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On generalized asymptotically quasi-ϕ-nonexpansive mappings and a Ky Fan inequality

Jianmin Song* and Minjiang Chen

*Correspondence:
hbsongjm@yeah.net
School of Mathematics and Sciences, Shijiazhuang University of Economics, Shijiazhuang, Hebei 050031, China

Abstract

Generalized asymptotically quasi-ϕ-nonexpansive mappings and a Ky Fan inequality are investigated. A strong convergence theorem for common solutions to a fixed point problem of generalized asymptotically quasi-ϕ-nonexpansive mappings and a Ky Fan inequality is established in a Banach space.

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1 Introduction

Iterative algorithms have been studied by many authors. The applications of iterative algorithms are found in a wide range of areas, including economics, image recovery and signal processing. Many well-known problems can be studied by using algorithms which are iterative in their nature; see [1–14] and the references therein. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set, in which the required solution lies. The problem of finding a point in the intersection of these convex subsets is then of crucial interest, and it cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such a point.

Mann iteration, introduced by Mann [15], is an efficient tool to study fixed point problems of asymptotical nonexpansive mappings. However, Mann iteration is only weak convergence in infinite-dimensional spaces; see [10] and the references therein. The importance of strong convergence is underlined in [16], where a convex function f is minimized via the proximal-point algorithm: it is shown that the rate of convergence of the value sequence $\{f(x_n)\}$ is better when $\{x_n\}$ converges strongly than when it converges weakly. Such properties have a direct impact when the process is executed directly in the underlying infinite-dimensional space. To obtain strong convergence of Mann iteration, projection methods, which were first introduced by Haugazeau [17], have been considered for modifying Mann iteration to obtain strong convergence. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions.

The organization of this paper is as follows. In Section 2, we provide some necessary concepts and lemmas. In Section 3, fixed point problems of generalized asymptotically quasi-

ϕ-nonexpansive mappings and solutions of a Ky Fan inequality are investigated. A strong convergence theorem is established in a Banach space.

2 Preliminaries

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth iff

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth iff the above limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex. Recall that E is said to be strictly convex iff $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex iff $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Recall that E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on the Kadec-Klee property, readers can refer to [18] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that in a Hilbert space H, the equality is reduced to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [19] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J; see, for example, [18]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of a function ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \tag{2.1}$$

and

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x,y) \leq \big(\|y\| + \|x\|\big)^2, \quad \forall x,y \in E. \tag{2.2}$$

Remark 2.1 If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$; for more details, see [18] and the reference therein.

Let C be a nonempty subset of E and $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n\to\infty}\big\{\big\|T^{n+1}x - T^n x\big\| : x \in K\big\} = 0.$$

T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote strong convergence and weak convergence, respectively. Recall that a point p in C is said to be an asymptotic fixed point of T iff C contains a sequence $\{x_n\}$ which converges weakly to p so that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

Recall that T is said to be relatively nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p,Tx) \leq \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

Recall that T is said to be relatively asymptotically nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi\big(p,T^n x\big) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Recall that a mapping T is said to be quasi-ϕ-nonexpansive iff

$$F(T) \neq \emptyset, \qquad \phi(p,Tx) \leq \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

Recall that a mapping T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \qquad \phi\big(p,T^n x\big) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

Remark 2.2 The class of asymptotically quasi-ϕ-nonexpansive mappings was considered in Zhou *et al.* [20] and Qin *et al.* [21]; see also [22] and [23].

Remark 2.3 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings [24]. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the restriction $F(T) = \widetilde{F}(T)$.

Remark 2.4 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces.

Recall that T is said to be generalized asymptotically quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$, and there exists a sequence $\{\mu_n\} \subset [1,\infty)$ with $\mu_n \to 1$ as $n \to \infty$ and a sequence $\{\nu_n\} \subset [0,\infty)$ with $\nu_n \to 0$ as $n \to \infty$ such that $\phi(p,Tx) \leq \mu_n \phi(p,x) + \nu_n$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 2.5 The class of generalized asymptotically quasi-ϕ-nonexpansive mappings was considered in Qin *et al.* [25]; see also [26].

