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Fixed and periodic points of generalized contractions in metric spaces

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Abstract
Wardowski (Fixed Point Theory Appl. 2012:94, 2012, doi:10.1186/1687-1812-2012-94) introduced a new type of contraction called F-contraction and proved a fixed point result in complete metric spaces, which in turn generalizes the Banach contraction principle. The aim of this paper is to introduce F-contractions with respect to a self-mapping on a metric space and to obtain common fixed point results. Examples are provided to support results and concepts presented herein. As an application of our results, periodic point results for the F-contractions in metric spaces are proved.
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1 Introduction and preliminaries

The Banach contraction principle [1] is a popular tool in solving existence problems in many branches of mathematics (see, *e.g.*, [2–4]). Extensions of this principle were obtained either by generalizing the domain of the mapping or by extending the contractive condition on the mappings [5–9]. Initially, existence of fixed points in ordered metric spaces was investigated and applied by Ran and Reurings [10]. Since then, a number of results have been proved in the framework of ordered metric spaces (see [11–18]). Contractive conditions involving a pair of mappings are further additions to the metric fixed point theory and its applications (for details, see [19–23]).

Recently, Wardowski [24] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle [1]. In this paper, we introduce an F-contraction with respect to a self-mapping on a metric space and obtain common fixed point results in an ordered metric space. In the last section, we give some results on periodic point properties of a mapping and a pair of mappings in a metric space. We begin with some basic known definitions and results which will be used in the sequel. Throughout this article, $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}$ denote the set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

Definition 1 Let f and g be self-mappings on a set X. If $fx = gx = w$ for some x in X, then x is called a coincidence point of f and g and w is called a coincidence point of f and g. Furthermore, if $fgx = gfx$ whenever x is a coincidence point of f and g, then f and g are called weakly compatible mappings [22].

Let $C(f,g) = \{x \in X : fx = gx\}$ ($F(f,g) = \{x \in X : x = fx = gx\}$) denote the set of all coincidence points (the set of all common fixed points) of self-mappings f and g.

Definition 2 ([25]) Let (X,d) be a metric space and $f,g : X \to X$. The mapping f is called a g-contraction if there exists $\alpha \in (0,1)$ such that

$$d(fx,fy) \leq \alpha d(gx,gy)$$

holds for all $x,y \in X$.

In 1976, Jungck [25] obtained the following useful generalization of the Banach contraction principle.

Theorem 1 *Let g be a continuous self-mapping on a complete metric space (X,d). Then g has a fixed point in X if and only if there exists a g-contraction mapping $f : X \to X$ such that f commutes with g and $g(X) \subseteq f(X)$.*

Let $\mathcal{F}$ be the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ that satisfy the following conditions:

(C1) F is strictly increasing, that is, for all $\alpha,\beta \in \mathbb{R}_+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$.

(C2) For every sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive real numbers, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent.

(C3) There exists $k \in (0,1)$ such that

$$\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0.$$

Definition 3 ([24]) Let (X,d) be a metric space and $F \in \mathcal{F}$. A mapping $f : X \to X$ is said to be an F-contraction on X if there exists $\tau > 0$ such that

$$d(fx,fy) > 0 \quad \text{implies that} \quad \tau + F\big(d(fx,fy)\big) \leq F\big(d(x,y)\big) \tag{1}$$

for all $x,y \in X$.

Note that every F-contraction is continuous (see [24]). We extend the above definition to two mappings.

Definition 4 Let (X,d) be a metric space, $F \in \mathcal{F}$ and $f,g : X \to X$. The mapping f is said to be an F-contraction with respect to g on X if there exists $\tau > 0$ such that

$$\tau + F\big(d(fx,fy)\big) \leq F\big(d(gx,gy)\big) \tag{2}$$

for all $x,y \in X$ satisfying $\min\{d(fx,fy), d(gx,gy)\} > 0$.

By different choices of mappings F in (1) and (2), one obtains a variety of contractions [24].

