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Convergence analysis of a general iteration schema of nonlinear mappings in hyperbolic spaces

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Abstract

Iterative schemas are ubiquitous in the area of abstract nonlinear analysis and still remain as a main tool for approximation of fixed points of generalizations of nonexpansive maps. The analysis of general iterative schemas, in a more general setup, is a problem of interest in theoretical numerical analysis. Therefore, we propose and analyze a general iterative schema for two finite families of asymptotically quasi-nonexpansive maps in hyperbolic spaces. Results concerning Δ-convergence as well as strong convergence of the proposed iteration are proved. It is instructive to compare the proposed general iteration schema and the consequent convergence results with that of several recent results in CAT(0) spaces and uniformly convex Banach spaces.

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1 Introduction and preliminaries

Iterative schemas play a key role in approximating fixed points for nonlinear mappings. Structural properties of the space under consideration are very important in establishing the fixed point property of the space, for example, strict convexity, uniform convexity and uniform smoothness *etc.* Hyperbolic spaces are general in nature and have rich geometrical structures for different results with applications in topology, graph theory, multivalued analysis and metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory. Throughout the paper, we work in the setting of hyperbolic spaces, introduced by Kohlenbach [1], which are prominent among non-positively curved spaces and play a significant role in many branches of mathematics.

Nonexpansive mappings are Lipschitzian mappings with the Lipschitz constant equal to 1. Moreover, the class of nonexpansive mappings is closely related to the class of strict pseudo-contractions as nonexpansive mappings are 0-strictly pseudo-contractive. The class of nonexpansive mappings enjoys the fixed point property and the approximate fixed point property in various settings of spaces. The importance of this class lies in its powerful applications in initial value problems of differential equations, game-theoretic model, image recovery and minimax problems.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] as an important generalization of the class of nonexpansive mappings. Therefore, it is natural to extend such powerful results to the class of asymptotically nonexpansive mappings as a means of testing the limit of the theory of nonexpansive mappings. Most of the results in fixed point theory guarantee that a fixed point exists, but they do not help in finding the fixed point. As a consequence, iterative construction of fixed points emerged as the most powerful tool for solving such nonlinear problems. It is worth mentioning that iteration schemas are the only main tool for approximation of fixed points of various generalizations of nonexpansive mappings. Several authors have studied approximation of fixed points of several generalizations of nonexpansive mappings using Mann and Ishikawa iterations (see, *e.g.*, [3–15]).

Moreover, finding common fixed points of a finite family of mappings acting on a Hilbert space is a problem that often arises in applied mathematics, for instance, in convex minimization problems and systems of simultaneous equations. One of the most elegant ways to prove that a partial differential equation or integral equation has a solution is to pose it as a fixed point problem. Hence, the analysis of a general iteration schema, in a more general setup, is a problem of interest in theoretical numerical analysis. Therefore, considerable research efforts have been devoted to developing iterations for the approximation of common fixed points of several classes of nonlinear mappings with a nonempty set of common fixed points.

In 1991, Schu [11] established weak and strong convergence results for asymptotically nonexpansive mappings using a modified Mann iteration. A unified treatment regarding weak convergence theorems for asymptotically nonexpansive mappings was analyzed by Chang *et al.* [16] and consequently improved and generalized the results of Schu [11] and many more. See, for example, Bose [17], Tan and Xu [14] and many others.

In 2000, Osilike and Aniagbosor [18] obtained weak and strong convergence results for asymptotically nonexpansive mappings using a modified Ishikawa iteration. Since the case for two mappings has a direct link to minimization problems [19], so this fact motivated Khan and Takahashi [7] to approximate common fixed points of two asymptotically nonexpansive mappings. For this purpose, they used a modified Ishikawa iteration. See also [9] and [13].

In 2008, Khan *et al.* [20] introduced a general iteration schema for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. Khan *et al.* [21] also proposed and analyzed a general iteration schema for strong convergence results in CAT(0) spaces. Inspired by the work of Khan *et al.* [20], Kettapun *et al.* [6] introduced a new iterative schema for finding a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. Quite recently, Sahin and Basarir [10] approximated common fixed points of a finite family of asymptotically quasi-nonexpansive mappings by a modified general iteration schema in CAT(0) spaces. Recently, Yildirim and Özdemir [15] approximated a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings using a new general iteration in a Banach space setting as follows.

Let $\{T_m\}_{m=1}^{r}$ be a family of asymptotically quasi-nonexpansive self-mappings on K. Suppose that $\{\alpha_{mn}\}$ is a real sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Define a sequence $\{x_n\}$

by

$$\begin{aligned} x_{n+1} &= (1-\alpha_{1n})y_{n+r-2} + \alpha_{1n}T_1^n y_{n+r-2}, \\ y_{n+r-2} &= (1-\alpha_{2n})y_{n+r-3} + \alpha_{2n}T_2^n y_{n+r-3}, \\ y_{n+r-3} &= (1-\alpha_{3n})y_{n+r-4} + \alpha_{3n}T_3^n y_{n+r-4}, \\ &\vdots \\ y_{n+1} &= (1-\alpha_{(r-1)n})y_n + \alpha_{(r-1)n}T_{r-1}^n y_n, \\ y_n &= (1-\alpha_{rn})x_n + \alpha_{rn}T_r^n x_n. \end{aligned} \tag{1.1}$$

Let (X,d) be a metric space and K be a nonempty subset of X. Let T be a self-mapping on K. Denote by $F(T) = \{x \in K : T(x) = x\}$ the set of fixed points of T. A self-mapping T on K is said to be

(i) nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for $x,y \in K$;
(ii) quasi-nonexpansive if $d(Tx,p) \leq d(x,p)$ for $x \in K$ and for $p \in F(T) \neq \emptyset$;
(iii) asymptotically nonexpansive if there exists a sequence $k_n \subset [0,\infty)$ and $\lim_{n\to\infty} k_n = 0$ and $d(T^n x, T^n y) \leq (1+k_n)d(x,y)$ for $x,y \in K$, $n \geq 1$;
(iv) asymptotically quasi-nonexpansive if there exists a sequence $k_n \subset [0,\infty)$ and $\lim_{n\to\infty} k_n = 0$ and $d(T^n x, p) \leq (1+k_n)d(x,p)$ for $x \in K$, $p \in F(T)$, $n \geq 1$;
(v) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq Ld(x,y)$ for $x,y \in K$ and $n \geq 1$.

It follows from the above definitions that a nonexpansive mapping is quasi-nonexpansive and that an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. Moreover, an asymptotically nonexpansive mapping is uniformly L-Lipschitzian. However, the converse of these statements is not true, in general.

