Let* C *be a nonempty closed convex subset of a Banach space* E *and let* $P : C \to C$ *be an* η*-strictly pseudo-contractive mapping with* $F(P) \neq \emptyset$. *Then* $F(P) = S(C, I-P)$.

Proof It is easy to see that $F(P) \subseteq S(C, I-P)$. Put $A = I - P$ and $z^* \in F(P)$. Let $z_0 \in S(C, I-P)$, then there exists $j(x - z_0) \in J(x - z_0)$ such that

$$\langle (I-P)z_0, j(x-z_0)\rangle \geq 0, \quad \forall x \in C. \tag{4.1}$$

Since P is an η-strictly pseudo-contractive mapping, then there exists $j(z_0 - z^*)$ such that

$$\begin{aligned} \langle Pz_0 - Pz^*, j(z_0 - z^*)\rangle &= \langle (I-A)z_0 - (I-A)z^*, j(z_0 - z^*)\rangle \\ &= \langle z_0 - z^* - (Az_0 - Az^*), j(z_0 - z^*)\rangle \\ &= \langle z_0 - z^*, j(z_0 - z^*)\rangle - \langle Az_0 - Az^*, j(z_0 - z^*)\rangle \\ &= \|z_0 - z^*\|^2 - \langle Az_0, j(z_0 - z^*)\rangle \\ &\leq \|z_0 - z\|^2 - \eta\|(I-P)z_0\|^2. \end{aligned} \tag{4.2}$$

From (4.1), (4.2), we have

$$\eta\|z_0 - Pz_0\|^2 \leq \langle Az_0, j(z_0 - z^*)\rangle = -\langle Az_0, j(z^* - z_0)\rangle \leq 0.$$

It implies that $z_0 = Pz_0$, that is, $z_0 \in F(P)$. Then we have $S(C, I-P) \subseteq F(P)$. Hence, we have $S(C, I-P) = F(P)$. □

Remark 4.2 If C is a closed convex subset of a smooth Banach space E and Q_C is a sunny nonexpansive retraction from E onto C, from Remark 1.1, Lemma 2.6 and 4.1, we have

$$F(P) = S(C, I-P) = F(Q_C(I - \lambda(I-P))) \tag{4.3}$$

for all $\lambda > 0$.

Theorem 4.3 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $S_i : C \to E$ be an η_i-strictly pseudo-contractive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i(I - S_i))x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\eta_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $T((1-\alpha)I + \alpha S)x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \geq 1,$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c,d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Proof Since S_i is an η_i-strictly pseudo-contractive mapping, then we have $(I - S_i)$ is an η_i-inverse strongly accretive mapping for every $i = 1,2,\dots,N$. For every $i = 1,2,\dots,N$, putting $A_i = I - S_i$ in Theorem 3.1, from Remark 4.2 and Theorem 3.1, we can conclude the desired results. □

Next corollaries are derived from Theorem 4.3. We, therefore, omit the proof.

Corollary 4.4 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $S_i : C \to E$ be an η_i-strictly pseudo contractive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i(I - S_i))x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\eta_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F} = F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n T x_n, \quad \forall n \geq 1,$$

where $f : C \to C$ *is a contractive mapping and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c,d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q - p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Corollary 4.5 *Let* C *be a nonempty closed convex subset of a uniformly convex and* 2-*uniformly smooth Banach space* E. *Let* Q_C *be the sunny nonexpansive retraction from* E *onto* C. *For every* $i = 1,2,\dots,N$, *let* $S_i : C \to E$ *be an* η_i-*strictly pseudo contractive mapping. Define a mapping* $G_i : C \to C$ *by* $Q_C(I - \lambda_i(I - S_i))x = G_i x$ *for all* $x \in C$ *and* $i = 1,2,\dots,N$, *where* $\lambda_i \in (0, \frac{\eta_i}{K^2})$, K *is the* 2-*uniformly smooth constant of* E. *Let* $B : C \to C$ *be the* K-*mapping generated by* $G_1, G_2, \dots, G_N$ *and* $\rho_1, \rho_2, \dots, \rho_N$, *where* $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ *and* $\rho_N \in (0,1]$. $S : C \to C$ *be an* η-*strictly pseudo contractive mapping with* $\mathcal{F} = F(S) \cap \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. *Define a mapping* $B_A : C \to C$ *by* $(1 - \alpha)x + \alpha Sx = B_A x$, $\forall x \in C$ *and* $\alpha \in (0, \frac{\eta}{K^2})$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \geq 1,$$

where $f : C \to C$ *is a contractive mapping and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c,d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q - p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Competing interests
The author declares that they have no competing interests.

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