Example 1 Let $F_1 : \mathbb{R}_+ \to \mathbb{R}$ be given by $F_1(\alpha) = \ln(\alpha)$. It is clear that $F \in \digamma$. Suppose that $f : X \to X$ is an F-contraction with respect to a self-mapping g on X. From (2) we have

$$\tau + \ln\bigl(d(fx, fy)\bigr) \le \ln\bigl(d(gx, gy)\bigr),$$

which implies that

$$d(fx, fy) \le e^{-\tau} d(gx, gy).$$

Therefore an F_1-contraction map f with respect to g reduces to a g-contraction mapping.

Now we give an example of an F-contraction with respect to a self-mapping g on X which is not a g-contraction on X.

Example 2 Consider the following sequence of partial sums $\{S_n\}_{n\in\mathbb{N}}$ [24, Example 2.5]:

$$S_1 = 1,$$
$$S_2 = 1 + 2,$$
$$S_3 = 1 + 2 + 3,$$
$$\dots$$
$$S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}.$$

Let $X = \{S_n : n \in \mathbb{N}\}$ and d be the usual metric on X. Let $f : X \to X$ and $g : X \to X$ be defined as

$$fS_n = \begin{cases} S_{n-1}, & \text{if } n > 1, \\ S_1, & \text{if } n = 1, \end{cases} \qquad gS_n = \begin{cases} S_{n+1}, & \text{if } n > 1, \\ S_1, & \text{if } n = 1. \end{cases}$$

Let $F_1 : \mathbb{R}_+ \to \mathbb{R}$ be given by $F_1(\alpha) = \ln(\alpha)$. As

$$\lim_{n\to\infty} \frac{d(fS_n, fS_1)}{d(gS_n, gS_1)} = \lim_{n\to\infty} \frac{S_{n-1} - S_1}{S_{n+1} - S_1} = 1,$$

so f is not a g-contraction. If we take $F_2(\alpha) = \ln(\alpha) + \alpha$, then $F_2 \in \digamma$ and f is an F_2-contraction with respect to a mapping g (taking $\tau = 2$). Indeed, the following holds:

$$\frac{d(fS_n, fS_1)}{d(gS_n, gS_1)} e^{d(fS_n, fS_1) - d(gS_n, gS_1)} = \frac{S_{n-1} - S_1}{S_{n+1} - S_1} e^{S_{n-1} - S_1 - S_{n+1} + S_1} = \frac{n^2 - n - 2}{n^2 + 3n} e^{-4n-2} \le e^{-2}$$

for all $n > 1$. For all $m, n \in \mathbb{N}$ with $m > n > 1$, we have

$$\begin{aligned} &\frac{d(fS_m, fS_n)}{d(gS_m, gS_n)} e^{d(fS_m, fS_n) - d(gS_m, gS_n)} \\ &\quad = \frac{S_{m-1} - S_{n-1}}{S_{m+1} - S_{n+1}} e^{S_{m-1} - S_{n-1} - S_{m+1} + S_{n+1}} \\ &\quad = \frac{m^2 + m - n^2 - n}{m^2 + 3m - n^2 - 3n} e^{-2(m-n)} \le e^{-2}. \end{aligned}$$

Definition 5 ([26], Dominance condition) Let $(X,\preceq)$ be a partially ordered set. A self-mapping f on X is said to be (i) a dominated map if $fx \preceq x$ for each x in X, (ii) a dominating map if $x \preceq fx$ for each x in X.

Example 3 Let $X = [0,1]$ be endowed with the usual ordering and $f,g : X \to X$ defined by $gx = x^n$ for some $n \in \mathbb{N}$ and $fx = kx$ for some real number $k \geq 1$. Note that

$$gx = x^n \leq x \quad \text{and} \quad x \leq kx = fx$$

for all x in X. Thus g is dominated and f is a dominating map.

Definition 6 Let $(X,\preceq)$ be a partially ordered set. Two mappings $f,g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all x in X (see [27]).

Definition 7 Let X be a nonempty set. Then $(X,d,\preceq)$ is called an ordered metric space if (X,d) is a metric space and $(X,\preceq)$ is a partially ordered set.