A hyperbolic space [1] is a metric space (X,d) together with a mapping $W : X^2 \times [0,1] \to X$ satisfying

$$\begin{aligned} &(1)\quad d\big(u, W(x,y,\alpha)\big) \leq \alpha d(u,x) + (1-\alpha)d(u,y), \\ &(2)\quad d\big(W(x,y,\alpha), W(x,y,\beta)\big) = |\alpha-\beta| d(x,y), \\ &(3)\quad W(x,y,\alpha) = W\big(y,x,(1-\alpha)\big), \\ &(4)\quad d\big(W(x,z,\alpha), W(y,w,\alpha)\big) \leq (1-\alpha)d(x,y) + \alpha d(z,w) \end{aligned}$$

for all $x,y,z,w \in X$ and $\alpha, \beta \in [0,1]$.

The class of hyperbolic spaces in the sense of Kohlenbach [1] contains all normed linear spaces and convex subsets thereof but also Hadamard manifolds and CAT(0) spaces. An important example of a hyperbolic space is the open unit ball B in a complex domain $\mathbb{C}$ w.r.t. the Poincare metric (also called 'Poincare distance')

$$d_B(x,y) := \arg\tanh\left|\frac{x-y}{1-x\bar{y}}\right| = \arg\tanh\big(1-\sigma(x,y)\big)^{\frac{1}{2}},$$

where

$$\sigma(x,y) := \frac{(1-|x|^2)(1-|y|^2)}{|1-x\bar{y}|^2} \quad \text{for all } x,y \in B.$$

Note that the above example can be extended from $\mathbb{C}$ to general complex Hilbert spaces $(H, \langle\cdot\rangle)$ as follows.

Let B_H be an open unit ball in H. Then

$$k_{B_H}(x,y) := \arg\tanh\bigl(1-\sigma(x,y)\bigr)^{\frac{1}{2}},$$

where

$$\sigma(x,y) = \frac{(1-\|x\|^2)(1-\|x\|^2)}{|1-\langle x,y\rangle|^2} \quad \text{for all } x,y \in B_H,$$

defines a metric on B_H (also known as the Kobayashi distance). The open unit ball B_H together with this metric is coined as a Hilbert ball. Since (B_H, k_{B_H}) is a unique geodesic space, so one can define W in a similar way for the corresponding hyperbolic space (B_H, k_{B_H}, W).

A metric space (X,d) satisfying only (1) is a convex metric space introduced by Takahashi [22]. A subset K of a hyperbolic space X is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$. For more on hyperbolic spaces and a comparison between different notions of hyperbolic space present in the literature, we refer to [1, p.384].

A hyperbolic space (X,d,W) is uniformly convex [23] if for all $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $d(W(x,y,\frac{1}{2}),u) \le (1-\delta)r$ whenever $d(x,u) \le r$, $d(y,u) \le r$, $d(x,y) \ge \epsilon r$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$ is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ϵ). CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r,\epsilon) = \frac{\epsilon^2}{8}$ [24]. Therefore, the class of uniformly convex hyperbolic spaces includes both uniformly convex normed spaces and CAT(0) spaces as special cases.

Inspired and motivated by Khan and Takahashi [7], Sahin and Basarir [10], Shahzad and Udomene [13], Yildirim and Özdemir [15] and Khan *et al.* [21], we introduce a general iteration schema in hyperbolic spaces and approximate common fixed points of two finite families of asymptotically quasi-nonexpansive mappings as follows.

Let $\{T_m\}_{m=1}^r$ and $\{S_m\}_{m=1}^r$ be two finite families of asymptotically quasi-nonexpansive self-mappings on K. Suppose that $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are two double real sequences in $[a,b]$ for some $a,b \in (0,1)$. Define a sequence $\{x_n\}$ by

$$\begin{aligned}
x_{n+1} &= W\bigl(T_1^n y_{n+r-2}, W\bigl(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}\bigr), \alpha_{1n}\bigr),\\
y_{n+r-2} &= W\bigl(T_2^n y_{n+r-3}, W\bigl(y_{n+r-3}, S_2^n y_{n+r-3}, \theta_{2n}\bigr), \alpha_{2n}\bigr),\\
y_{n+r-3} &= W\bigl(T_3^n y_{n+r-4}, W\bigl(y_{n+r-4}, S_3^n y_{n+r-4}, \theta_{3n}\bigr), \alpha_{3n}\bigr),\\
&\vdots\\
y_{n+1} &= W\bigl(T_{r-1}^n y_n, W\bigl(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}\bigr), \alpha_{(r-1)n}\bigr),\\
y_n &= W\bigl(T_r^n x_n, W\bigl(x_n, S_r^n x_n, \theta_{rn}\bigr), \alpha_{rn}\bigr), \quad r \ge 2, n \ge 1,
\end{aligned} \tag{1.2}$$

where $\theta_{mn} := \frac{\beta_{mn}}{1-\alpha_{mn}}$ for each $m = 1,2,3,\ldots,r$.

In 1976, Lim [25] introduced the notion of asymptotic center and, consequently, coined the concept of $\triangle$-convergence in a general setting of a metric space. In 2008, Kirk and Panyanak [26] proposed an analogous version of convergence in geodesic spaces, namely $\triangle$-convergence, which was originally introduced by Lim [25]. They showed that $\triangle$-convergence coincides with the usual weak convergence in Banach spaces and both concepts share many useful properties.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K\big(\{x_n\}\big) = \big\{x \in X : r\big(x, \{x_n\}\big) \le r\big(y, \{x_n\}\big) \text{ for any } y \in K\big\}.$$

This is the set of minimizers of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that 'bounded sequences have unique asymptotic centers with respect to closed convex subsets.' The following lemma is due to Leustean [24] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 [24] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Recall that a sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_n x_n = x$ and call x a $\triangle$-limit of $\{x_n\}$. A mapping $T : K \to K$ is *semi-compact* if every bounded sequence $\{x_n\} \subset K$, satisfying $d(x_n, Tx_n) \to 0$, has a convergent subsequence.

Let f be a nondecreasing self-mapping on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$. Then the two finite families $\{T_m\}_{m=1}^r$ and $\{S_m\}_{m=1}^r$, with $F = \bigcap_{i=1}^N (F(T_i) \cap F(S_i)) \neq \emptyset$, are said to satisfy condition (A) on K if

$$d(x, Tx) \ge f\big(d(x, F)\big) \quad \text{or} \quad d(x, Sx) \ge f\big(d(x, F)\big) \quad \text{for } x \in K$$

holds for at least one $T \in \{T_m\}_{m=1}^r$ or one $S \in \{S_m\}_{m=1}^r$, where $d(x, F) = \inf\{d(x, y) : y \in F\}$.

In the sequel, we shall need the following results.

Lemma 1.2 [27] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If*

$\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\longrightarrow\infty} d(x_n, x) \le c$, $\limsup_{n\longrightarrow\infty} d(y_n, x) \le c$ and $\lim_{n\longrightarrow\infty} d(W(x_n, y_n, \alpha_n), x) = c$ for some $c \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.3 [27] *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space, and let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 1.4 [16] *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \le a_n + b_n$$

for all $n \ge 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. Moreover, if there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} \to 0$ as $j \to \infty$, then $a_n \to 0$ as $n \to \infty$.