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, and let $A : C \to E^*$ be a mapping. Consider the following Ky Fan inequality which is known as a generalized equilibrium problem. Find $p \in C$ such that

$$f(p,q) + \langle Ap, q - p\rangle \geq 0, \quad \forall q \in C. \tag{2.3}$$

We use $S(f,A)$ to denote the solution set of inequality (2.3). That is,

$$S(f) = \{p \in C : f(p,q) + \langle Ap, q - p\rangle \geq 0, \forall q \in C\}.$$

If $A = 0$, then problem (2.3) is reduced to the following Ky Fan inequality which is known as an equilibrium problem. Find $p \in C$ such that

$$f(p,q) \geq 0, \quad \forall q \in C. \tag{2.4}$$

We use $S(f)$ to denote the solution set of inequality (2.4). That is,

$$S(f) = \{p \in C : f(p,q) \geq 0, \forall q \in C\}.$$

If $f = 0$, then problem (2.3) is reduced to the classical variational inequality. Find $p \in C$ such that

$$\langle Ap, q - p\rangle \geq 0, \quad \forall q \in C. \tag{2.5}$$

We use $VI(C,A)$ to denote the solution set of inequality (2.5). That is,

$$VI(C,A) = \{p \in C : \langle Ap, q - p\rangle \geq 0, \forall q \in C\}.$$

Recall that a mapping $A : C \to E^*$ is said to be α-inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2.$$

For solving problem (2.3), let us assume that the nonlinear mapping $A : C \to E^*$ is α-inverse-strongly monotone and the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;
(A3)

$$\limsup_{t\downarrow 0} F\bigl(tz + (1-t)x,y\bigr) \leq F(x,y), \quad \forall x,y,z \in C;$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semicontinuous.

Recently, many authors investigated the solutions of problems (2.3), (2.4) and (2.5) based on iterative methods; see [27–37]. In this paper, we investigate generalized asymptotically quasi-ϕ-nonexpansive mappings and problem (2.3). A strong convergence theorem for common solutions to a fixed point problem of generalized asymptotically quasi-ϕ-nonexpansive mappings and problem (2.3) is established in a Banach space.

In order to state our main results, we need the following lemmas, which play an import role in the paper.

Lemma 2.6 [28] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping and f be a bifunction satisfying conditions* (A1)-(A4). *Let $r > 0$ be any given number and $x \in E$ be any given point. Then there exists $p \in C$ such that*

$$f(p,q) + \langle Ap, q-p\rangle + \frac{1}{r}\langle q-p, Jp - Jx\rangle \geq 0, \quad \forall q \in C.$$

Lemma 2.7 [28] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping and f be a bifunction satisfying conditions* (A1)-(A4). *Let $r > 0$ be any given number and $x \in E$ define a mapping $K_r : C \to C$ as follows*: *for any $x \in C$,*

$$K_r x = \left\{p \in C : f(p,q) + \langle Ap, q-p\rangle + \frac{1}{r}\langle q-p, Jp - Jx\rangle \geq 0, \forall q \in C\right\}.$$

Then the following conclusions hold:
(1) *K_r is single-valued*;
(2) *K_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle S_r x - S_r y, Jx - Jy\rangle;$$

(3) $F(K_r) = S(f,A)$;
(4) *K_r is quasi-ϕ-nonexpansive*;
(5)

$$\phi(q, K_r x) + \phi(K_r x, x) \leq \phi(q,x), \quad \forall q \in F(K_r);$$

(6) *$S(f,A)$ is closed and convex.*

Lemma 2.8 [19] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.9 [19] *Let E be a reflexive, strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \forall y \in C.$$

Lemma 2.10 [25] *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed generalized asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.11 [38] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r = \{x \in E : \|x\| \le r\}$ and $t \in [0, 1]$.

