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Hybrid extragradient method for generalized mixed equilibrium problems and fixed point problems in Hilbert space

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Abstract

In this paper, we introduce iterative schemes based on the extragradient method for finding a common element of the set of solutions of a generalized mixed equilibrium problem and the set of fixed points of a nonexpansive mapping, and the set of solutions of a variational inequality problem for inverse strongly monotone mapping. We obtain some strong convergence theorems for the sequences generated by these processes in Hilbert spaces. The results in this paper generalize, extend and unify some well-known convergence theorems in literature.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. Let $B: C \to H$ be a nonlinear mapping and let $\varphi: C \to R \cup \{+\infty\}$ be a function and F be a bifunction from $C \times C$ to R, where R is the set of real numbers. Peng and Yao [1] considered the following generalized mixed equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x,y) + \varphi(y) + \langle Bx, y - x\rangle \geq \varphi(x). \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(F,\varphi,B)$. It is easy to see that x is a solution of problem (1.1) implying that $x \in \operatorname{dom}\varphi = \{x \in C : \varphi(x) < +\infty\}$.

If $B = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following mixed equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x,y) + \varphi(y) \geq \varphi(x). \tag{1.2}$$

Problem (1.2) was studied by Ceng and Yao [2] and Peng and Yao [3, 4]. The set of solutions of (1.2) is denoted by $MEP(F,\varphi)$.

If $\varphi = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following generalized equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x,y) + \langle Bx, y - x\rangle \geq 0. \tag{1.3}$$

Problem (1.3) was studied by Takahashi and Takahashi [5]. The set of solutions of (1.3) is denoted by $GEP(F,B)$.

If $\varphi = 0$ and $B = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x,y) \geq 0. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(F)$.

If $F(x,y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following generalized variational inequality problem:

$$\text{Finding } x \in C \text{ such that } \varphi(y) + \langle Bx, y - x\rangle \geq \varphi(x). \tag{1.5}$$

The set of solutions of (1.5) is denoted by $GVI(C,\varphi,B)$.

If $\varphi = 0$ and $F(x,y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following variational inequality problem:

$$\text{Finding } x \in C \text{ such that } \langle Bx, y - x\rangle \geq 0. \tag{1.6}$$

The set of solutions of (1.6) is denoted by $VI(C,B)$.

If $B = 0$ and $F(x,y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following minimization problem:

$$\text{Finding } x \in C \text{ such that } \varphi(y) \geq \varphi(x). \tag{1.7}$$

Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others, see for instance, [1–7].

For solving the variational inequality problem in the finite-dimensional Euclidean spaces, in 1976, Korpelevich [8] introduced the following so-called extragradient method:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda B x_n), \\ x_{n+1} = P_C(x_n - \lambda B y_n) \end{cases} \tag{1.8}$$

for every $n = 0, 1, 2, \ldots$, $\lambda \in (0, \frac{1}{k})$, where C is a closed convex subset of R^n, $B : C \to R^n$ is a monotone and k-Lipschitz continuous mapping, and P_C is the metric projection of R^n into C. She showed that if $VI(C,B)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.8), converge to the same point $x \in VI(C,A)$. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces, see, *e.g.*, the recent papers of He *et at.* [9], Gárciga Otero and Iuzem [10], Solodov and Svaiter [11], Solodov [12]. Moreover, Zeng and Yao [13] and Nadezhkina and Takahashi [14] introduced some iterative processes based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. Yao and Yao [15] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality problem for a k-inverse strongly monotone mapping. Plubtieng and Punpaeng [16] introduced an iterative process, based on the extragradient method, for finding the common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem for α-inverse strongly monotone mappings.