Definition 8 Let $(X,\preceq)$ be a partial ordered set, then x, y in X are called comparable elements if either $x \preceq y$ or $y \preceq x$ holds true. Moreover, we define $\Delta \subseteq X \times X$ by

$$\Delta = \{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

Definition 9 An ordered metric space $(X,d,\preceq)$ is said to have the sequential limit comparison property if for every non-decreasing sequence (non-increasing sequence) $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \to x$ implies that $x_n \preceq x$ ($x \preceq x_n$).

2 Common fixed point results in ordered metric spaces

We present the following theorem as a generalization of results in [25] and [24, Theorem 2.1].

Theorem 2 *Let $(X,\preceq)$ be a partially ordered set such that there exists a metric d on X, and let $f : X \to X$ be an F-contraction with respect to $g : X \to X$ on Δ with $f(X) \subseteq g(X)$. Assume that f is dominating and g is dominated. Then*

(a) *f and g have a coincidence point in X provided that $g(X)$ is complete and has the sequential limit comparison property.*
(b) *$C(f,g)$ is well ordered if and only if $C(f,g)$ is a singleton.*
(c) *f and g have a unique common fixed point if f and g are weakly compatible and $C(f,g)$ is well ordered.*

Proof (a) Let x_0 be an arbitrary point of X. Since the range of g contains the range of f, there exists a point x_1 in X such that $f(x_0) = g(x_1)$. As f is dominating and g is dominated, so we have

$$x_0 \preceq fx_0 = gx_1 \preceq x_1.$$

Hence $(x_0,x_1) \in \Delta$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that

$$x_n \preceq fx_n = gx_{n+1} \preceq x_{n+1}.$$

So, we obtain $(x_n, x_{n+1}) \in \Delta$ for every $n \in \mathbb{N} \cup \{0\}$. For the sake of simplicity, take

$$\gamma_n = d(gx_n, gx_{n+1}) \tag{3}$$

for all $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $x_{n_0+1} = x_{n_0}$, then $fx_{n_0} = gx_{n_0+1}$ implies that $fx_{n_0+1} = gx_{n_0+1}$, that is, $x_{n_0+1} \in C(f,g)$. Now we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. As f is an F-contraction with respect to g on Δ, so we obtain

$$\begin{aligned} F(\gamma_n) &= F\big(d(gx_n, gx_{n+1})\big) = F\big(d(fx_{n-1}, fx_n)\big) \\ &\leq F\big(d(gx_{n-1}, gx_n)\big) - \tau \\ &= F\big(d(fx_{n-2}, fx_{n-1})\big) - \tau \\ &\leq F\big(d(gx_{n-2}, gx_{n-1})\big) - 2\tau \leq \cdots \\ &\leq F\big(d(gx_1, gx_2)\big) - (n-1)\tau = F(\gamma_1) - (n-1)\tau. \end{aligned}$$

That is,

$$F(\gamma_n) \leq F(\gamma_1) - (n-1)\tau.$$

On taking limit as $n \to \infty$, we obtain $\lim_{n\to\infty} F(\gamma_n) = -\infty$. Hence $\lim_{n\to\infty} \gamma_n = 0$ by (C2). Now, by (C3), there exists $k \in (0,1)$ such that $\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0$. Note that

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_1) \leq \gamma_n^k \big(F(\gamma_1) - (n-1)\tau\big) - \gamma_n^k F(\gamma_1) = -\gamma_n^k (n-1)\tau \leq 0. \tag{4}$$

Taking limit as $n \to \infty$ in (4), we have $\lim_{n\to\infty} (n-1)\gamma_n^k = 0$. Consequently, $\lim_{n\to\infty} n\gamma_n^k = 0$. Thus there exists n_1 in $\mathbb{N}$ such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$, that is, $\gamma_n \leq 1/n^{1/k}$ for all $n \geq n_1$. Now, for integers $m > n \geq 1$, we obtain

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &< \sum_{i=n}^{\infty} \gamma_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty. \end{aligned}$$

