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Characterizations of weak-*ANODD* set-valued mappings with applications to an approximate solution of GNMOQV inclusions involving ⊕ operator in ordered Banach spaces

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Abstract
In ordered Banach spaces, characterizations of ordered (α_A, λ)-weak-*ANODD* set-valued mappings are introduced and studied, which is applied to giving an approximate solution for a new class of general nonlinear mixed-order quasi-variational inclusions involving ⊕ operator. By using the resolvent operator associated with an (α_A, λ)-weak-*ANODD* set-valued mapping and fixed point theory, an existence theorem of solutions and an approximation algorithm for this kind of inclusions are established and discussed in ordered Banach spaces, and the relation between the first-valued point x_0 and the solution of the problems is shown. The results obtained seem to be general in nature.

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Keywords: ordered (α_A, λ)-weak-*ANODD* set-valued mappings; general nonlinear mixed-order quasi-variational inclusions involving ⊕ operator; existence theorem of solutions; strongly comparison mapping; ordered Banach space; approximate solution of GNMOQVI

1 Introduction

Generalized nonlinear ordered variational inequalities (ordered equations) have wide applications in many fields including, for example, mathematics, physics, optimization and control, nonlinear programming, economics, and engineering sciences.

The variational inclusion, which was introduced and studied by Hassouni and Moudafi [1], is a useful and important extension of the variational inequality. In 1989, Chang and Zhu [2] introduced and investigated a class of variational inequalities for fuzzy mappings. Afterwards, Chang and Huang [3], Ding and Jong [4], Jin [5], Li [6] and others studied several kinds of variational inequalities (inclusions) for fuzzy mappings. A generalized random multivalued quasi-complementarity problem was introduced and studied by Chang and Huang [7], and the random variational inclusion (inequalities, equalities, quasi-variational inclusions, quasi-complementarity) problems were studied by Ahmad and Bazán [8], Chang [9], Cho *et al.* [10]. In recent years, Huang and Fang [11] introduced the concept of generalized m-accretive mapping, studied the properties of the resolvent operator with the generalized m-accretive mapping; and furthermore, Huang and Fang [12] studied a class of generalized monotone mappings, maximal η-monotone mappings, and defined an associated resolvent operator in 2003. Using resolvent operator methods, they developed some iterative algorithms to approximate the solution of a class of general variational inclusions involving maximal η-monotone operators. Huang and Fang's method extended the resolvent operator method associated with an η-subdifferential operator due to Ding and Luo [13]. In [14], Fang and Huang introduced another class of generalized monotone operators, H-monotone operators, defined an associated resolvent operator, established the Lipschitz continuity of the resolvent operator, and studied a class of variational inclusions in Hilbert spaces using the resolvent operator associated with H-monotone operators. In a recent paper [15], Fang, Huang and Thompson further introduced a new class of generalized monotone operators, (H, η)-monotone operators, which provided a unifying framework for classes of maximal monotone operators, maximal η-monotone operators, and H-monotone operators. Very recently, Lan *et al.* [16] introduced a new concept of (A, η)-accretive mappings, which generalized the existing monotone or accretive operators, and studied some properties of mappings. They also studied a class of variational inclusions using the resolvent operator associated with (A, η)-accretive mappings.

On the other hand, in 1972, a number of solutions of nonlinear equations were introduced and studied by Amann [17], and in recent years, the nonlinear mapping fixed point theory and application have been intensively studied in ordered Banach spaces [18–20]. Therefore, it is very important and natural that generalized nonlinear ordered variational inequalities (ordered equation) are studied and discussed.