In 2003, Takahashi and Toyoda [17], introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Tx_n), \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They proved that if $F(S) \cap VI(A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.9) converges weakly to some $z \in F(S) \cap VI(A)$. Recently, Zeng and Yao [18] introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)SP_C(x_n - \lambda_n Ay_n), \end{cases} \tag{1.10}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions: (i) $\lambda_n k \subset (0, 1-\delta)$ for some $\delta \in (0,1)$ and (ii) $\alpha_n \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$. They proved that the sequence $\{x_n\}$ and $\{y_n\}$ generated by (1.10) converges strongly to the same point $P_{F(S)\cap VI(C,A)}x_0$ provided that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

In 2006, Nadezhkina and Takahashi [19] also considered the extragradient method (1.9) for finding a common element of a fixed point of nonexpansive mapping and a set of solutions of variational inequalities, but the convergence result was still the weak convergence. The question posed by Takahashi and Toyoda [17] on whether the strong convergence result can be proved by the same iteration scheme Algorithm (1.9) remains open.

In 2010, with the techniques adopted by Noor and Rassias [20], Huang, Noor and Al-Said [21] set the projected residual function by

$$R_\lambda(x) = x - P_C(x - \lambda Ax), \tag{1.11}$$

it is well known that $x \in C$ is a solution of variational inequality (1.6) if and only if $x \in C$ is a zero of the projected residual function (1.11). They proved the strong convergence result of the iteration scheme (1.9) using the error analysis technique.

In this paper, inspired and motivated by the above researches and Huang, Noor and Al-Said [21], we introduce a new iterative scheme based on the extragradient method for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of nonexpansive mappings and the set of solutions of an inverse strongly monotone mapping, as follows:

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n Au_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)S[\beta_n x_n + (1-\beta_n)P_C(y_n - \lambda_n Ay_n)], \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{\lambda_n\}$ satisfy some parameters controlling conditions. We will obtain some strong convergence theorems using the error analysis technique as in [21]. The results in this paper generalize, extend and unify some well-known convergence theorems in the literature.

2 Preliminaries

Let C be a closed convex subset of a Hilbert space H for every point $x \in H$. There exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} &\langle x - P_C x, y - P_C y\rangle \leq 0, \\ &\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \end{aligned} \tag{2.1}$$

for all $x \in H$, $y \in C$.

A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

for all $x, y \in C$. A mapping A of C into H is called inverse strongly monotone with a modulus α (in short, α-inverse strongly monotone) if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

Recall that a mapping S of C into itself is nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

A mapping T of C into itself is pseudocontractive if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2$$

for all $x, y \in C$. Obviously, the class of pseudocontractive mappings is more general than the class of nonexpansive mappings.

Let A be a monotone mapping from C into H. In the context of the variational inequality problem, the characterization of projection (2.1) implies the following:

$$u \in VI(A, C) \quad \Longleftrightarrow \quad u = P_C(u - \lambda Au), \quad \lambda > 0.$$

It is also known that H satisfies Opial's condition; for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

For solving the generalized mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function φ and the set C:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) for each $y \in C$, $x \vdash F(x,y)$ is weakly upper semicontinuous;

(A4) for each $x \in C$, $y \vdash F(x,y)$ is convex;

(A5) for each $x \in C$, $y \vdash F(x,y)$ is lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D(x) \subset C$ and $y_x \in C \cap \operatorname{dom}(\varphi)$ such that for any $z \in C - D_x$,

$$F(z,y_x) + \varphi(y_x) + \langle Bz, y_x - z\rangle + \frac{1}{r}\langle y_x - z, z - x\rangle \leq \varphi(z);$$

(B2) C is a bounded set.

Lemma 2.1 [1] *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)-(A5) *and let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r = \left\{ z \in C : F(z,y) + \varphi(y) + \langle Bz, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \leq \varphi(z), \forall y \in C \right\}$$

for all $x \in H$. Then the following conclusions hold:

(1) *For each $x \in H$, $T_r(x) \neq \emptyset$*;

(2) *T_r is single-valued*;

(3) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$*,

$$\left\| T_r(x) - T_r(y) \right\|^2 \leq \left\langle T_r(x) - T_r(y), x - y \right\rangle;$$

(4) $\operatorname{Fix}(T_r(I - rB)) = GMEP(F,\varphi,B)$;

(5) *$GMEP(F,\varphi,B)$ is closed and convex.*

Lemma 2.2 [22] *For any $x^* \in VI(C,A)$, if $A : C \to H$ is α-inverse strongly monotone, then $R_\lambda(x)$ is $(1 - \frac{\lambda}{4\alpha})$-inverse strongly monotone for any $\lambda \in [0, 4\alpha]$ and*

$$\left\langle x - x^*, R_\lambda(x) \right\rangle \geq \left(1 - \frac{\lambda}{4\alpha}\right) \left\| R_\lambda(x) \right\|^2,$$

where $R_\lambda(x) = x - P_C(x - \lambda Ax)$.