This shows that $\{gx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, so there exists q in $g(X)$ such that $\lim_{n\to\infty} gx_n = q$. Let $p \in X$ be such that $g(p) = q$. The sequential limit comparison property implies that $gx_{n+1} \preceq q$. As $x_n \preceq fx_n = gx_{n+1} \preceq q = g(p) \preceq p$ so $(x_n, p) \in \Delta$. Hence from (2) we have

$$F\big(d(gx_n, fp)\big) = F\big(d(fx_{n-1}, fp)\big) \leq F\big(d(gx_{n-1}, gp)\big) - \tau.$$

Since $\lim_{n\to\infty} d(gx_{n-1}, gp) = 0$, therefore by (C2) we have $\lim_{n\to\infty} F(d(gx_{n-1}, gp)) = -\infty$. Hence $\lim_{n\to\infty} F(d(gx_n, fp)) = -\infty$ implies that $\lim_{n\to\infty} d(gx_n, fp) = 0$. That is, $\lim_{n\to\infty} gx_n = fp$. Uniqueness of limit implies $fp = gp$, that is, $p \in C(f,g)$.

(b) Now suppose that $C(f,g)$ is well ordered. We prove that $C(f,g)$ is a singleton. Assume on the contrary that there exists another point w in X such that $fw = gw$ with $w \neq p$. Since $C(f,g)$ is well ordered, so $(w,p) \in \Delta$. Now from (2) we have

$$\tau \leq F\big(d(gw, gp)\big) - F\big(d(fw, fp)\big) = 0,$$

a contradiction. Therefore $w = p$. Hence f and g have a unique coincidence point p in X. The converse follows immediately.

(c) Now if f and g are weakly compatible mappings, then we have $fq = fgp = gfp = gq$, that is, q is the coincidence point of f and g. But q is the only point of coincidence of f and g, so $fq = gq = q$. Hence q is the unique common fixed point of f and g. □

Example 4 Let $X = [0,5]$ be endowed with usual metric and usual order. Define mappings $f,g : X \to X$ by

$$gx = \begin{cases} 0 & \text{if } x \in [0,3), \\ 3 & \text{if } x \in [3,5), \\ 5 & \text{if } x = 5, \end{cases} \qquad fx = \begin{cases} 3 & \text{if } x \in [0,3), \\ 5 & \text{if } x \in [3,5]. \end{cases}$$

Clearly, g is dominated and f is dominating. Define $F : \mathbb{R}_+ \to \mathbb{R}$ as $F(x) = \ln(x)$. If $x \in [0,3)$ and $y \in [3,5)$, then

$$\begin{aligned} F\big(d(fx,fy)\big) &= F\big(d(3,5)\big) = F(2) = \ln(2) \approx 0.693 \\ &< F\big(d(gx,gy)\big) = F\big(d(0,3)\big) \\ &= F(3) = \ln(3) \approx 1.098. \end{aligned}$$

Hence, for $\tau \in (0,0.40]$, inequality (2) is satisfied. Similarly, for $x \in [0,3)$ and $y = 5$, we have

$$\begin{aligned} F\big(d(fx,fy)\big) &= F\big(d(3,5)\big) = F(2) = \ln(2) \approx 0.693 \\ &< F\big(d(gx,gy)\big) = F\big(d(0,5)\big) \\ &= F(5) = \ln(5) \approx 1.6094. \end{aligned}$$

Hence, for $\tau \in (0,0.9164]$, inequality (2) is satisfied. We can take a $\tau \in (0,0.40]$ so that

$$\tau + F\big(d(fx,fy)\big) \le F\big(d(gx,gy)\big)$$

is satisfied for all $x,y \in [0,5]$, whenever $\min\{d(fx,fy),d(gx,gy)\} > 0$. Hence f is an F-contraction with respect to g on $[0,5]$. Hence all the conditions of Theorem 2 are satisfied. Moreover, $x = 5$ is the coincidence point of f and g. Also note that f and g are weakly compatible and $x = 5$ is the common fixed point of g and f as well.

Now we give a common fixed point result without imposing any type of commutativity condition for self-mappings f and g on X. Moreover, we relax the dominance conditions on f and g as well.