3 Main results

Theorem 3.1 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a generalized asymptotically quasi-ϕ-nonexpansive mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and $A : C \to E^*$ be an α-inverse-strongly monotone mapping. Assume that T is closed and asymptotically regular on C, and $F(T) \cap S(f, A)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}((1-\alpha_n)Jx_n + \alpha_n JT^n x_n), \\ u_n \in C \text{ such that } f(u_n, q) + \langle Au_n + q - u_n \rangle + \frac{1}{r_n}\langle q - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall q \in C, \\ C_{n+1} = \{k \in C_n : \phi(k, u_n) \le \phi(k, x_n) + (\mu_n - 1)W_n + \nu_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $W_n = \sup\{\phi(p, x_n) : p \in F(T) \cap S(f, A)\}$, $\{\alpha_n\}$ is a real number sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\}$ is a real number sequence such that $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap S(f,A)} x_1$, where $\Pi_{F(T)\cap S(f,A)}$ is the generalized projection from E onto $F(T) \cap S(f, A)$.

Proof First, we prove C_n is closed and convex so that the projection is well defined. We see that $C_1 = C$ is closed and convex. Assume that C_m is closed and convex for some positive integer m. For $k \in C_m$, we find that

$$\phi(k, u_m) \le \phi(k, x_m) + (\mu_m - 1)W_m + \nu_m,$$

which is equivalent to

$$2\langle k, Jx_m - Ju_m\rangle \le \|x_m\|^2 - \|u_m\|^2 + (\mu_m - 1)W_m + \nu_m.$$

It is easy to see that C_{m+1} is closed and convex. This proves that C_n is closed and convex so that $\Pi_{C_{n+1}}x_1$ is well defined. Set $u_n = K_{r_n}y_n$. It follows from Lemma 2.7 that K_{r_n} is quasi-ϕ-nonexpansive.

Now, we are in a position to prove that $F(T) \cap S(f, A) \subset C_n$. Indeed, $F(T) \cap S(f, A) \subset C_1 = C$ is obvious. Assume that $F(T) \cap S(f, A) \subset C_m$ for some positive integer m. Then, for $\forall e \in F(T) \cap S(f, A) \subset C_m$, we have

$$\begin{aligned}
\phi(e, u_m) &= \phi(e, S_{r_m}y_m) \\
&\le \phi(e, y_m) \\
&= \phi\big(e, J^{-1}\big((1-\alpha_m)Jx_h + \alpha_m JT^m x_m\big)\big) \\
&= \|e\|^2 - 2\big\langle e, (1-\alpha_m)Jx_m + \alpha_m JT^m x_m\big\rangle + \big\|(1-\alpha_m)Jx_h + \alpha_m JT^m x_m\big\|^2 \\
&\le \|e\|^2 - 2(1-\alpha_m)\langle e, Jx_m\rangle - 2\alpha_m\big\langle e, JT^m x_m\big\rangle + (1-\alpha_m)\|x_m\|^2 + \alpha_m\big\|T^m x_m\big\|^2 \\
&= (1-\alpha_m)\phi(e, x_m) + \alpha_m\phi\big(e, T^m x_m\big) \\
&\le (1-\alpha_m)\phi(e, x_m) + \alpha_m\mu_m\phi(e, x_m) + \alpha_m\nu_m \\
&\le \phi(e, x_m) + \alpha_m(\mu_m - 1)\phi(e, x_m) + \alpha_m\nu_m \\
&\le \phi(e, x_m) + (\mu_m - 1)W_m + \nu_m,
\end{aligned} \tag{3.1}$$

which proves that $e \in C_{m+1}$. This implies that $F(T) \cap S(f, A) \subset C_n$. Notice that $x_n = \Pi_{C_n}x_1$. We find from Lemma 2.8 that $\langle x_n - z, Jx_1 - Jx_n\rangle \ge 0$ for any $z \in C_n$. Since $F(T) \cap S(f, A) \subset C_n$, we therefore find that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \ge 0, \quad \forall w \in F(T) \cap S(f). \tag{3.2}$$