Lemma 2.3 [21] *For all $x \in H$ and $\lambda' \geq \lambda > 0$, it holds that*

$$\|R_{\lambda'}(x)\| \geq \|R_{\lambda}(x)\|,$$

where $R_{\lambda}(x) = x - P_C(x - \lambda Ax)$.

Lemma 2.4 [23] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-negative numbers, such that $a_{n+1} \leq a_n + b_n$ for all $n \in N$. If $\sum_{n=1}^{\infty} b_n < +\infty$, and if $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to 0, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.5 [24] *Let H be a real Hilbert space, and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for any $x^* \in C$,*

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| \quad (n \in N).$$

Then $\lim_{n\to\infty} P_C(x_n) = z$ for some $z \in C$.

3 Main results

Theorem 3.1 *Let C be a closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to R$ satisfying* (A1)-(A5) *and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse strongly monotone mapping from C into H and B be an β-inverse strongly monotone mapping from C into H. Let S be a nonexpansive mapping of C into itself, such that $\Omega = \mathrm{Fix}(S) \cap VI(C,A) \cap GMEP(F,\varphi,B) \neq \emptyset$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n,y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S[\beta_n x_n + (1-\beta_n) P_C(y_n - \lambda_n A y_n)] \end{cases} \tag{3.1}$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$, $\{r_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions: (i) $0 < r_n < 2\beta$, *$\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$ and* (ii) *$\{\alpha_n\} \subset [c,d]$, $\{\beta_n\} \subset [e,f]$ for some $c,d,e,f \in (0,1)$, then $\{x_n\}$ converges strongly to $p^* \in \Omega$, where $p^* = \lim_{n\to\infty} P_{\Omega}(x_n)$.*

Proof We divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded and $\lim_{n\to\infty} R_a(u_n) = \lim_{n\to\infty} R_{\lambda_n}(u_n) = 0$.

Put

$$v_n = P_C(y_n - \lambda_n A y_n), \qquad w_n = \beta_n x_n + (1-\beta_n) v_n,$$
$$R_{\lambda_n}(u_n) = u_n - P_C(u_n - \lambda_n A u_n), \qquad R_{\lambda_n}(y_n) = y_n - P_C(y_n - \lambda_n A y_n)$$

for every $n = 1, 2, \ldots$. Take any $p \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1, then $p = P_C(p - \lambda_n A p) = T_{r_n}(p - r_n B p)$. From $u_n = T_{r_n}(x_n - r_n B x_n) \in C$, the β-inverse strongly monotonicity of B and $0 < r_n < 2\beta$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n B x_n) - T_{r_n}(p - r_n B p)\|^2 \\ &\leq \|x_n - r_n B x_n - (p - r_n B p)\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bx_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bx_n - Bp\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bp\|^2 \\
&\leq \|x_n - p\|^2, \qquad (3.2)
\end{aligned}
$$

and from Lemma 2.2, we have

$$
\begin{aligned}
\|y_n - p\|^2 &= \left\|u_n - R_{\lambda_n}(u_n) - p\right\|^2 \\
&= \|u_n - p\|^2 - 2\left\langle u_n - p, R_{\lambda_n}(u_n)\right\rangle + \left\|R_{\lambda_n}(u_n)\right\|^2 \\
&\leq \|u_n - p\|^2 - 2\left(1 - \frac{\lambda_n}{4\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 + \left\|R_{\lambda_n}(u_n)\right\|^2 \\
&= \|u_n - p\|^2 - \left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2, \qquad (3.3)
\end{aligned}
$$

which implies from (3.2) that

$$
\|y_n - p\|^2 \leq \|x_n - p\|^2 - \left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2. \qquad (3.4)
$$