2 Some preparatory lemmas

From now onward, we denote $F = \bigcap_{i=1}^{N}(F(T_i) \cap F(S_i)) \neq \emptyset$ for two finite families $\{T_m\}_{m=1}^{r}$ and $\{S_m\}_{m=1}^{r}$ of asymptotically quasi-nonexpansive self-mappings on K with sequences $\{u_{mn}^{(1)}\}_{m=1}^{\infty}$ and $\{u_{mn}^{(2)}\}_{m=1}^{\infty}$ respectively. If we put $u_{mn} = \max\{u_{mn}^{(1)}, u_{mn}^{(2)}\}$, then $\{u_{mn}\}_{m=1}^{\infty}$ is a sequence in $[0,1)$ and $\lim_{n\to\infty} u_{mn} = 0$.

We start with the following lemma.

Lemma 2.1 *Let K be a nonempty, closed and convex subset of a hyperbolic space X, and let $\{T_m\}_{m=1}^{r}$ and $\{S_m\}_{m=1}^{r}$ be two finite families of asymptotically quasi-nonexpansive self-mappings on K with a sequence $\{u_{mn}\}_{m=1}^{\infty}$ satisfying $\sum_{n=1}^{\infty} u_{mn} < \infty$, $m = 1, 2, \dots, r$. Then, for the sequence $\{x_n\}$ in* (1.2), *$\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.*

Proof Let $s_n = \max_{1\le m\le r} u_{mn}$ for $n \ge 1$. For any $p \in F$, it follows from (1.2) that

$$\begin{aligned}
d(y_n, p) &= d\big(W\big(T_r^n x_n, W\big(x_n, S_r^n x_n, \theta_{rn}\big), \alpha_{rn}\big), p\big) \\
&\le \alpha_{rn} d\big(T_r^n x_n, p\big) + (1 - \alpha_{rn}) d\big(W\big(x_n, S_r^n x_n, \theta_{rn}\big), p\big) \\
&\le \alpha_{rn} d\big(T_r^n x_n, p\big) + \beta_{rn} d(x_n, p) + (1 - \alpha_{rn} - \beta_{rn}) d\big(S_r^n x_n, p\big) \\
&\le \alpha_{rn}(1 + u_{rn}) d(x_n, p) + \beta_{rn} d(x_n, p) + (1 - \alpha_{rn} - \beta_{rn})(1 + u_{rn}) d(x_n, p) \\
&\le (1 + u_{rn}) d(x_n, p)
\end{aligned}$$

and

$$\begin{aligned}
d(y_{n+1}, p) &= d\big(W\big(T_{r-1}^n y_n, W\big(y_n, S_{r-1}^n x_n, \theta_{(r-1)n}\big), \alpha_{(r-1)n}\big), p\big) \\
&\le \alpha_{(r-1)n} d\big(T_{r-1}^n y_n, p\big) + (1 - \alpha_{(r-1)n}) d\big(W\big(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}\big), p\big) \\
&\le \alpha_{(r-1)n} d\big(T_{r-1}^n y_n, p\big) + \beta_{(r-1)n} d(y_n, p) + (1 - \alpha_{(r-1)n} - \beta_{(r-1)n}) d\big(S_{r-1}^n y_n, p\big) \\
&\le (1 + u_{(r-1)n}) d(y_n, p).
\end{aligned}$$

Similarly, we have

$$d(y_{n+r-2}, p) \le (1 + u_{2n}) d(y_{n+r-3}, p).$$

Therefore

$$\begin{aligned}
d(x_{n+1},p) &\leq (1+u_{1n})(1+u_{2n})\cdots(1+u_{rn})d(x_n,p) \\
&= d(x_n,p)(1+u_{1n}+u_{2n}+\cdots+u_{rn}+u_{1n}u_{2n}+u_{1n}u_{3n}+\cdots+u_{1n}u_{rn} \\
&\quad +u_{1n}u_{2n}u_{3n}+u_{1n}u_{2n}u_{4n}+\cdots+u_{1n}u_{2n}u_{3n}\cdots u_{rn}) \\
&\leq d(x_n,p)\left[1+\binom{r}{1}s_n+\binom{r}{2}s_n^2+\binom{r}{3}s_n^3+\cdots+\binom{r}{r}s_n^r\right] \\
&\leq (1+a_r s_n)d(x_n,p) \\
&\leq e^{a_r s_n}d(x_n,p) \\
&\quad \vdots \\
&\leq e^{a_r\sum_{k=1}^{n}s_k}d(x_1,p) \\
&\leq e^{a_r\sum_{k=1}^{\infty}s_k}d(x_1,p)<\infty,
\end{aligned}$$

where $a_r=\binom{r}{1}+\binom{r}{2}+\binom{r}{3}+\cdots+\binom{r}{r}$.

Hence $\{x_n\}$ is bounded. Moreover, it follows from the above that

$$d(x_{n+1},p)\leq d(x_n,p)+a_r s_n M.$$

Taking infimum on $p\in F$ on both sides in the above inequality, we have

$$d(x_{n+1},F)\leq d(x_n,F)+a_r s_n M.$$

Applying Lemma 1.4 to the above inequality, we have $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F$. Consequently, $\lim_{n\to\infty} d(x_n,F)$ exists. □

Lemma 2.2 *Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_m\}_{m=1}^r$ and $\{S_m\}_{m=1}^r$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of K with a sequence $\{u_{mn}\}_{n=1}^\infty$ satisfying $\sum_{n=1}^\infty u_{mn}<\infty$, $m=1,2,\dots,r$. Then, for the sequence $\{x_n\}$ in* (1.2), *we have*

$$\lim_{n\to\infty} d(x_n,T_m x_n)=\lim_{n\to\infty} d(x_n,S_m x_n)=0 \quad \textit{for each } m=1,2,\dots,r.$$

Proof It follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F$. Assume that $\lim_{n\to\infty} d(x_n,p)=c>0$. Otherwise the proof is trivial.