In 2008 the author introduced and studied the approximation algorithm and the approximation solution for a class of generalized nonlinear ordered variational inequalities and ordered equations to find $x \in X$ such that $A(g(x)) \geq \theta$ ($A(x)$ and $g(x)$ are single-valued mappings) in ordered Banach spaces [21]. By using the B-restricted-accretive method of the mapping A with constants α_1, α_2, the author introduced and studied a new class of general nonlinear ordered variational inequalities and equations to find $x \in X$ such that $A(x) \oplus F(x, g(x)) \geq \theta$ ($A(x)$, $g(x)$ and $F(\cdot, \cdot)$ are single-valued mappings), and established an existence theorem and an approximation algorithm of solutions for this kind of generalized nonlinear ordered variational inequalities (equations) in ordered Banach spaces [22]. By using the resolvent operator associated with an RME set-valued mapping, the author introduced and studied a class of nonlinear inclusion problems for ordered MR set-valued mappings to find $x \in X$ such that $0 \in M(x)$ ($M(x)$ is a set-valued mapping), and the existence theorem of solutions and an approximation algorithm for this kind of nonlinear inclusion problems for ordered extended set-valued mappings in ordered Hilbert spaces [23]. In 2012, the author introduced and studied a class of nonlinear inclusion problems to find $x \in X$ such that $0 \in M(x)$ ($M(x)$ is a set-valued mapping) for ordered (α, λ)-*NODM* set-valued mappings, and then, applying the resolvent operator associated with (α, λ)-*NODM* set-valued mappings, established the existence theorem on the solvability and a general algorithm applied to the approximation solvability of the nonlinear inclusion problem of this class of nonlinear inclusion problems, based on the existence theorem and the new (α, λ)-*NODM* model in an ordered Hilbert space [24]. In Banach spaces, the author proved sensitivity analysis of the solution for a new class of general nonlinear ordered parametric variational inequalities to find $x = x(\lambda) : \Omega \to X$ such that $A(g(x, \lambda), \lambda) + f(x, \lambda) \geq \theta$ ($A(x)$,

$g(x)$ and $F(\cdot,\cdot)$ are single-valued mappings) in 2012 [25]. In this field, the obtained results seem to be general in nature. Now, it is of excellent interest that we are studying the characterizations of ordered (α_A,λ)-weak-*ANODD* set-valued mappings, which is applied to solving an approximate solution for a new class of general nonlinear mixed-order quasi-variational inclusions involving $\oplus$ operator in ordered Banach spaces. For details, we refer the reader to [1–48] and the references therein.

Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ, a normal cone $\mathbf{P}$, a normal constant N and a partial ordered relation $\leq$ defined by the cone $\mathbf{P}$ [21]. Let $f: X\to X$ be a single-valued ordered compression mapping, and let $M: X\to 2^X$ and

$$f(x)\oplus M(x)=\{y|y=f(x)\oplus u,\forall x\in X,u\in M(x)\}:X\to 2^X$$

be two set-valued mappings. We consider the following problem:

For $w\in X$, find $x\in X$ such that

$$w\in f(x)\oplus M(x). \tag{1.1}$$

Problem (1.1) is called a general nonlinear mixed-order quasi-variational inclusions (GNMOQVI) involving $\oplus$ operator in an ordered Banach space.

In recent years, the nonlinear mapping fixed point theory and its applications have been intensively studied in ordered Banach spaces [21–26]. And very recently, the author introduced and studied the approximation theory, the approximation algorithm and the approximation solution for generalized nonlinear ordered variational inequalities (ordered equations, inclusion problems) in ordered Hilbert spaces or ordered Banach spaces [21–25].

Inspired by recent research work in this field, we introduce a new class of general nonlinear mixed-order quasi-variational inclusion problems involving $\oplus$ operator in an ordered Banach space. By using the resolvent operator techniques [27–37, 44, 45, 48] associated with an ordered (α,λ)-weak-*ANODD* set-valued mapping with a strongly comparison mapping A, an existence theorem of solutions and an approximation algorithm for this kind of problems are studied, and the relation between the first-valued point x_0 and the solution of the problems is discussed. The results obtained seem to be general in nature.

2 Preliminaries

Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ, a normal cone $\mathbf{P}$, a normal constant N and a partial ordered relation $\leq$ defined by the cone $\mathbf{P}$. For arbitrary $x,y\in X$, $lub\{x,y\}$ and $glb\{x,y\}$ express the least upper bound of the set $\{x,y\}$ and the greatest lower bound of the set $\{x,y\}$ on the partial ordered relation $\leq$, respectively. Suppose that $lub\{x,y\}$ and $glb\{x,y\}$ exist. Let us recall some concepts and results.