Theorem 3 *Let $(X,\preceq)$ be a partially ordered set such that there exists a complete metric d on X. If self-mappings f and g on X are weakly increasing and for some $\tau > 0$ satisfy*

$$\tau + F\big(d(fx,gy)\big) \le F\big(d(x,y)\big) \tag{5}$$

for all $(x,y) \in \Delta$ *such that* $\min\{d(fx,gy), d(x,y)\} > 0$, *then* $F(f,g) \neq \emptyset$, *provided that* X *has the sequential limit comparison property. Further,* f *and* g *have a unique common fixed point if and only if* $F(f,g)$ *is well ordered.*

Proof Let x_0 be an arbitrary point of X. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X as follows: $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. Since f and g are weakly increasing, we have $x_{2n+1} = fx_{2n} \preceq gfx_{2n} = gx_{2n+1} = x_{2n+2}$ and $x_{2n+2} = gx_{2n+1} \preceq fgx_{2n+1} = fx_{2n+2} = x_{2n+3}$. Hence $(x_{2n+1}, x_{2n+2}) \in \Delta$ and $(x_{2n+2}, x_{2n+3}) \in \Delta$ for every $n \in \mathbb{N} \cup \{0\}$. Now define

$$\gamma_{2n} = d(x_{2n+1}, x_{2n+2}) \tag{6}$$

for all $n \in \mathbb{N} \cup \{0\}$. Using (5) the following holds for every $n \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned} F(\gamma_{2n}) &= F\bigl(d(x_{2n+1}, x_{2n+2})\bigr) = F\bigl(d(fx_{2n}, gx_{2n+1})\bigr) \\ &\leq F\bigl(d(x_{2n}, x_{2n+1})\bigr) - \tau = F(\gamma_{2n-1}) - \tau. \end{aligned}$$

Similarly,

$$\begin{aligned} F(\gamma_{2n+1}) &= F\bigl(d(x_{2n+3}, x_{2n+2})\bigr) = F\bigl(d(fx_{2n+2}, gx_{2n+1})\bigr) \\ &\leq F\bigl(d(x_{2n+1}, x_{2n+2})\bigr) - \tau = F(\gamma_{2n}) - \tau. \end{aligned}$$

Therefore, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} F(\gamma_n) &\leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \\ &\leq F\bigl(d(x_1, x_2)\bigr) - n\tau = F(\gamma_0) - n\tau. \end{aligned}$$

Thus

$$F(\gamma_n) \leq F(\gamma_0) - n\tau. \tag{7}$$

Taking limit as $n \to \infty$ in (7), we get

$$\lim_{n\to\infty} F(\gamma_n) = -\infty.$$

By (C2) and (C3) we get $\lim_{n\to\infty} \gamma_n = 0$ and $k \in (0,1)$ such that $\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0$. Note that

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq \gamma_n^k \bigl(F(\gamma_0) - n\tau\bigr) - \gamma_n^k F(\gamma_0) = -\gamma_n^k n\tau \leq 0. \tag{8}$$

By taking limit as $n \to \infty$ in (8), we get $\lim_{n\to\infty} n\gamma_n^k = 0$. This implies that there exists n_1 such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. Consequently, we obtain $\gamma_n \leq 1/n^{1/k}$ for all $n \geq n_1$. Now, for integers $m > n \geq 1$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) < \sum_{i=n}^{\infty} \gamma_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty.$$

This shows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X, so there exists p in X such that $\lim_{n\to\infty} x_n = p$. As X has the sequential limit comparison property, so $(x_n, p), (x_{2n}, p), (x_{2n+1}, p) \in \Delta$. Therefore

$$\lim_{n\to\infty} F\big(d(x_{2n+1}, gp)\big) = \lim_{n\to\infty} F\big(d(fx_{2n}, gp)\big) \leq F\big(d(x_{2n}, p)\big) - \tau.$$