It follows from Lemma 2.9 that

$$\begin{aligned}
\phi(x_n, x_1) &\le \phi(\Pi_{F(T)\cap S(f,A)}x_1, x_1) - \phi(\Pi_{F(T)\cap S(f,A)}x_1, x_n) \\
&\le \phi(\Pi_{F(T)\cap S(f,A)}x_1, x_1).
\end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_1)\}$ is bounded. This in turn implies that the sequence $\{x_n\}$ is bounded. Since E is a uniform space, we find that E is reflexive. We may assume, without loss of generality, that $x_n \rightharpoonup \hat{x}$. Next, we prove that $\hat{x} \in F(T) \cap S(f, A)$. Since C_n is closed and convex, we find that $\hat{x} \in C_n$. This implies from $x_n = \Pi_{C_n}x_1$ that $\phi(x_n, x_1) \le \phi(\hat{x}, x_1)$. On the other hand, we see from the weakly lower semicontinuity of the norm $\|\cdot\|$ that

$$\begin{aligned}
\phi(\hat{x}, x_1) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx_1\rangle + \|x_1\|^2 \\
&\le \liminf_{n\to\infty}\big(\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2\big) \\
&= \liminf_{n\to\infty}\phi(x_n, x_1)
\end{aligned}$$

$$\begin{aligned}&\leq \limsup_{n\to\infty}\phi(x_n,x_1)\\&\leq \phi(\hat{x},x_1),\end{aligned}$$

which implies that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(\hat{x},x_1)$. Hence, we have $\lim_{n\to\infty}\|x_n\|=\|\hat{x}\|$. Since E enjoys the Kadec-Klee property, we find that $x_n\to\hat{x}$ as $n\to\infty$. In the light of $x_n=\Pi_{C_n}x_1$ and $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}\subset C_n$, we find that $\phi(x_n,x_1)\leq\phi(x_{n+1},x_1)$. This shows that $\{\phi(x_n,x_1)\}$ is nondecreasing. We obtain that $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. It follows that

$$\begin{aligned}\phi(x_{n+1},x_n)&=\phi(x_{n+1},\Pi_{C_n}x_1)\\&\leq\phi(x_{n+1},x_1)-\phi(\Pi_{C_n}x_1,x_1)\\&=\phi(x_{n+1},x_1)-\phi(x_n,x_1).\end{aligned}$$

This implies that $\lim_{n\to\infty}\phi(x_{n+1},x_n)=0$. In view of $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}$, we find that

$$\phi(x_{n+1},u_n)\leq\phi(x_{n+1},x_n)+(\mu_n-1)W_n+\nu_n.$$

It follows that

$$\lim_{n\to\infty}\phi(x_{n+1},u_n)=0.$$

In view of (2.2), we see that $\lim_{n\to\infty}(\|x_{n+1}\|-\|u_n\|)=0$. This implies that $\lim_{n\to\infty}\|u_n\|=\|\hat{x}\|$. That is,

$$\lim_{n\to\infty}\|Ju_n\|=\lim_{n\to\infty}\|u_n\|=\|J\bar{x}\|. \tag{3.3}$$

This implies that $\{Ju_n\}$ is bounded. Since both E and E^* are uniform, we find that both E and E^* are reflexive. We may assume, without loss of generality, that $Ju_n\rightharpoonup u^*\in E^*$. In view of the reflexivity of E, we see that $J(E)=E^*$. This shows that there exists an element $u\in E$ such that $Ju=u^*$. It follows that

$$\begin{aligned}\phi(x_{n+1},u_n)&=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_n\rangle+\|u_n\|^2\\&=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_n\rangle+\|Ju_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}0&\geq\|\hat{x}\|^2-2\langle\hat{x},u^*\rangle+\|u^*\|^2\\&=\|\hat{x}\|^2-2\langle\hat{x},Ju\rangle+\|Ju\|^2\\&=\|\hat{x}\|^2-2\langle\hat{x},Ju\rangle+\|u\|^2\\&=\phi(\hat{x},u).\end{aligned}$$

That is, $\hat{x}=u$, which in turn implies that $u^*=J\hat{x}$. It follows that $Ju_n\rightharpoonup J\hat{x}\in E^*$. Since E is uniformly smooth, we know that E^* is uniformly convex. Therefore, E^* enjoys the Kadec-Klee property, we obtain that $\lim_{n\to\infty}Ju_n=J\hat{x}$. Since $J^{-1}:E^*\to E$ is demicontinuous and