Definition 2.1 [21, 26] Let X be a real Banach space with a norm $\|\cdot\|$, θ be a zero element in X.

(i) A nonempty closed convex subset $\mathbf{P}$ of X is said to be a cone if
  (1) for any $x\in\mathbf{P}$ and any $\lambda>0$, $\lambda x\in\mathbf{P}$ holds,
  (2) if $x\in\mathbf{P}$ and $-x\in\mathbf{P}$, then $x=\theta$;

(ii) **P** is said to be a normal cone if and only if there exists a constant $N > 0$, a normal constant of **P**, such that for $\theta \leq x \leq y$, $\|x\| \leq N\|y\|$ holds;

(iii) For arbitrary $x, y \in X$, $x \leq y$ if and only if $x - y \in \mathbf{P}$;

(iv) For $x, y \in X$, x and y are said to be comparative to each other if and only if $x \leq y$ (or $y \leq x$) holds (denoted by $x \propto y$ for $x \leq y$ and $y \leq x$).

Lemma 2.2 [18] *If $x \propto y$, then lub$\{x, y\}$ and glb$\{x, y\}$ exist, $x - y \propto y - x$, and $\theta \leq (x - y) \vee (y - x)$.*

Lemma 2.3 [18] *If for any natural number n, $x \propto y_n$ and $y_n \to y^*$ $(n \to \infty)$, then $x \propto y^*$.*

Lemma 2.4 [21, 22, 24, 25] *Let X be an ordered Banach space, let **P** be a cone of X, let $\leq$ be a relation defined by the cone **P** in Definition* 2.1(iii). *For $x, y, v, u \in X$, the following relations hold*:

(1) *the relation $\leq$ in X is a partial ordered relation in X*;

(2) $x \oplus y = y \oplus x$;

(3) $x \oplus x = \theta$;

(4) $\theta \leq x \oplus \theta$;

(5) *let λ be real, then* $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;

(6) *if x, y, and w can be comparative to each other, then* $(x \oplus y) \leq x \oplus w + w \oplus y$;

(7) *let $(x + y) \vee (u + v)$ exist, and if $x \propto u, v$ and $y \propto u, v$, then*
$(x + y) \oplus (u + v) \leq (x \oplus u + y \oplus v) \wedge (x \oplus v + y \oplus u)$;

(8) *if x, y, z, w can be compared with each other, then*
$(x \wedge y) \oplus (z \wedge w) \leq ((x \oplus z) \vee (y \oplus w)) \wedge ((x \oplus w) \vee (y \oplus z))$;

(9) *if $x \leq y$ and $u \leq v$, then $x + u \leq y + v$*;

(10) *if $x \propto \theta$, then $-x \oplus \theta \leq x \leq x \oplus \theta$*;

(11) *if $x \propto y$, then* $(x \oplus \theta) \oplus (y \oplus \theta) \leq (x \oplus y) \oplus \theta = x \oplus y$;

(12) $(x \oplus \theta) - (y \oplus \theta) \leq (x - y) \oplus \theta$;

(13) *if $\theta \leq x$ and $x \neq \theta$, and $\alpha > 0$, then $\theta \leq \alpha x$ and $\alpha x \neq \theta$.*

Proof (1)-(8) come from Lemma 2.5 in [21] and Lemma 2.3 in [22], and (8)-(13) directly follow from (1)-(8). □

Definition 2.5 [24] Let X be a real ordered Banach space, let $A : X \to X$ be a single-valued mapping, and let $M : X \to 2^X$ be a set-valued mapping.