Since $u_{mn}\to 0$ as $n\to\infty$, therefore taking lim sup on both sides of the first two inequalities in the proof of Lemma 2.1, we have $\limsup_{n\to\infty} d(y_n,p)\leq c$ and $\limsup_{n\to\infty} d(y_{n+1},p)\leq c$. Similarly, we get that $\limsup_{n\to\infty} d(y_{n+r-2},p)\leq c$, and so in total

$$\limsup_{n\to\infty} d(y_{n+k-1},p)\leq c \quad \text{for each } k=1,2,\dots,r-1. \tag{2.1}$$

Since $s_n=\max_{1\leq m\leq r} u_{mn}$ for $n\geq 1$, therefore

$$d(x_{n+1},p)\leq (1+s_n)^{r-1}d(y_n,p).$$

This implies

$$c \le \liminf_{n\to\infty} d(y_{n+k-1}, p) \quad \text{for each } k = 1, 2, \ldots, r-1. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$\lim_{n\to\infty} d(y_{n+k-1}, p) = c \quad \text{for each } k = 1, 2, \ldots, r-1. \tag{2.3}$$

For $k = 1$ in (2.3), we have

$$\lim_{n\to\infty} d\big(W\big(T_r^n x_n, W\big(x_n, S_r^n x_n, \theta_{rn}\big), \alpha_{rn}\big), p\big) = c. \tag{2.4}$$

Moreover,

$$\begin{aligned} d\big(W\big(x_n, S_r^n x_n, \theta_{rn}\big), p\big) &\le \theta_{rn} d(x_n, p) + (1-\theta_{rn}) d\big(S_r^n x_n, p\big) \\ &\le \theta_{rn} d(x_n, p) + (1-\theta_{rn})(1+u_{rn}) d(x_n, p) \\ &\le (1+u_{rn}) d(x_n, p) \end{aligned}$$

implies that

$$\limsup_{n\to\infty} d\big(W\big(x_n, S_r^n x_n, \theta_{rn}\big), p\big) \le c. \tag{2.5}$$

Obviously,

$$\limsup_{n\to\infty} d\big(T_r^n x_n, p\big) \le c. \tag{2.6}$$

With the help of (2.4)-(2.6) and Lemma 1.2, we have

$$\lim_{n\to\infty} d\big(T_r^n x_n, W\big(x_n, S_r^n x_n, \theta_{rn}\big)\big) = 0. \tag{2.7}$$

Again, for $k = 2, 3, \ldots, r-1$, (2.3) is expressed as

$$\lim_{n\to\infty} d\big(W\big(T_{r-(k-1)}^n y_{n+k-2}, W\big(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}\big), \alpha_{(r-k+1)n}\big), p\big) = c. \tag{2.8}$$

With the help of (2.1) and the inequality

$$\begin{aligned} & d\big(W\big(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}\big), p\big) \\ &\quad \le \theta_{(r-k+1)n} d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n}) d\big(S_{r-(k-1)}^n y_{n+k-2}, p\big) \\ &\quad \le \theta_{(r-k+1)n} d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n})(1+u_{(r-k+1)n}) d(y_{n+k-2}, p) \\ &\quad \le (1+u_{(r-k+1)n}) d(y_{n+k-2}, p), \end{aligned}$$

we get that

$$\limsup_{n\to\infty} d\big(W\big(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}\big), p\big) \le c. \tag{2.9}$$

Further,

$$\limsup_{n\to\infty} d\big(T^n_{r-(k-1)}y_{n+k-2}, p\big) \le c \quad \text{for } k = 2, 3, \ldots, r-1. \tag{2.10}$$

By (2.8)-(2.10) and Lemma 1.2, we have

$$\lim_{n\to\infty} d\big(T^n_{r-(k-1)}y_{n+k-2}, W\big(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}\big)\big) = 0 \tag{2.11}$$

for $k = 2, 3, \ldots, r-1$. For $k = r$, we have

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d\big(W\big(T^n_1 y_{n+r-2}, W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big), \alpha_{1n}\big), p\big) = c. \tag{2.12}$$

Utilizing (2.1), the following estimate

$$\begin{aligned} d\big(W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big), p\big) &\le \theta_{1n} d(y_{n+r-2}, p) + (1-\theta_{1n}) d\big(S^n_1 y_{n+r-2}, p\big) \\ &\le \theta_{1n} d(y_{n+r-2}, p) + (1-\theta_{1n})(1+u_{2n}) d(y_{n+r-2}, p) \\ &\le (1+u_{2n}) d(y_{n+r-2}, p) \end{aligned}$$

implies

$$\limsup_{n\to\infty} d\big(W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big), p\big) \le c. \tag{2.13}$$

Also,

$$\limsup_{n\to\infty} d\big(T^n_1 y_{n+r-2}, p\big) \le c. \tag{2.14}$$

Hence (2.12)-(2.14) and Lemma 1.2 imply that

$$\lim_{n\to\infty} d\big(T^n_1 y_{n+r-2}, W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big)\big) = 0. \tag{2.15}$$

Observe that

$$\begin{aligned} d\big(x_{n+1}, T^n_1 y_{n+r-2}\big) &= d\big(W\big(T^n_1 y_{n+r-2}, W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big), \alpha_{1n}\big), T^n_1 y_{n+r-2}\big) \\ &\le (1-\alpha_{1n}) d\big(W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big), T^n_1 y_{n+r-2}\big) \\ &\quad + \alpha_{1n} d\big(T^n_1 y_{n+r-2}, T^n_1 y_{n+r-2}\big). \end{aligned}$$

On utilizing (2.15), this implies

$$\lim_{n\to\infty} d\big(x_{n+1}, T^n_1 y_{n+r-2}\big) = 0. \tag{2.16}$$

Since $a \le \alpha_{mn}, \beta_{mn} \le b$, therefore (reasoning as above)

$$\begin{aligned} d(x_{n+1}, p) &= d\big(W\big(T^n_1 y_{n+r-2}, W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big), \alpha_{1n}\big), p\big) \\ &\le \alpha_{1n} d\big(T^n_1 y_{n+r-2}, p\big) + (1-\alpha_{1n}) d\big(W\big(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}\big), p\big) \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_{1n} d(x_{n+1}, p) + \alpha_{1n} d\big(x_{n+1}, T_1^n y_{n+r-2}\big) \\
&\quad + (1-\alpha_{1n}) d\big(W\big(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}\big), p\big) \\
&= \frac{\alpha_{1n}}{1-\alpha_{1n}} d\big(x_{n+1}, T_1^n y_{n+r-2}\big) + d\big(W\big(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}\big), p\big) \\
&\leq \frac{b}{1-b} d\big(x_{n+1}, T_1^n y_{n+r-2}\big) + d\big(W\big(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}\big), p\big).
\end{aligned}$$

Taking lim inf on both sides of the above estimate and using (2.16), we have

$$c \leq \liminf_{n\to\infty} d\big(W\big(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}\big), p\big). \tag{2.17}$$

Combining (2.13) and (2.17), we have

$$\lim_{n\to\infty} d\big(W\big(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}\big), p\big) = c. \tag{2.18}$$

By Lemma 1.2 and (2.18), we get

$$\lim_{n\to\infty} d\big(y_{n+r-2}, S_1^n y_{n+r-2}\big) = 0. \tag{2.19}$$

In a similar way, for $k = 2, 3, \ldots, r-1$, we compute

$$\begin{aligned}
&d\big(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}\big) \\
&\quad = d\big(W\big(T_{r-(k-1)}^n y_{n+k-2}, W\big(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}\big), \alpha_{(r-k+1)n}\big), T_{r-(k-1)}^n y_{n+k-2}\big) \\
&\quad \leq (1-\alpha_{(r-k+1)n}) d\big(W\big(y_{n+k-2}, S_{r-(k-1)}^n y_{r+k-2}, \theta_{(r-k+1)n}\big), T_{r-(k-1)}^n y_{n+k-2}\big) \\
&\qquad + \alpha_{(r-k+1)n} d\big(T_{r-(k-1)}^n y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}\big).
\end{aligned}$$