By the same process as in (3.3), we also have from (3.4) that

$$
\begin{aligned}
\|v_n - p\|^2 &\leq \|y_n - p\|^2 - \left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(y_n)\right\|^2 \\
&\leq \|y_n - p\|^2 - \left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(y_n)\right\|^2 \\
&\leq \|x_n - p\|^2 - \left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 - \left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(y_n)\right\|^2. \qquad (3.5)
\end{aligned}
$$

Further, from (3.1) and (3.5), we get

$$
\begin{aligned}
\|w_n - p\|^2 &= \beta_n^2\|x_n - p\|^2 + 2\beta_n(1 - \beta_n)\langle x_n - p, v_n - p\rangle + (1 - \beta_n)^2\|v_n - p\|^2 \\
&\leq \beta_n^2\|x_n - p\|^2 + 2\beta_n(1 - \beta_n)\|x_n - p\|^2 + (1 - \beta_n)^2\|x_n - p\|^2 \\
&\quad - (1 - \beta_n)^2\left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 \\
&\leq \|x_n - p\|^2 - (1 - \beta_n)^2\left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 \\
&\quad - (1 - \beta_n)^2\left(1 - \frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(y_n)\right\|^2. \qquad (3.6)
\end{aligned}
$$

Hence, from (3.6), the nonexpansive property of the mapping S and $0 < \lambda_n < 2\alpha$, we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \alpha_n^2\|x_n - p\|^2 + (1 - \alpha_n)^2\|Sw_n - p\|^2 + 2\alpha_n(1 - \alpha_n)\langle Sw_n - Sp, x_n - p\rangle \\
&\leq \alpha_n^2\|x_n - p\|^2 + (1 - \alpha_n)^2\|w_n - p\|^2 + 2\alpha_n(1 - \alpha_n)\|x_n - p\|^2 \\
&\leq \alpha_n^2\|x_n - p\|^2 + (1 - \alpha_n)^2\|x_n - p\|^2 + 2\alpha_n(1 - \alpha_n)\|x_n - p\|^2
\end{aligned}
$$

$$\begin{aligned} &- (1-\alpha_n)^2(1-\beta_n)^2\left(1-\frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 \\ &= \|x_n - p\|^2 - (1-\alpha_n)^2(1-\beta_n)^2\left(1-\frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 \\ &\quad - (1-\alpha_n)^2(1-\beta_n)^2\left(1-\frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(y_n)\right\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.7}$$

Since the sequence $\{\|x_n - p\|\}$ is a bounded and nonincreasing sequence, $\lim_{n\to\infty}\|x_n - p\|$ exists. Hence $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{y_n\}$ are also bounded. By (3.7), we have

$$(1-\alpha_n)^2(1-\beta_n)^2\left(1-\frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

From the conditions (i) and (ii), there must exist a constant $M_1 > 0$ such that

$$M_1\left\|R_{\lambda_n}(u_n)\right\|^2 \le (1-\alpha_n)^2(1-\beta_n)^2\left(1-\frac{\lambda_n}{2\alpha}\right)\left\|R_{\lambda_n}(u_n)\right\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2,$$

from which it follows that

$$M_1\sum_{n=1}^{\infty}\left\|R_{\lambda_n}(u_n)\right\|^2 \le \sum_{n=1}^{\infty}\left[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right] = \|x_1 - p\|^2 < \infty.$$

Hence, $\lim_{n\to\infty} R_{\lambda_n}(u_n) = \lim_{n\to\infty}\|R_{\lambda_n}(u_n)\| = 0$. Since $R_{\lambda_n}(u_n) = u_n - P_C(u_n - \lambda_n A u_n) = u_n - y_n$, $\lim_{n\to\infty}\|u_n - y_n\| = 0$. Notice that $\lambda_n \ge a$, then by Lemma 2.3, $\|R_a(u_n)\| \le \|R_{\lambda_n}(u_n)\|$. Therefore,

$$\lim_{n\to\infty} R_a(u_n) = \lim_{n\to\infty} R_{\lambda_n}(u_n) = 0. \tag{3.8}$$

By the same way, we also get that

$$\lim_{n\to\infty}\left\|R_{\lambda_n}(y_n)\right\| = \lim_{n\to\infty}\|y_n - v_n\| = 0,$$

and thus

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{3.9}$$

Step 2. We show that $\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|Sx_n - x_n\| = 0$.