(1) A single-valued mapping A is said to be a γ-order non-extended mapping if there exists a constant $\gamma > 0$ such that

$$\gamma(x \oplus y) \leq A(x) \oplus A(y) \quad \forall x, y \in X;$$

(2) A single-valued mapping A is said to be a strongly comparison mapping if A is a comparison mapping, and $A(x) \propto A(y)$ if and only if $x \propto y$ for any $x, y \in X$.

3 Characterizations of ordered (α, λ)-weak-*ANODD* set-valued mappings in ordered Banach spaces

Definition 3.1 Let X be a real ordered Banach space, let $A : X \to X$ be a single-valued mapping, and let $M : X \to 2^X$ be a set-valued mapping.

(1) A set-valued mapping M is said to be a weak-comparison mapping if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then there exist $v_x \in M(x)$ and $v_y \in M(y)$ such that $v_x \propto v_y$ $(\forall x, y \in X)$.

(2) A weak-comparison mapping M is said to be an α-week-non-ordinary difference mapping with respect to A if for each $x, y \in X$, there exist a constant $\alpha > 0$ and $v_x \in M(A(x))$ and $v_y \in M(A(y))$ such that

$$(v_x \oplus v_y) \oplus \alpha\bigl(A(x) \oplus A(y)\bigr) = \theta.$$

(3) A weak-comparison mapping M is said to be a λ-order different weak-comparison mapping with respect to B if there exists a constant $\lambda > 0$, $\forall x, y \in X$, and there exist $v_x \in M(B(x))$, $v_y \in M(B(y))$ such that

$$\lambda(v_x - v_y) \propto x - y.$$

(4) A weak-comparison mapping M is said to be an ordered (α_A, λ)-weak-*ANODD* mapping with respect to B if M is an α-weak-non-ordinary difference with respect to A and a λ-order different weak-comparison mapping with respect to B, and $(A + \lambda M)(X) = X$ for $\alpha, \lambda > 0$.

Remark 3.2 Let X be a real ordered Banach space, let $A : X \to X$ be a single-valued mapping, and let $M : X \to 2^X$ be a set-valued mapping, then the following properties hold obviously:

(i) A λ-order different comparison mapping must be a λ-order monotone mapping;
(ii) An ordered (α, λ)-*ANODD* mapping must be an ordered (α, λ)-*ANODM* mapping [24];
(iii) A comparison, an α-non-ordinary difference mapping, a λ-order different comparison mapping, or an ordered (α, λ)-*ANODD* mapping, a set-valued mapping M must be a weak-comparison mapping, an α-weak-non-ordinary difference mapping, a λ-order different weak-comparison mapping, or an ordered (α, λ)-weak-*ANODD* mapping, respectively.

Definition 3.3 [24] Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, let A be a γ-order non-extended mapping, and let M be an α-non-ordinary difference mapping with respect to A. The resolvent operator $J^A_{M,\lambda} : X \to X$ of M is defined by

$$J^A_{M,\lambda}(x) = (A + \lambda M)^{-1}(x) \quad \text{for all } x \in X,$$

where $\gamma, \alpha, \lambda > 0$ are three constants.

Lemma 3.4 *Let X be a real ordered Banach space. If A is a γ-order non-extended mapping, M is an α-weak-non-ordinary difference mapping with respect to A and $\alpha\lambda \neq 1$, then $M_\theta = \{\theta \oplus x | x \in M\}$ is an α-weak-non-ordinary difference mapping with respect to A and an inverse mapping $J^A_{M_\theta,\lambda} = (A + \lambda M_\theta)^{-1}$ of $(A + \lambda M_\theta)$ is a single-valued mapping $(\alpha, \lambda > 0)$, that is, the resolvent operator $J^A_{M_\theta,\lambda} : X \to X$ of M_θ exists.*