Utilizing (2.11), we have

$$\lim_{n\to\infty} d\big(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}\big) = 0, \quad \text{for } k = 2, 3, \ldots, r-1. \tag{2.20}$$

For $k = r$, we calculate

$$\begin{aligned}
d\big(y_n, T_r^n x_n\big) &= d\big(W\big(T_r^n x_n, W\big(x_n, S_r^n x_n, \theta_{rn}\big), \alpha_{rn}\big), T_r^n x_n\big) \\
&\leq \alpha_{rn} d\big(T_r^n x_n, T_r^n x_n\big) + (1-\alpha_{rn}) d\big(W\big(x_n, S_r^n x_n, \theta_{rn}\big), T_r^n x_n\big).
\end{aligned}$$

Now, utilizing (2.7), we have

$$\lim_{n\to\infty} d\big(y_n, T_r^n x_n\big) = 0. \tag{2.21}$$

Reasoning as above, we get that

$$d(y_n, p) \leq \frac{b}{1-b} d\big(T_r^n x_n, y_n\big) + d\big(W\big(x_n, S_r^n x_n, \theta_{rn}\big), p\big).$$

Applying lim inf on both sides of the above estimate and utilizing (2.3) and (2.21), we have

$$c \leq \liminf_{n\to\infty} d\big(W\big(x_n, S_r^n x_n, \theta_{rn}\big), p\big). \tag{2.22}$$

Inequalities (2.5) and (2.22) collectively imply that

$$\lim_{n\to\infty} d\bigl(W\bigl(x_n, S_r^n x_n, \theta_{rn}\bigr), p\bigr) = c. \tag{2.23}$$

Consequently, Lemma 1.2 and (2.23) imply that

$$\lim_{n\to\infty} d\bigl(x_n, S_r^n x_n\bigr) = 0. \tag{2.24}$$

Note that

$$\begin{aligned} d\bigl(x_n, T_r^n x_n\bigr) &\le d(x_n, y_n) + d\bigl(y_n, T_r^n x_n\bigr) \\ &= d\bigl(x_n, W\bigl(T_r^n x_n, W\bigl(x_n, S_r^n x_n, \theta_{rn}\bigr), \alpha_{rn}\bigr)\bigr) + d\bigl(y_n, T_r^n x_n\bigr) \\ &\le \alpha_{rn} d\bigl(x_n, T_r^n x_n\bigr) + (1-\alpha_{rn}) d\bigl(W\bigl(x_n, S_r^n x_n, \theta_{rn}\bigr), x_n\bigr) + d\bigl(y_n, T_r^n x_n\bigr) \\ &\le (1-\theta_{rn}) d\bigl(x_n, S_r^n x_n\bigr) + \frac{1}{1-\alpha_{rn}} d\bigl(y_n, T_r^n x_n\bigr) \\ &\le \left(\frac{1-2a}{1-b}\right) d\bigl(x_n, S_r^n x_n\bigr) + \frac{1}{1-b} d\bigl(y_n, T_r^n x_n\bigr). \end{aligned}$$

Utilizing (2.21) and (2.24), we have

$$\lim_{n\to\infty} d\bigl(x_n, T_r^n x_n\bigr) = 0. \tag{2.25}$$

Moreover,

$$\begin{aligned} d(x_n, y_n) &= d\bigl(x_n, W\bigl(T_r^n x_n, W\bigl(x_n, S_r^n x_n, \theta_{rn}\bigr), \alpha_{rn}\bigr)\bigr) \\ &\le \alpha_{rn} d\bigl(x_n, T_r^n x_n\bigr) + (1-\alpha_{rn}) d\bigl(x_n, W\bigl(x_n, S_r^n x_n, \theta_{rn}\bigr)\bigr) \\ &\le \alpha_{rn} d\bigl(x_n, T_r^n x_n\bigr) + (1-\alpha_{rn}-\beta_{rn}) d\bigl(x_n, S_r^n x_n\bigr) \\ &\le b d\bigl(x_n, T_r^n x_n\bigr) + (1-2a) d\bigl(x_n, S_r^n x_n\bigr). \end{aligned}$$

By (2.24) and (2.25), we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.26}$$

Again, reasoning as above, we have

$$\begin{aligned} d(y_{n+k-1}, p) &\le d\bigl(W\bigl(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}\bigr), p\bigr) \\ &\quad + \frac{b}{1-b} d\bigl(T_{r-(k-1)}^n y_{n+k-2}, y_{n+k-1}\bigr). \end{aligned}$$

Now, utilizing (2.3) and (2.20), we get

$$c \le \liminf_{n\to\infty} d\bigl(W\bigl(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}\bigr), p\bigr). \tag{2.27}$$

Thus (2.9) and (2.27) imply in total

$$\lim_{n\to\infty} d\bigl(W\bigl(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}\bigr), p\bigr) = c,$$

and by Lemma 1.2, we conclude that

$$\lim_{n\to\infty} d\bigl(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}\bigr) = 0 \quad \text{for } k = 2, 3, \dots, r-1. \tag{2.28}$$

Also,

$$\begin{aligned}
&d\bigl(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}\bigr)\\
&\quad\le d(y_{n+k-2}, y_{n+k-1}) + d\bigl(y_{n+k-1}, T^n_{r-(k-1)}y_{n+k-2}\bigr)\\
&\quad= d\bigl(y_{n+k-2}, W\bigl(T^n_{r-(k-1)}y_{n+k-2}, W\bigl(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}\bigr), \alpha_{(r-k+1)n}\bigr)\bigr)\\
&\qquad+ d\bigl(y_{n+k-1}, T^n_{r-(k-1)}y_{n+k-2}\bigr)\\
&\quad\le d\bigl(y_{n+k-1}, T^n_{r-(k-1)}y_{n+k-2}\bigr) + \alpha_{(r-k+1)n} d\bigl(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}\bigr)\\
&\qquad+ (1-\alpha_{(r-k+1)n}) d\bigl(y_{n+k-2}, W\bigl(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}\bigr)\bigr)\\
&\quad\le d\bigl(y_{n+k-1}, T^n_{r-(k-1)}y_{n+k-2}\bigr) + \alpha_{(r-k+1)n} d\bigl(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}\bigr)\\
&\qquad+ (1-\alpha_{(r-k+1)n} - \beta_{(r-k+1)n}) d\bigl(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}\bigr)\\
&\quad\le \frac{1}{1-b} d\bigl(y_{n+k-1}, T^n_{r-(k-1)}y_{n+k-2}\bigr) + \frac{1-2a}{1-b} d\bigl(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}\bigr).
\end{aligned}$$

Now, utilizing (2.20) and (2.28), we have

$$\lim_{n\to\infty} d\bigl(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}\bigr) = 0, \quad \text{for } k = 2, 3, \dots, r-1. \tag{2.29}$$