E enjoys the Kadec-Klee property, we obtain that $u_n \to \hat{x}$ as $n \to \infty$. Note that

$$\|x_n - u_n\| \le \|x_n - \hat{x}\| + \|\hat{x} - u_n\|.$$

It follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.4}$$

Since E is uniformly smooth, we know that E^* is uniformly convex. In the light of Lemma 2.11, we find that

$$\begin{aligned}
\phi(e, u_n) &= \phi(e, S_{r_n} y_n) \\
&\le \phi(e, y_n) \\
&= \phi\big(e, J^{-1}\big((1-\alpha_n)Jx_n + \alpha_n JT^n x_n\big)\big) \\
&= \|e\|^2 - 2\big\langle e, (1-\alpha_n)Jx_n + \alpha_n JT^n x_n\big\rangle + \big\|(1-\alpha_n)Jx_n + \alpha_n JT^n x_n\big\|^2 \\
&\le \|e\|^2 - 2(1-\alpha_n)\langle e, Jx_n\rangle - 2\alpha_n\big\langle e, JT^n x_n\big\rangle + (1-\alpha_n)\|x_n\|^2 \\
&\quad + \alpha_n\big\|T^n x_n\big\|^2 - \alpha_n(1-\alpha_n) g\big(\big\|Jx_n - JT^n x_n\big\|\big) \\
&= (1-\alpha_n)\phi(e, x_n) + \alpha_n\phi\big(e, T^n x_n\big) - \alpha_n(1-\alpha_n) g\big(\big\|Jx_n - JT^n x_n\big\|\big) \\
&\le (1-\alpha_n)\phi(e, x_n) + \alpha_n\mu_n\phi(e, x_n) + \alpha_n\nu_n \\
&\quad - \alpha_n(1-\alpha_n) g\big(\big\|Jx_n - JT^n x_n\big\|\big) \\
&\le \phi(e, x_n) + \alpha_n(\mu_n - 1)\phi(e, x_n) + \alpha_n\nu_n \\
&\quad - \alpha_n(1-\alpha_n) g\big(\big\|Jx_n - JT^n x_n\big\|\big) \\
&\le \phi(e, x_n) + (\mu_n - 1)W_n + \nu_n - \alpha_n(1-\alpha_n) g\big(\big\|Jx_n - JT^n x_n\big\|\big).
\end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n) g\big(\big\|Jx_n - JT^n x_n\big\|\big) \le \phi(e, x_n) - \phi(e, u_n) + (\mu_n - 1)W_n + \nu_n. \tag{3.5}$$

Notice that

$$\begin{aligned}
\phi(e, x_n) - \phi(e, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle e, Jx_n - Ju_n\rangle \\
&\le \|x_n - u_n\|\big(\|x_n\| + \|u_n\|\big) + 2\|e\|\|Jx_n - Ju_n\|.
\end{aligned}$$

We find from (3.4) that

$$\lim_{n\to\infty}\big(\phi(e, x_n) - \phi(e, u_n)\big) = 0.$$

In view of the restriction on the sequences, we find from (3.5) that $\lim_{n\to\infty} g(\|Jx_n - JT^n x_n\|) = 0$. Notice that

$$\big\|JT^n x_n - J\hat{x}\big\| \le \big\|JT^n x_n - Jx_n\big\| + \|Jx_n - J\hat{x}\|.$$

It follows that

$$\lim_{n\to\infty} \|JT^n x_n - J\hat{x}\| = 0.$$

The demicontinuity of $J^{-1} : E^* \to E$ implies that $T_i^n x_n \rightharpoonup \hat{x}$. Note that

$$\big| \|T^n x_n\| - \|\hat{x}\| \big| = \big| \|JT^n x_n\| - \|J\hat{x}\| \big| \le \|JT^n x_n - J\hat{x}\|.$$

This implies that $\lim_{n\to\infty} \|T^n x_n\| = \|\hat{x}\|$. Since E has the Kadec-Klee property, we obtain that $\lim_{n\to\infty} \|T^n x_n - \hat{x}\| = 0$. Notice that

$$\|T^{n+1} x_n - \hat{x}\| \le \|T^{n+1} x_n - T^n x_n\| + \|T^n x_n - \hat{x}\|.$$