Proof Let X be a real ordered Banach space, let A be a γ-order non-extended mapping, and let M be an α-weak-non-ordinary difference mapping with respect to A, then for each $x, y \in X$, there exist a constant $\alpha > 0$ and $v_x \in M(A(x))$ and $v_y \in M(A(y))$ such that

$$(v_x \oplus v_y) \oplus \alpha\big(A(x) \oplus A(y)\big) = \theta.$$

By using Lemma 2.4(11), we consider

$$\begin{aligned}\theta &\le \big((v_x \oplus \theta) \oplus (v_y \oplus \theta)\big) \oplus \alpha\big(A(x) \oplus A(y)\big) \\ &\le \big(\theta \oplus (v_x \oplus v_y)\big) \oplus \alpha\big(A(x) \oplus A(y)\big) \\ &= \theta \oplus \big((v_x \oplus v_y) \oplus \alpha\big(A(x) \oplus A(y)\big)\big) \\ &= \theta \oplus \theta \\ &= \theta.\end{aligned}$$

Therefore, M_θ is surely an α-weak-non-ordinary difference mapping with respect to A.

Let $u \in X$, and let x and y be two elements in $(A + \lambda M_\theta)^{-1}(u)$. It follows that $u - A(x) \in \lambda M_\theta(x)$ and $u - A(y) \in \lambda M_\theta(y)$ from $x, y \in (A + \lambda M_\theta)^{-1}(u)$. And

$$\frac{1}{\lambda}\big(u - A(x)\big) \oplus \frac{1}{\lambda}\big(u - A(y)\big) = \frac{1}{\lambda}\big(A(x) \oplus A(y)\big).$$

Since M_θ is an α-weak-non-ordinary difference mapping with respect to A, and A is a γ-order non-extended mapping, it follows

$$\begin{aligned}\theta &= \left(\frac{1}{\lambda}\big(u - A(x)\big) \oplus \frac{1}{\lambda}\big(u - A(y)\big)\right) \oplus \alpha\big(A(x) \oplus A(y)\big) \\ &= \frac{1}{\lambda}\big(A(x) \oplus A(y)\big) \oplus \alpha\big(A(x) \oplus A(y)\big) \\ &= \left|\frac{1}{\lambda} - \alpha\right|\big(A(x) \oplus A(y)\big)\end{aligned}$$

and $A(x) \oplus A(y) = \theta$ from Lemma 2.4(5)(11)(13). It follows that $x = y$ from Definition 2.5(1). Thus $(A + \lambda M_\theta)^{-1}(u)$ is a single-valued mapping and the resolvent operator $J^A_{M_\theta,\lambda}$ of M_θ exists. The proof is completed. □

Lemma 3.5 *Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, let $\le$ be an ordered relation defined by the cone $\mathbf{P}$, let the operator $\oplus$ be an XOR operator. If the resolvent operator $J^A_{M,\lambda}$ of M exists, and M is a λ-order different weak-comparison mapping with respect to $J^A_{M,\lambda}$ and A is a strongly comparison mapping, then the resolvent operator $J^A_{M,\lambda}$ is a comparison mapping.*

Proof Let X be a real ordered Banach space, and let the resolvent operator $J^A_{M,\lambda}$ of M exist. If $M : X \to 2^X$ is a λ-order different weak-comparison mapping with respect to $J^A_{M,\lambda}$, and for any $x, y \in X$, $x \propto y$, it follows from Definition 3.1(1) that there exist $v_x = \frac{1}{\lambda}(x - A(J^A_{M,\lambda}(x))) \in M(J^A_{M,\lambda}(x))$ and $v_y = \frac{1}{\lambda}(y - A(J^A_{M,\lambda}(y))) \in M(J^A_{M,\lambda}(y))$ such that $\lambda(v_x - v_y) \propto x - y$.