For $k = 2, 3, \dots, r-1$, we have

$$d(y_{n+k-2}, y_{n+k-1}) \le d\bigl(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}\bigr) + d\bigl(T^n_{r-(k-1)}y_{n+k-2}, y_{n+k-1}\bigr).$$

Hence, (2.20) and (2.29) imply that

$$\lim_{n\to\infty} d(y_{n+k-2}, y_{n+k-1}) = 0. \tag{2.30}$$

Additionally,

$$d(x_n, y_{n+k-1}) \le d(x_n, y_n) + d(y_n, y_{n+1}) + \cdots + d(y_{n+r-3}, y_{n+r-2}).$$

By (2.26) and (2.30), we have

$$\lim_{n\to\infty} d(x_n, y_{n+k-1}) = 0, \quad \text{for } k = 1, 2, 3, \dots, r-1. \tag{2.31}$$

Let $L = \max_{1\le j\le r} L_j$, where L_j is a Lipschitz constant for T_j. Since each T_j is uniformly L-Lipschitzian, therefore we have

$$\begin{aligned}
d\bigl(x_n, T^n_m x_n\bigr) &\le d(x_n, y_{n+r-m-1}) + d\bigl(y_{n+r-m-1}, T^n_m x_n\bigr)\\
&\le d(x_n, y_{n+r-m-1}) + d\bigl(y_{n+r-m-1}, T^n_m y_{n+r-m-1}\bigr) + d\bigl(T^n_m y_{n+r-m-1}, T^n_m x_n\bigr)\\
&\le (1+L) d(x_n, y_{n+r-m-1}) + d\bigl(y_{n+r-m-1}, T^n_m y_{n+r-m-1}\bigr) \quad \text{for } 2 \le m \le r-1.
\end{aligned}$$

Now, it follows from (2.29) and (2.31) that

$$\lim_{n\to\infty} d\bigl(x_n, T_m^n x_n\bigr) = 0. \tag{2.32}$$

Moreover,

$$\begin{aligned} d(x_{n+1}, T_m x_{n+1}) &\leq d\bigl(x_{n+1}, T_m^{n+1} x_{n+1}\bigr) + d\bigl(T_m^{n+1} x_{n+1}, T_m^{n+1} y_{n+r-m}\bigr) \\ &\quad + d\bigl(T_m^{n+1} y_{n+r-m}, T_m x_{n+1}\bigr) \\ &\leq d\bigl(x_{n+1}, T_m^{n+1} x_{n+1}\bigr) + Ld(x_{n+1}, y_{n+r-m}) + Ld\bigl(T_m^n y_{n+r-m}, x_{n+1}\bigr) \\ &\leq d\bigl(x_{n+1}, T_m^{n+1} x_{n+1}\bigr) + 2Ld(x_{n+1}, y_{n+r-m}) + Ld\bigl(T_m^n y_{n+r-m}, y_{n+r-m}\bigr). \end{aligned}$$

Hence (2.29), (2.31) and (2.32) imply that $d(x_{n+1}, T_m x_{n+1}) \to 0$ as $n \to \infty$ and hence

$$\lim_{n\to\infty} d(x_n, T_m x_n) = 0. \tag{2.33}$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, S_m x_n) = 0. \tag{2.34}$$

This completes the proof. □

Remark 2.3 (i) It is worth mentioning that the asymptotic regularity (2.33)-(2.34) of the iteration schema (1.2) can easily be extended to a more general class of weakly asymptotically quasi-nonexpansive (short: w.aq.n.) mappings. That is, $T : K \to K$ is a w.aq.n. mapping if for all $x \in K$, there exists $p \in K$ such that $d(T^n x, p) \leq (1 + k_n) d(x, p)$, where k_n is a sequence in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$. Obviously, all self-mappings having a zero vector and satisfying $\|T(x)\| \leq \|x\|$ are w.aq.n. mappings. On the other hand, if we define $T : [0,1] \to [0,1]$ by $T(x) = x^2$, then $F(T) = \{0,1\}$. However, quasi-nonexpansivity fails for 1, and hence quasi-nonexpansive mappings are properly included in the class of w.aq.n. mappings. The aforementioned class of w.aq.n. mappings was introduced by Kohlenbach and Lambov [28] as it has nice logical behavior w.r.t. metatheorems [29].

(ii) The above derived results (2.33)-(2.34) can also be achieved if the hypothesis regarding the existence of a common fixed point is weakened by the existence of common approximate fixed points in some neighborhood of the starting point $x_1 \in K$.

(iii) The seminal work of Kohlenbach and Leustean [30] gives a comprehensive logical treatment of asymptotically nonexpansive mappings in the more general setup of uniformly convex hyperbolic spaces and generalizes the corresponding results announced in [28]. They extract explicit rates Φ of metastability (in the sense of Tao) for the asymptotic regularity for the Krasnoselskii-Mann iteration schema. For more on rates of asymptotic regularity in the context of CAT(0) spaces, we refer to [31, 32]. Following the procedure in [28] and [30, Theorem 3.5], one should be able to get such rates Φ also for (2.33)-(2.34) which will - as the rates in [28, 30] - only depend on a (monotone) modulus of uniform convexity for X, an upper bound $b \geq d(x_1, p)$, the Lipschitz constant L, an upper bound $U \geq \sum_{n=1}^{\infty} u_{mn}$ and $N \in \mathbb{N}$ such that $\frac{1}{N} \leq \alpha_{mn}, \beta_{mn} \leq 1 - \frac{1}{N}$. Thus Φ will be largely independent of X, T or x_1. We intend to carry out the extraction of such Φ in another paper.

3 Convergence of approximants to fixed points

In this section, we approximate common fixed points of two finite families of asymptotically nonexpansive mappings in a hyperbolic space. More briefly, we establish $\triangle$-convergence and strong convergence of the iteration schema (1.2).

Theorem 3.1 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_m\}_{m=1}^r$ and $\{S_m\}_{m=1}^r$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings on K. Then the sequence $\{x_n\}$ defined in* (1.2) *$\triangle$-converges to a common fixed point of $p \in F$.*

Proof Since the sequence $\{x_n\}$ is bounded (by Lemma 2.1), therefore Lemma 1.1 asserts that $\{x_n\}$ has a unique asymptotic center. That is, $A(\{x_n\}) = \{x\}$ (say). Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. Then, by Lemma 2.2, we have

$$\lim_{n\to\infty} d(v_n, T_l v_n) = 0 = \lim_{n\to\infty} d(v_n, S_l v_n) \quad \text{for each } l \in I. \tag{3.1}$$

We claim that u is the common fixed point of $\{T_m\}_{m=1}^r$ *and* $\{S_m\}_{m=1}^r$.

For each $m \in \{1, 2, 3, \dots, r\}$, we define a sequence $\{z_n\}$ in K by $z_i = T_m^i v$.