Since $\lim_{n\to\infty} d(x_{2n}, p) = 0$, by (C2) we have $\lim_{n\to\infty} F(d(x_{2n}, p)) = -\infty$. This implies $\lim_{n\to\infty} F(d(x_{2n+1}, gp)) = -\infty$, which further implies that $\lim_{n\to\infty} d(x_{2n+1}, gp) = 0$. Hence $d(p, gp) = 0$ and $p = gp$. Similarly, we obtain $p = fp$. This shows that p is a common fixed point of g and f. Now suppose that $F(f, g)$ is well ordered. We prove that $F(f, g)$ is a singleton. Assume on the contrary that there exists another point q in X such that $q = fq = gq$ with $q \neq p$. Obviously, $(q, p) \in \Delta$. So, from (5) we have $\tau \leq F(d(q, p)) - F(d(fq, gp)) = 0$, a contradiction. Therefore $q = p$. Hence g and f have a unique common fixed point p in X. The converse follows immediately. □

3 Periodic point results in metric spaces

If x is a fixed point of the self-mapping f, then x is a fixed point of f^n for every $n \in \mathbb{N}$, but the converse is not true. In the sequel, we denote by $F(f)$ the set of all fixed points of f.

Example 5 Let $f : [0,1] \to [0,1]$ be given by

$$f(x) = 1 - x.$$

Then f has a unique fixed point $x = 1/2$. Note that $f^n x = x$ holds for every even natural number n and x in $[0,1]$. On the other hand, define a mapping $g : [0,\pi] \to [0,\pi]$ as

$$g(x) = \cos x.$$

Then g has the same fixed point as g^n for every n.

Definition 10 The self-mapping f is said to have the property P if $F(f^n) = F(f)$ for every $n \in \mathbb{N}$. A pair (f, g) of self-mappings is said to have the property Q if $F(f) \cap F(g) = F(f^n) \cap F(g^n)$.

For further details on these properties, we refer to [20, 28].

Let (X, d) be a metric space and $f : X \to X$ be a self-mapping. The set $O(x) = \{x, fx, \ldots, f^n x, \ldots\}$ is called the orbit of x [29]. A mapping f is called orbitally continuous at p if $\lim_{n\to\infty} f^n x = p$ implies that $\lim_{n\to\infty} f^{n+1} x = fp$. A mapping f is orbitally continuous on X if f is orbitally continuous for all $x \in X$.

In this section we prove some periodic point results for self-mappings on complete metric spaces.

Theorem 4 *Let X be a nonempty set such that there exists a complete metric d on X. Suppose that $f : X \to X$ satisfies*

$$\tau + F\big(d(fx, f^2 x)\big) \leq F\big(d(x, fx)\big) \tag{9}$$

for some $\tau > 0$ *and for all* x *in* X *such that* $d(fx, f^2x) > 0$. *Then* f *has the property P provided that* f *is orbitally continuous on* X.

Proof First we show that $F(f) \neq \emptyset$. Let $x_0 \in X$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, such that $x_{n+1} = fx_n$, for all $n \in \mathbb{N} \cup \{0\}$. Denote $\gamma_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $x_{n_0+1} = x_{n_0}$, then $fx_{n_0} = x_{n_0}$ and the proof is finished. Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Using (9), we obtain

$$\begin{aligned} F(\gamma_n) &= F\bigl(d(x_n, x_{n+1})\bigr) = F\bigl(d\bigl(fx_{n-1}, f^2x_{n-1}\bigr)\bigr) \\ &\leq F\bigl(d(x_{n-1}, fx_{n-1})\bigr) - \tau = F\bigl(d\bigl(fx_{n-2}, f^2x_{n-2}\bigr)\bigr) - \tau \\ &\leq F\bigl(d(x_{n-2}, fx_{n-2})\bigr) - 2\tau \leq \cdots \\ &\leq F\bigl(d(x_1, x_2)\bigr) - (n-1)\tau \\ &= F\bigl(d\bigl(fx_0, f^2x_1\bigr)\bigr) - (n-1)\tau \leq F\bigl(d(x_0, x_1)\bigr) - n\tau \\ &= F(\gamma_0) - n\tau \end{aligned}$$

for every $n \in \mathbb{N} \cup \{0\}$. By taking limit as $n \to \infty$ in the above inequality, we obtain that $\lim_{n\to\infty} F(\gamma_n) = -\infty$, which together with (C2) gives $\lim_{n\to\infty} \gamma_n = 0$. From (C3), there exists $k \in (0,1)$ such that $\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0$. Note that

$$\begin{aligned} \gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) &\leq \gamma_n^k \bigl(F(\gamma_0) - n\tau\bigr) - \gamma_n^k F(\gamma_0) \\ &= -\gamma_n^k n\tau \leq 0. \end{aligned}$$