Then we have

$$v_x - v_y = \frac{1}{\lambda}\big(x - y + A\big(J^A_{M,\lambda}(y)\big) - A\big(J^A_{M,\lambda}(x)\big)\big),$$

and

$$\lambda(v_x - v_y) - (x - y) = A\big(J^A_{M,\lambda}(y)\big) - A\big(J^A_{M,\lambda}(x)\big).$$

Therefore, $A(J^A_{M,\lambda}(y)) \propto A(J^A_{M,\lambda}(x))$ by using $\lambda(u_x - u_y) - (x - y) \in \mathbf{P}$ and Lemma 2.4(1). It follows that $J^A_{M,\lambda}(y) \propto J^A_{M,\lambda}(x)$ from strong comparability of A. The proof is completed. □

Lemma 3.6 *Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, and let $\leq$ be an ordered relation defined by the cone $\mathbf{P}$. Let A be a γ-order non-extended mapping, and M be an ordered (α_A, λ)-weak-ANODD mapping with respect to $J^A_{M,\lambda}$. If $\alpha_A > \frac{1}{\lambda} > 0$, then for the resolvent operator $J^A_{M,\lambda}$, the following relation holds*:

$$J^A_{M,\lambda}(x) \oplus J^A_{M,\lambda}(y) \leq \frac{1}{\gamma(\alpha_A\lambda - 1)}(x \oplus y). \tag{3.1}$$

Proof Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, and let $\leq$ be an ordered relation defined by the cone $\mathbf{P}$. Let A be a γ-order non-extended mapping and M be an α-weak-non-ordinary difference mapping with respect to A, it follows that $J^A_{M,\lambda}$ exists from Lemma 3.5.

For any $x, y \in X$, let $u_x = J^A_{M,\lambda}(x) \propto u_y = J^A_{M,\lambda}(y)$, then there exist $v_x = \frac{1}{\lambda}(x - A(u_x)) \in M(u_x)$ and $v_y = \frac{1}{\lambda}(y - A(u_y)) \in M(u_y)$ such that $v_x \propto v_y$ for $x \propto y$. Since M is an ordered (α_A, λ)-weak-*ANODM* mapping with respect to $J^A_{M,\lambda}$, the following relation holds by Lemma 2.4(7) and the condition $(v_x \oplus v_y) \oplus \alpha_A(A(u_x) \oplus A(u_y)) = \theta$:

$$\frac{1}{\lambda}\big((x \oplus y) + \big(A(u_x) \oplus A(u_y)\big)\big) \geq v_x \oplus v_y = \alpha_A\big(A(u_x) \oplus A(u_y)\big).$$

Since A is a γ-order non-extended mapping and $\alpha_A > \frac{1}{\lambda} > 0$, we have

$$(\lambda\alpha_A - 1)\big(A(u_x) \oplus A(u_y)\big) \leq (x \oplus y)$$

and

$$\gamma(u_x \oplus u_y) \leq A(u_x) \oplus A(u_y) \leq \frac{1}{(\alpha_A\lambda - 1)}(x \oplus y).$$

The proof is completed. □

Theorem 3.7 *Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, and let $\leq$ be an ordered relation defined by the cone $\mathbf{P}$. Let A be a γ-order non-extended mapping, and let M be an ordered (α_A, λ)-weak-ANODD mapping with respect to $J^A_{M,\lambda}$. If $\alpha_A > \frac{1}{\lambda} > 0$, then the resolvent operator $J^A_{M,\lambda}$ is continuous.*

Proof Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, let $\leq$ be an ordered relation defined by the cone $\mathbf{P}$. Let A be a γ-order non-extended mapping, and let $M : X \to 2^X$ be an ordered (α_A, λ)-weak-*ANODD* mapping with respect to $J^A_{M,\lambda}$. If $\alpha_A > \frac{1}{\lambda} > 0$, then

$$J^A_{M,\lambda}(x) \oplus J^A_{M,\lambda}(y) \leq \frac{1}{\gamma(\alpha_A\lambda - 1)}(x \oplus y)$$

holds.