So, we calculate

$$\begin{aligned} d(z_i, v_n) &\le d\big(T_m^i v, T_m^i v_n\big) + d\big(T_m^i v_n, T_m^{i-1} v_n\big) + \cdots + d(T_m v_n, v_n) \\ &\le (1 + u_{mn}) d(v, v_n) + \sum_{i=0}^{r-1} d\big(T_m^i v_n, T_m^{i+1} v_n\big). \end{aligned}$$

Since each T_m is uniformly L-Lipschitzian with the Lipschitz constant L, where $L = \max_{1\le m\le r} L_m$. Therefore, the above estimate yields

$$d(z_i, v_n) \le (1 + u_{mn}) d(v, v_n) + rLd(T_m v_n, v_n).$$

Taking lim sup on both sides of the above estimate and using (3.1), we have

$$\begin{aligned} r\big(z_i, \{v_n\}\big) &= \limsup_{n\to\infty} d(z_i, v_n) \\ &\le \limsup_{n\to\infty} d(v, v_n) = r\big(v, \{v_n\}\big). \end{aligned}$$

This implies that $|r(z_i, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $i \to \infty$. It follows from Lemma 1.3 that $\lim_{i\to\infty} T_m^i v = v$. Utilizing the uniform continuity of T_m, we have that $T_m(v) = T_m(\lim_{i\to\infty} T_m^i v) = \lim_{i\to\infty} T_m^{i+1} v = v$. From the arbitrariness of m, we conclude that v is the common fixed point of $\{T_m\}_{m=1}^r$. Similarly, we can show that v is the common fixed point of $\{S_m\}_{m=1}^r$. Therefore $v \in F$.

Next, we claim that the common fixed point 'v' is the unique asymptotic center for each subsequence $\{v_n\}$ of $\{x_n\}$.

Assume contrarily, that is, $x \neq v$.

Since $\lim_{n\to\infty} d(x_n, v)$ exists (by Lemma 2.1), therefore by the uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x)\\ &\leq \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

a contradiction. Hence $x = v$. Since $\{v_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{v_n\}) = \{v\}$ for all subsequences $\{v_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle$-converges to a common fixed point of $\{T_m\}_{m=1}^r$ and $\{S_m\}_{m=1}^r$. □

Remark 3.2 It follows from the uniqueness of the asymptotic center and a common fixed point of the two families of mappings that Theorem 3.1 can also be generalized to the class of mappings as mentioned in Remark 2.3(i).

Theorem 3.3 *Let K, X, $\{T_m\}_{m=1}^r$, $\{S_m\}_{m=1}^r$ and $\{x_n\}$ be as in Theorem* 3.1. *Then $\{x_n\}$ converges strongly to some $p \in F$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. It follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, F)$ exists. Now $\liminf_{n\to\infty} d(x_n, F) = 0$ reveals that $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, so for any given $\epsilon > 0$, there exists a positive integer n_0 such that

$$d(x_{n_0}, F) < \frac{\epsilon}{8} \quad \text{and} \quad \sum_{n=n_0}^{\infty} a_r s_n < \frac{\epsilon}{2M}. \tag{3.2}$$

The first inequality in (3.2) implies that there exists $p_0 \in F$ such that $d(x_{n_0}, p_0) < \frac{\epsilon}{4}$. Hence, for any $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}d(x_{n_0+m}, x_{n_0}) &\leq d(x_{n_0+m}, p_0) + d(x_{n_0}, p_0)\\ &\leq 2d(x_{n_0}, p_0) + M \sum_{n=n_0}^{n_0+m-1} a_r s_n\\ &< 2\frac{\epsilon}{4} + M\left(\frac{\epsilon}{2M}\right) = \epsilon.\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in X and so it must converge. Let $\lim_{n\to\infty} x_n = q$ (say). We claim that $q \in F$. Indeed, $d(x_n, F) \leq d(x_n, p_0)$ for any $p_0 \in F$. Assume that for each $\epsilon > 0$, there exists $p_n(\epsilon) \in F$ such that

$$d\bigl(x_n, p_n(\epsilon)\bigr) \leq d(x_n, F) + \frac{\epsilon}{2}.$$

This implies that $\lim_{n\to\infty} d(x_n, p_n(\epsilon)) \leq \frac{\epsilon}{2}$. Further, $d(p_n(\epsilon), q) \leq d(x_n, p_n(\epsilon)) + d(x_n, q)$, it follows that

$$\limsup_{n\to\infty} d\bigl(p_n(\epsilon), q\bigr) \leq \frac{\epsilon}{1+L}, \quad \text{where } L \text{ is the Lipschitz constant.}$$

Note that

$$\begin{aligned} d(q, T_m q) &\leq d\bigl(q, p_n(\epsilon)\bigr) + d\bigl(p_n(\epsilon), T_m q\bigr) \\ &= d\bigl(q, p_n(\epsilon)\bigr) + d\bigl(T_m p_n(\epsilon), T_m q\bigr) \\ &\leq (1+L) d\bigl(q, p_n(\epsilon)\bigr). \end{aligned}$$

Then we have $d(q, T_m q) \leq (1+L)\limsup_{n\to\infty} d(q, p_n(\epsilon)) \leq \epsilon$. Since ϵ is arbitrary, we have $d(T_m q, q) = 0$. Similarly, we can show that $d(S_m q, q) = 0$. Hence $q \in F$. □

We now establish strong convergence of the iteration schema (1.2) based on Lemma 2.2.

Theorem 3.4 *Let K, X, $\{T_m\}_{m=1}^r$, $\{S_m\}_{m=1}^r$ and $\{x_n\}$ be as in Theorem* 3.1. *Suppose that a pair of mappings T and S in $\{T_m\}_{m=1}^r$ and $\{S_m\}_{m=1}^r$, respectively, satisfies condition* (A). *Then the sequence $\{x_n\}$ defined in* (1.2) *converges strongly to some $p \in F$.*

Proof It follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, F)$ exists. Moreover, Lemma 2.2 implies that $\lim_{n\to\infty} d(x_n, T_l x_n) = d(x_n, S_l x_n) = 0$ for each $l \in I$. So, condition (A) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Then Theorem 3.3 implies that $\{x_n\}$ converges strongly to a point p in F. □

Theorem 3.5 *Let K, X, $\{T_m\}_{m=1}^r$, $\{S_m\}_{m=1}^r$ and $\{x_n\}$ be as in Theorem* 3.1. *Suppose that either $T_m \in \{T_m\}_{m=1}^r$ or $S_m \in \{S_m\}_{m=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in* (1.2) *converges strongly to $p \in F$.*

Proof Suppose that T_m is semi-compact for some positive integers $1 \leq m \leq r$. We have

$$\begin{aligned} d\bigl(T_m^i x_n, x_n\bigr) &\leq d\bigl(T_m^i x_n, T_m^i x_n\bigr) + d\bigl(T_m^i x_n, T_m^{i-1} x_n\bigr) + \cdots + d(T_m x_n, x_n) \\ &\leq rLd(T_m x_n, x_n). \end{aligned}$$