Indeed, for any $p \in \Omega$, it follows from (3.1) and (3.5) that

$$\begin{aligned} \|w_n - p\|^2 &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2 \\ &\le \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 - \alpha_n(1-\alpha_n)\|Sw_n - x_n\|^2 \\ &\le \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|Sw_n - x_n\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2. \end{aligned} \tag{3.10}$$

Thus, it follows from (3.10) that

$$\alpha_n(1-\alpha_n)\|Sw_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

From the condition (ii), there exists a constant $M_2 > 0$ such that

$$M_2\|Sw_n - x_n\|^2 \leq \alpha_n(1-\alpha_n)\|Sw_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2,$$

from which it follows that

$$M_2\sum_{n=1}^{\infty}\|Sw_n - x_n\|^2 \leq \sum_{n=1}^{\infty}\bigl[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\bigr] = \|x_1 - p\|^2 < \infty.$$

Hence

$$\lim_{n\to\infty}\|Sw_n - x_n\| = 0. \tag{3.11}$$

From (3.10), we also get that

$$\beta_n(1-\beta_n)\|x_n - v_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

By the same way, we obtain that

$$\lim_{n\to\infty}\|x_n - v_n\| = 0, \tag{3.12}$$

which combining (3.9) implies that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.13}$$

Since

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|Sx_n - Sv_n\| + \|Sv_n - Sw_n\| + \|Sw_n - x_n\|\\ &\leq \|x_n - v_n\| + \|v_n - w_n\| + \|Sw_n - x_n\|\\ &\leq \|x_n - v_n\| + \beta_n\|x_n - v_n\| + \|Sw_n - x_n\|,\end{aligned}$$

which implies from (3.11), (3.12) that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{3.14}$$

Further, it follows from (3.1) and (3.11) that

$$\|x_{n+1} - x_n\| = (1-\alpha_n)\|Sw_n - x_n\| \leq (1-c)\|Sw_n - x_n\| \to 0 \quad (n\to\infty). \tag{3.15}$$

Step 3. We claim that $\{x_n\}$ must have a convergent subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = p^*$ for some $p^* \in C$. Moreover, $p^* \in \Omega = \mathrm{Fix}(S) \cap VI(C,A) \cap GMEP(F,\varphi,B)$.

Since $\{x_n\}$ is a bounded sequence generated by Algorithm (3.1), then $\{x_n\}$ must have a weakly convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup p^*$ ($k \to \infty$), which implies from (3.11) and (3.13) that $Sw_{n_k} \rightharpoonup p^*$ ($k \to \infty$) and $u_{n_k} \rightharpoonup p^*$ ($k \to \infty$). Next we will show that $p^* \in \Omega = \mathrm{Fix}(S) \cap VI(C,A) \cap GMEP(F,\varphi,B)$.

Since A is inverse strongly monotone with the positive constant $\alpha > 0$, so A is $\frac{1}{\alpha}$-Lipschitz continuous. Indeed, it yields that $\|Ax - Ay\| \le \frac{1}{\alpha}\|x - y\|$ from the definition of the inverse strongly monotonicity of A, such that

$$\alpha\|Ax - Ay\|^2 \le \langle Ax - Ay, x - y\rangle \le \|Ax - Ay\|\|x - y\|.$$

From the $\frac{1}{\alpha}$-Lipschitz continuity of A and the continuity of P_C, it follows that $R_a(x) = x - P_C[x - aAx]$ is also continuous. Notice that $\rho_n \ge a$, then by Lemma 2.3, $\|R_x(x_n)\| \le \|R_{\rho_n}(x_n)\|$. Then from Step 1,

$$\lim_{k\to\infty}\left\|R_x(x_{n_k})\right\| = \lim_{n\to\infty}\left\|R_{\rho_n}(x_{n_k})\right\| = 0.$$