Let the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ and $y \in X$, then we have

$$\theta \leq J^A_{M,\lambda}(x_n) \oplus J^A_{M,\lambda}(y) \leq \frac{1}{\gamma(\alpha_A\lambda - 1)}(x_n \oplus y),$$

and

$$\left\|J^A_{M,\lambda}(x_n) \oplus J^A_{M,\lambda}(y)\right\| \leq \frac{1}{\gamma(\alpha_A\lambda - 1)}\left\|(x_n \oplus y)\right\|.$$

Therefore, $\|J^A_{M,\lambda}(x_n) - J^A_{M,\lambda}(y)\| \leq \frac{1}{\gamma(\alpha_A\lambda-1)}\|(x_n - y)\|$ by [21]. If $x_n \to y$, then $J^A_{M,\lambda}(x_n) \to J^A_{M,\lambda}(y)$ obviously. The proof is completed. □

4 Approximate solution for GNMOQVI problem (1.1)

In this section, we show the algorithm of approximation sequences for finding a solution of general nonlinear mixed-order quasi-variational inclusions problem (1.1) involving $\oplus$ operator in an ordered Banach space, and we discuss the convergence and the relation between the first-valued x_0 and the solution of problem (1.1) in X, real Banach spaces.

Theorem 4.1 *Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, let $\leq$ be an ordered relation defined by the cone $\mathbf{P}$, and let the operator $\oplus$ be an XOR operator. Let $A, f : X \to X$ be two single-valued ordered compression mappings and $A \propto f, f \propto \theta$. If A is a γ-order non-extended strongly comparison mapping and M_θ is an α-weak-non-ordinary difference mapping with respect to A, then inclusion problem* (1.1) *has a solution $x^* \in X$ if and only if*

$$x^* = J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)\bigl(x^*\bigr) \quad \bigl(x^* \in X\bigr).$$

Proof This directly follows from (1.1), Lemma 2.4, and the definition of the resolvent operator $J^A_{M_\theta,\lambda}$ of M_θ. □

Theorem 4.2 *Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in the X, and let $\leq$ be an ordered relation defined by the cone $\mathbf{P}$. Let $A, f : X \to X$ be two single-valued β, ξ ordered compression mappings, respectively, let A be a γ non-extended strongly comparison mapping, $A \propto f, f \propto \theta$, let M be a α_A-weak-non-ordinary difference mapping with respect to A and M_θ be a λ-order different weak-comparison mapping with respect to $J^A_{M_\theta,\lambda}$ and $(A + \lambda M_\theta)(X) = X$ for $\alpha_A, \lambda > 0$. Then M_θ is an ordered (α_A, λ)-weak-ANODD mapping with respect to $J^A_{M_\theta,\lambda}$. And if $\alpha_A > \frac{1}{\lambda} > 0$ and*

$$\beta + \xi + \gamma < \gamma\alpha_A\lambda \tag{4.1}$$

(*where* $\alpha_A \geq \gamma > 0$, $\beta, \lambda, \xi > 0$, *and they are constants*), *then a sequence* $\{x_n\}$ *converges strongly to* x^* *solution of problem* (1.1), *which is generated by the following algorithm*:

For any given $x_0 \in X$, *let* $x_1 = J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x_0)$, *set*

$$x_{n+1} = J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x_n) \quad (n = 0, 1, 2, \ldots);$$

thus the following holds:

$$\|x^* - x_0\| \leq \left(1 + \frac{N(\beta + \xi)}{\alpha_A \gamma \lambda - (\beta + \gamma + \xi)}\right) \|J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x_0) - x_0\|. \tag{4.2}$$

Proof Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with normal constant N in X, let $\leq$ be an ordered relation defined by the cone $\mathbf{P}$. By using Lemma 3.4-Lemma 3.6 and Theorem 3.7, M_θ is an ordered (α_A, λ)-weak-*ANODD* mapping with respect to $J^A_{M_\theta,\lambda}$.

For any $x_0 \in X$, let $x_1 = J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x_0)$, let M_θ be an ordered (α, λ)-weak-*ANODD* mapping with respect to $J^A_{M_\theta,\lambda}$, $(A + \lambda M_\theta)(X) = X$, and the comparability of $J^A_{M_\theta,\lambda}$, we know that $x_1 \propto x_0$ by Lemma 2.2. Further, we can obtain a sequence $\{x_n\}$, and $x_{n+1} \propto x_n$ (where $n = 0, 1, 2, \cdot, \cdot, \cdot$). Using Lemma 2.4 and (2) in Lemma 3.6, we have