Then by Lemma 2.2, we have $\lim_{n\to\infty} d(T_m^i x_n, x_n) = 0$. Since $\{x_n\}$ is bounded and T_m is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. By continuity of T_m and Lemma 2.2, we obtain

$$d(q, T_m q) = \lim_{j\to\infty} d(x_{n_j}, T_m x_{n_j}) = 0 \quad \text{for each } m = 1, 2, 3, \ldots, r.$$

This implies that q is the common fixed point of $\{T_m\}_{m=1}^r$. Similarly, we can show that q is the common fixed point of $\{S_m\}_{m=1}^r$. Therefore $q \in F$. The rest of the proof is similar to Theorem 3.1 and is, therefore, omitted. □

Remark 3.6 Compactness of the underlying sequence space is useful for establishing strong convergence of an approximant of a fixed point. Sequential compactness (every sequence has a convergent subsequence), among other notions of compactness, is a widely used tool in this regard. Moreover, if K (or just $T(K)$) is compact, then the approximate sequence $\{x_n\}$ strongly converges to a fixed point. Using a logical analysis of the classical compactness argument, it is shown in [33, Theorem 4.7] how to convert an approximate fixed point bound Φ for Krasnoselskii-Mann iteration schema $\{x_n\}$ of asymptotically nonexpansive mappings (and hence *a fortiori* any rate of metastability for the asymptotic regularity of $\{x_n\}$) into a rate Ψ of metastability for the strong convergence of $\{x_n\}$ in the case of compact K. This rate Ψ depends - in addition to the data on which Φ depends (see Remark 2.3(iii)) - only on a modulus of total boundedness for K. We intend to carry out the extraction of a suitable Ψ for our more general iteration schema with two finite families of mappings in another paper. Combined with the Φ discussed in Remark 2.3(iii), this then yields a highly uniform rate of metastability of Theorem 3.5.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

1. Kohlenbach, U: Some logical metatheorems with applications in functional analysis. Trans. Am. Math. Soc. **357**, 89-128 (2005)
2. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically non-expansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)
3. Khan, MAA: Convergence analysis of a multi-step iteration for a finite family of asymptotically quasi-nonexpansive mappings. J. Inequal. Appl. **2013**, 423 (2013)
4. Fukhar-ud-din, H, Khan, AR, Khan, MAA: A new implicit algorithm of asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces. IAENG Int. J. Appl. Math. **42**(3), 171-175 (2012)
5. Fukhar-ud-din, H, Khan, AR: Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces. Comput. Math. Appl. **53**, 1349-1360 (2007)
6. Kettapun, A, Kananthai, A, Suantai, S: A new approximation method for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. Comput. Math. Appl. **60**, 1430-1439 (2010)
7. Khan, SH, Takahashi, W: Approximating common fixed points of two asymptotically nonexpansive mappings. Sci. Math. Jpn. **53**, 143-148 (2001)
8. Kuhfittig, PKF: Common fixed points of nonexpansive mappings by iteration. Pac. J. Math. **97**(1), 137-139 (1981)
9. Qin, X, Kang, SM, Agarwal, RP: On the convergence of an implicit iterative process for generalized asymptotically quasi-nonexpansive mappings. Fixed Point Theory Appl. **2010**, 714860 (2010)
10. Sahin, A, Basarır, M: On the strong convergence of a modified *S*-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space. Fixed Point Theory Appl. **2013**, 10 (2013). doi:10.1186/1687-1812-2013-12
11. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**, 153-159 (1991)
12. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
13. Shahzad, N, Udomene, A: Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2006**, 18909 (2006)
14. Tan, KK, Xu, HK: Fixed point iteration process for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **122**(3), 733-739 (1994)

15. Yildirim, I, Özdemir, M: Approximating common fixed points of asymptotically quasi-nonexpansive mappings by a new iterative process. Arab. J. Sci. Eng. **36**, 393-403 (2011)
16. Chang, SS, Cho, YJ, Zhou, H: Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings. J. Korean Math. Soc. **38**, 1245-1260 (2001)
17. Bose, SC: Weak convergence to the fixed point of an asymptotically nonexpansive map. Proc. Am. Math. Soc. **68**, 305-308 (1978)
18. Osilike, MO, Aniagbosor, SC: Weak and strong convergence theorems for fixed points of asymptotically non-expansive mappings. Math. Comput. Model. **32**, 1181-1191 (2000)
19. Takahashi, W: Iterative methods for approximation of fixed points and their applications. J. Oper. Res. Soc. Jpn. **43**(1), 87-108 (2000)
20. Khan, AR, Domlo, AA, Fukhar-ud-din, H: Common fixed point Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **341**, 1-11 (2008)
21. Khan, AR, Khamsi, MA, Fukhar-ud-din, H: Strong convergence of a general iteration scheme in CAT(0)-spaces. Nonlinear Anal. **74**, 783-791 (2011)
22. Takahashi, W: A convexity in metric spaces and nonexpansive mappings. Kodai Math. Semin. Rep. **22**, 142-149 (1970)
23. Shimizu, T, Takahashi, W: Fixed points of multivalued mappings in certain convex metric spaces. Topol. Methods Nonlinear Anal. **8**, 197-203 (1996)
24. Leustean, L: Nonexpansive iterations in uniformly convex *W*-hyperbolic spaces. In: Leizarowitz, A, Mordukhovich, BS, Shafrir, I, Zaslavski, A (eds.) Nonlinear Analysis and Optimization I: Nonlinear Analysis. Contemp Math., vol. 513, pp. 193-209. Am. Math. Soc., Providence (2010)
25. Lim, TC: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179-182 (1976)
26. Kirk, W, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689-3696 (2008)
27. Khan, AR, Fukhar-ud-din, H, Khan, MAA: An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. Fixed Point Theory Appl. **2012**, 54 (2012). doi:10.1186/1687-1812-2012-54
28. Kohlenbach, U, Lambov, B: Bounds on iterations of asymptotically quasi-nonexpansive mappings. In: Falset, JG, Llorens Fuster, E, Sims, B (eds.) International Conference on Fixed Point Theory and Applications (Valencia, 2003), pp. 143-172. Yokohama Publ., Yokohama (2004) Zbl 1089.47053 MR 2144170
29. Gerhardy, P, Kohlenbach, U: General logical metatheorems for functional analysis. Trans. Am. Math. Soc. **360**, 2615-2660 (2008)
30. Kohlenbach, U, Leustean, L: Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces. J. Eur. Math. Soc. **12**, 71-92 (2010)
31. Leustean, L: A quadratic rate of asymptotic regularity for CAT(0)-spaces. J. Math. Anal. Appl. **325**, 386-399 (2007)
32. Schade, K, Kohlenbach, U: Effective metastability for modified Halpern iterations in CAT(0) spaces. Fixed Point Theory Appl. **2012**, 19 (2012). doi:10.1186/1687-1812-2012-19
33. Kohlenbach, U: Some computational aspects of metric fixed point theory. Nonlinear Anal. **61**, 823-837 (2005)

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