Therefore from the continuity of $R_a(x)$,

$$R_a\bigl(p^*\bigr) = \lim_{n\to\infty} R_a(x_{n_k}) = 0.$$

This shows that p^* is a solution of the variational inequality (1.6), that is $p^* \in VI(C,A)$. From (3.12), $\lim_{n\to\infty}\|x_{n_k} - p^*\| = 0$ and the property of the nonexpansive mapping S, it follows that $p^* = Sp^*$, that is $p^* \in \mathrm{Fix}(S)$. Finally, by the same argument as in the proof of [7, Theorem 3.1], we prove that $p^* \in GMEP(F,\varphi,B)$. Thus $p^* \in \Omega = \mathrm{Fix}(S) \cap VI(C,A) \cap GMEP(F,\varphi,B)$.

Next, we will prove that $x_{n_k} \to p^*$ ($k \to \infty$).

From (3.1), (3.6) and (3.7) we can calculate

$$\begin{aligned}
\left\|x_{n+1} - p^*\right\|^2 &= \bigl\langle \alpha_n x_n + (1-\alpha_n)Sw_n - p^*, x_{n+1} - p^*\bigr\rangle \\
&= \alpha_n\bigl\langle x_n - p^*, x_{n+1} - p^*\bigr\rangle + (1-\alpha_n)\bigl\langle Sw_n - p^*, x_{n+1} - p^*\bigr\rangle \\
&\le \alpha_n\left\|x_n - p^*\right\|^2 + (1-\alpha_n)\bigl\langle Sw_n - p^*, x_{n+1} - p^*\bigr\rangle \\
&\le \alpha_n\left\|x_n - p^*\right\|^2 + (1-\alpha_n)\bigl\langle Sw_n - p^*, x_{n+1} - x_n\bigr\rangle \\
&\quad + (1-\alpha_n)\bigl\langle Sw_n - p^*, x_n - p^*\bigr\rangle \\
&\le \alpha_n\left\|x_n - p^*\right\|^2 + (1-\alpha_n)\left\|x_n - p^*\right\|^2 + (1-\alpha_n)\bigl\langle Sw_n - p^*, x_{n+1} - x_n\bigr\rangle \\
&= \left\|x_n - p^*\right\|^2 + (1-\alpha_n)\bigl\langle Sw_n - p^*, x_{n+1} - x_n\bigr\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
\left\|x_{n+1} - p^*\right\|^2 - \left\|x_n - p^*\right\|^2 &\le (1-\alpha_n)\bigl\langle Sw_n - p^*, x_{n+1} - x_n\bigr\rangle \\
&\le (1-c)\bigl\langle Sw_n - p^*, x_{n+1} - x_n\bigr\rangle. \qquad (3.16)
\end{aligned}$$

From $Sw_{n_k} \rightharpoonup p^*$ and $x_{n_{k+1}} - x_{n_k} \to 0$ as $k \to \infty$, it follows from (3.16) that

$$\left\|x_{n_{k+1}} - p^*\right\| \to \left\|x_{n_k} - p^*\right\| \quad (k \to \infty).$$

Using the Kadec-Klee property of H, we obtain that $\lim_{k\to\infty} x_{n_k} = p^*$.

Step 4. We claim that the sequence $\{x_n\}$ generated by Algorithm (3.1) converges strongly to $p^* \in \Omega = \mathrm{Fix}(S) \cap VI(C,A) \cap GMEP(F,\varphi,B)$.

In fact, from the result of Step 3, $p^* \in \Omega$. Let $p = p^*$ in (3.7). Consequently, $\|x_{n+1} - p^*\| \le \|x_n - p^*\|$. Meanwhile, $\lim_{k\to\infty} \|x_{n_k} - p^*\| = 0$ from Step 3. Then from Lemma 2.4, we have $\lim_{n\to\infty} \|x_n - p^*\| = 0$. Therefore, $\lim_{n\to\infty} x_n = p^*$.

Step 5. We claim that $p^* = \lim_{n\to\infty} P_\Omega x_n$.