It follows from the uniformly asymptotic regularity of T that

$$\lim_{n\to\infty} \|T^{n+1} x_n - \hat{x}\| = 0.$$

That is, $TT^n x_n \to \hat{x}$. From the closedness of T, we find $\hat{x} = T\hat{x}$. This proves $\hat{x} \in F(T)$. Next, we show that $\hat{x} \in S(f, A)$. It follows from Lemma 2.9 and (3.1) that

$$\begin{aligned} \phi(u_n, y_n) &\le \phi(e, y_n) - \phi(e, u_n) \\ &\le \phi(e, x_n) + (\mu_n - 1) W_n + \nu_n - \phi(e, u_n). \end{aligned}$$

This yields that $\lim_{n\to\infty} \phi(u_n, y_n) = 0$. This implies from (2.2) that $\lim_{n\to\infty}(\|u_n\| - \|y_n\|) = 0$. It follows that

$$\lim_{n\to\infty} \|y_n\| = \|\hat{x}\|.$$

We, therefore, find that

$$\lim_{n\to\infty} \|Jy_n\| = \lim_{n\to\infty} \|y_n\| = \|\hat{x}\| = \|J\hat{x}\|.$$

This shows that $\{Jy_n\}$ is bounded. Since E^* is reflexive, we may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of $JE = E^*$, we see that there exists $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned} \phi(u_n, y_n) &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|y_n\|^2 \\ &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|Jy_n\|^2. \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above yields that

$$\begin{aligned} 0 &\ge \|\hat{x}\|^2 - 2\langle \hat{x}, y^*\rangle + \|y^*\|^2 \\ &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jy\rangle + \|Jy\|^2 \\ &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jy\rangle + \|y\|^2 \\ &= \phi(\hat{x}, y). \end{aligned}$$

That is, $\hat{x} = y$, which in turn implies that $y^* = J\hat{x}$. It follows that $Jy_n \rightharpoonup J\hat{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Jy_n - J\hat{x} \to 0$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is demicontinuous. It follows that $y_n \rightharpoonup \hat{x}$. Since E enjoys the Kadec-Klee property, we obtain that $y_n \to \hat{x}$ as $n \to \infty$. Note that

$$\|u_n - y_n\| \leq \|u_n - \hat{x}\| + \|\hat{x} - y_n\|.$$

This implies that $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0$. In view of the restriction $\liminf_{n\to\infty} r_n > 0$, we see that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0.$$

Since $u_n = K_{r_n} y_n$, we find that

$$F(u_n, q) + \frac{1}{r_n}\langle q - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall q \in C,$$

where

$$F(u_n, q) = f(u_n, q) + \langle Au_n, q - u_n\rangle.$$

It follows from (A2) that

$$\|q - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} \geq \frac{1}{r_n}\langle q - u_n, Ju_n - Jy_n\rangle \geq F(q, u_n), \quad \forall q \in C.$$

In view of (A4), we find that

$$F(q, \bar{x}) \leq 0, \quad \forall q \in C.$$

For $0 < t < 1$ and $q \in C$, define $q_t = tq + (1-t)\hat{x}$. It follows that $q_t \in C$, which yields that $F(q_t, \hat{x}) \leq 0$. It follows from (A1) and (A4) that

$$0 = F(q_t, q_t) \leq tF(q_t, q) + (1-t)F(q_t, \hat{x}) \leq tF(q_t, q).$$

That is,

$$F(q_t, q) = f(q_t, q) + \langle Aq_t, q - u_n\rangle \geq 0.$$

Letting $t \downarrow 0$, we obtain from (A3) that $F(\hat{x}, q) \geq 0$, $\forall q \in C$. This implies that $\hat{x} \in S(f, A)$. This completes the proof $\hat{x} \in F(T) \cap S(f, A)$.

Finally, what we need to prove is $\hat{x} = \Pi_{F(T)\cap S(f,A)} x_1$.