$$\begin{aligned}
\theta &\leq x_{n+1} \oplus x_n \leq J^A_{M_\oplus,\lambda}(A + \lambda w \oplus f)(x_n) \oplus J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x_{n-1}) \\
&\leq \frac{1}{\gamma(\alpha_A \lambda - 1)}\big((A + \lambda w \oplus f)(x_n) \oplus (A + \lambda w \oplus f)(x_{n-1})\big) \\
&\leq \frac{1}{\gamma(\alpha_A \lambda - 1)}\big(A(x_n) \oplus A(x_{n-1}) + \big(w \oplus f(x_n)\big) \oplus \big(w \oplus f(x_{n-1})\big)\big) \\
&\leq \frac{1}{\gamma(\alpha_A \lambda - 1)}\big(A(x_n) \oplus A(x_{n-1}) + \big(\theta \oplus f(x_n) \oplus f(x_{n-1})\big)\big) \\
&\leq \frac{1}{\gamma(\alpha_A \lambda - 1)}\big(A(x_n) \oplus A(x_{n-1}) + \big(f(x_n) \oplus f(x_{n-1})\big)\big) \\
&\leq \frac{1}{\gamma(\alpha_A \lambda - 1)}\big(\beta(x_n \oplus x_{n-1}) + \xi(x_n \oplus x_{n-1})\big) \\
&\leq \frac{1}{\gamma(\alpha_A \lambda - 1)}(\beta + \xi)(x_n \oplus x_{n-1}).
\end{aligned} \tag{4.3}$$

By Definition 2.1(ii), we obtain

$$\|x_n - x_{n-1}\| \leq \delta^n N \|x_1 - x_0\|, \tag{4.4}$$

where $\delta = \frac{1}{\gamma(\alpha_A \lambda - 1)}(\beta + \xi)$. Hence, for any $m > n > 0$, we have

$$\|x_m - x_n\| \leq \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \leq N\|x_1 - x_0\| \sum_{i=n}^{m-1} \delta^i.$$

It follows from condition (4.1) that $0 < \delta < 1$ and $\|x_m - x_n\| \to 0$, as $n \to \infty$, and so $\{x_n\}$ is a Cauchy sequence in the complete space X. Let $x_n \to x^*$ as $n \to \infty (x^* \in X)$. By the conditions, we can have

$$x^* = \lim_{n \to \infty} x_{n+1}$$

$$= \lim_{n\to\infty} J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x_n)$$
$$= J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x^*).$$

We know that x^* is a solution of inclusion problem (1.1). It is follows that $(J^A_{M_\theta,\lambda}(A + \lambda w \oplus f)(x_n)) \propto x^*$ $(n = 0, 1, 2, \ldots)$ from Lemma 2.3, and (4.1)

$$\begin{aligned} \|x^* - x_0\| &= \lim_{n\to\infty} \|x_n - x_0\| \\ &\le \lim_{n\to\infty} \sum_{i=1}^{n} \|x_{i+1} - x_i\| \le \lim_{n\to\infty} N \sum_{i=2}^{n} \delta^{n-1} \|x_1 - x_0\| + \|x_1 - x_0\| \\ &\le \left(\frac{1 + (N-1)\delta}{1-\delta}\right) \left\|J^A_{M,\lambda}(A + \lambda w \oplus f)(x_0) - x_0\right\| \\ &\le \left(1 + \frac{N(\beta + \xi)}{\alpha_A \gamma \lambda - (\beta + \gamma + \xi)}\right) \left\|J^A_{M,\lambda}(A + \lambda w \oplus f)(x_0) - x_0\right\| \end{aligned}$$

holds. This completes the proof. □

Remark 4.3 Though the method of solving the problem by the resolvent operator is the same as that in [31–37], or [47] for the nonlinear inclusion problem, but the character of an ordered (α_A, λ)-*ANODD* set-valued mapping is different from the one of an (A, η)-accretive mapping [31–37], or an (A, η)-maximal monotone mapping [47].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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