From (2.1), we have

$$\langle x_n - P_\Omega x_n, p^* - P_\Omega x_n \rangle \le 0. \tag{3.17}$$

By (3.7) and Lemma 2.5, $\lim_{n\to\infty} P_\Omega x_n = q^*$ for some $q^* \in \Omega$. Then in (3.13), let $n \to \infty$, since $\lim_{n\to\infty} x_n = p^*$ by Step 4, we have

$$\langle p^* - q^*, p^* - q^* \rangle \le 0,$$

and, consequently, we have $p^* = q^*$. Hence, $p^* = \lim_{n\to\infty} P_\Omega x_n$.

This completes the proof of Theorem 3.1. □

The following theorems can be obtained from Theorem 3.1 immediately.

Theorem 3.2 *Let C, H, S be as in Theorem* 3.1. *Assume that* $\Omega = \mathrm{Fix}(S) \cap VI(C,A) \neq \emptyset$, *let* $\{x_n\}$, $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S[\beta_n x_n + (1-\beta_n) P_C(y_n - \lambda_n A y_n)] \end{cases}$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions: (i) $\{\lambda_n\} \subset [a,b]$ *for some $a, b \in (0, 2\alpha)$ and* (ii) $\{\alpha_n\} \subset [c,d]$, $\{\beta_n\} \subset [e,f]$ *for some $c, d, e, f \in (0,1)$, then $\{x_n\}$ converges strongly to $p^* \in \Omega$, where $p^* = \lim_{n\to\infty} P_\Omega(x_n)$.*

Proof Putting $B = F = \varphi = 0$, $r_n = 1$ in Theorem 3.1, the conclusion of Theorem 3.2 can be obtained from Theorem 3.1. □

Remark 3.1 The main result of Nadezhkina and Takahashi [14] is a special case of our Theorem 3.2. Indeed, if we take $\beta_n = 0$ in Theorem 3.2, then we obtain the result of [14].

Theorem 3.3 *Let C, H, F, A, B, S be as in Theorem* 3.1. *Assume* $\Omega = \mathrm{Fix}(S) \cap VI(C,A) \cap GEP(F,B) \neq \emptyset$; *let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle B x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S[\beta_n x_n + (1-\beta_n) P_C(y_n - \lambda_n A y_n)] \end{cases}$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$, $\{r_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy conditions (i) *and* (ii) *as in Theorem* 3.1, *then $\{x_n\}$ converges strongly to $p^* \in \Omega$, where $p^* = \lim_{n\to\infty} P_\Omega(x_n)$.*

Proof Putting $\varphi = 0$ in Theorem 3.1, the conclusion of Theorem 3.3 is obtained. □

Remark 3.2 Theorem 3.3 can be viewed as an improvement of Theorem 3.1 of Inchan [25] because of removing the iterative step C_n in the algorithm of Theorem 3.1 of [25].

Theorem 3.4 *Let C, H, F, A, S be as in Theorem* 3.1. *Assume that* $\Omega = \mathrm{Fix}(S) \cap VI(C,A) \cap EP(F) \neq \emptyset$; *let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(y_n - \lambda_n A y_n) \end{cases}$$

for every $n = 1, 2, \ldots$, *where* $\{\lambda_n\}$, $\{r_n\}$, $\{\alpha_n\}$ *satisfy the following conditions*: $0 < r_n < 2\beta$, $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$, $\{\alpha_n\} \subset [c,d]$ *for some* $c, d \in (0,1)$, *then* $\{x_n\}$ *converges strongly to* $p^* \in \Omega$, *where* $p^* = \lim_{n\to\infty} P_\Omega(x_n)$.

Proof Taking $B = \varphi = 0$, $\beta_n = 0$ in Theorem 3.1, the conclusion of Theorem 3.4 is obtained. □

Remark 3.3 Theorem 3.4 is the strong convergence result of Theorem 3.1 of Jaiboon, Kumam and Humphries [26].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SL and LL carried out the proof of convergence of the theorems. LC, XH and XY carried out the check of the manuscript. All authors read and approved the final manuscript.

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