Letting $n \to \infty$ in (3.2), we obtain that

$$\langle \hat{x} - w, Jx_1 - J\hat{x}\rangle \geq 0, \quad \forall w \in F(T) \cap S(f, A).$$

From Lemma 2.8, we immediately find that $\hat{x} = \Pi_{F(T)\cap S(f,A)} x_1$. This completes the whole proof. □

Remark 3.2 Since the class of generalized asymptotically quasi-ϕ-nonexpansive mappings is a generalization of the class of asymptotically quasi-ϕ-nonexpansive mappings, Theorem 3.1 includes Kim's [36] results as a special case.

Remark 3.3 Notice that every uniformly smooth and uniformly convex space is a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and every uniformly convex Banach space enjoys the Kadec-Klee property. We find that Theorem 3.1 is still valid in the framework of every uniformly smooth and uniformly convex space.

Next, we consider the solution of problem (2.4).

If the mapping T is closed quasi-ϕ-nonexpansive, which is more general than relatively nonexpansive mappings, we have the following.

Corollary 3.4 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a quasi-ϕ-nonexpansive mapping and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Assume that T is closed and $F(T) \cap S(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}((1-\alpha_n)Jx_n + \alpha_n JTx_n), \\ u_n \in C \text{ such that } f(u_n, q) + \frac{1}{r_n}\langle q - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall q \in C, \\ C_{n+1} = \{k \in C_n : \phi(k, u_n) \leq \phi(k, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\}$ is a real number sequence such that $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap S(f)} x_1$, where $\Pi_{F(T)\cap S(f)}$ is the generalized projection from E onto $F(T) \cap S(f)$.

In the framework of Hilbert spaces, we find from Theorem 3.1 the following.

Theorem 3.5 *Let E be a Hilbert space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $A : C \to E$ be an α-inverse-strongly monotone mapping. Assume that T is closed and asymptotically regular on C, and $F(T) \cap$*

$S(f,A)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ u_n \in C \text{ such that } f(u_n,q) + \langle Au_n, q-u_n\rangle + \frac{1}{r_n}\langle q-u_n, u_n-y_n\rangle \ge 0, \quad \forall q \in C, \\ C_{n+1} = \{k \in C_n : \|k-u_n\|^2 \le \|k-x_n\|^2 + (\mu_n-1)W_n + \nu_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases}$$

where $W_n = \sup\{\|p-x_n\|^2 : p \in F(T)\cap S(f,A)\}$, $\{\alpha_n\}$ *is a real number sequence in* $(0,1)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\}$ *is a real number sequence such that* $\liminf_{n\to\infty} r_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_{F(T)\cap S(f,A)}x_1$, *where* $P_{F(T)\cap S(f,A)}$ *is the metric projection from E onto* $F(T)\cap S(f,A)$.

Proof In the framework of Hilbert spaces, we see that $\phi(x,y) = \|x-y\|^2$ and the mapping J is reduced to the identity mapping. The desired conclusion can be immediately drawn from Theorem 3.1. □

For problem (2.4), we have the following result.

Corollary 3.6 *Let E be Hilbert space and C be a nonempty closed convex subset of E. Let* $T : C \to C$ *be a generalized asymptotically quasi-nonexpansive mapping. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4). *Assume that T is closed and asymptotically regular on C, and* $F(T)\cap S(f)$ *is nonempty and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ u_n \in C \text{ such that } f(u_n,q) + \frac{1}{r_n}\langle q-u_n, u_n-y_n\rangle \ge 0, \quad \forall q \in C, \\ C_{n+1} = \{k \in C_n : \|k-u_n\|^2 \le \|k-x_n\|^2 + (\mu_n-1)W_n + \nu_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases}$$

where $W_n = \sup\{\|p-x_n\|^2 : p \in F(T)\cap S(f)\}$, $\{\alpha_n\}$ *is a real number sequence in* $(0,1)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\}$ *is a real number sequence such that* $\liminf_{n\to\infty} r_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_{F(T)\cap S(f)}x_1$, *where* $P_{F(T)\cap S(f)}$ *is the metric projection from E onto* $F(T)\cap S(f)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JS design the algorithm and give the main convergence analysis. MC participated in the design of the study. Both authors read and approved the final manuscript.

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