On taking limit as $n \to \infty$, we get $\lim_{n\to\infty} n\gamma_n^k = 0$. Hence there exists n_1 such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. Consequently $\gamma_n \leq 1/n^{1/k}$ for all $n \geq n_1$. Now, for integers $m > n \geq 1$ such that

$$\begin{aligned} d\bigl(f^n x_0, f^m x_0\bigr) &= d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &< \sum_{i=n}^{\infty} \gamma_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty. \end{aligned}$$

This shows that $\{f^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $\{f^n x_0 : n \in \mathbb{N}\} \subseteq O(x_0) \subseteq X$ and X is complete, which implies that there exists x in X such that $\lim_{n\to\infty} f^n x_0 = x$. Since f is orbitally continuous at x, so $x = \lim_{n\to\infty} f^n x_0 = f(\lim_{n\to\infty} f^{n-1} x_0) = fx$. Hence f has a fixed point and $F(f^n) = F(f)$ is true for $n = 1$. Now assume $n > 1$. Suppose on the contrary that $u \in F(f^n)$ but $u \notin F(f)$, then $d(u, fu) = \alpha > 0$. Now consider

$$\begin{aligned} F(\alpha) &= F\bigl(d(u, fu)\bigr) = F\bigl(d\bigl(f\bigl(f^{n-1}u\bigr), f^2\bigl(f^{n-1}u\bigr)\bigr)\bigr) \\ &\leq F\bigl(d\bigl(f^{n-1}u, f^n u\bigr)\bigr) - \tau \\ &\leq F\bigl(d\bigl(f^{n-2}u, f^{n-1}u\bigr)\bigr) - 2\tau \leq \cdots \\ &\leq F\bigl(d(u, fu)\bigr) - n\tau. \end{aligned}$$

Thus $F(\alpha) \leq \lim_{n\to\infty} F(d(u, fu)) - n\tau = -\infty$. Hence $F(\alpha) = -\infty$. By (C2) $\alpha = 0$, a contradiction. So $u \in F(f)$. □

Theorem 5 *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X and f, g self-mappings on X. Further assume that f, g are weakly increasing and satisfy*

$$\tau + F\bigl(d(fx, gy)\bigr) \leq F\bigl(d(x, y)\bigr)$$

for some $\tau > 0$, for all x, y in X such that $\min\{d(fx, gy), d(x, y)\} > 0$. Then f and g have the property Q provided that X has the sequential limit comparison property.

Proof By Theorem 3, f and g have a common fixed point. Suppose on the contrary that

$$u \in F\bigl(f^n\bigr) \cap F\bigl(g^n\bigr)$$

but $u \notin F(f) \cap F(g)$, then there are three possibilities (a) $u \in F(f) \setminus F(g)$, (b) $u \in F(g) \setminus F(f)$, (c) $u \notin F(f)$ and $u \notin F(g)$. Without loss of generality, let $u \notin F(g)$, that is, $d(u, gu) = \alpha > 0$, so we get

$$\begin{aligned} F(\alpha) &= F\bigl(d(u, gu)\bigr) = F\bigl(d\bigl(f\bigl(f^{n-1}u\bigr), g\bigl(g^n u\bigr)\bigr)\bigr) \\ &\leq F\bigl(d\bigl(f^{n-1}u, g^n u\bigr)\bigr) - \tau \\ &\leq F\bigl(d\bigl(f^{n-2}u, g^{n-1}u\bigr)\bigr) - 2\tau \leq \cdots \\ &\leq F\bigl(d(u, gu)\bigr) - n\tau. \end{aligned}$$

As $\lim_{n\to\infty} F(d(u, gu)) - n\tau = -\infty$, so we have $F(\alpha) = -\infty$. By (C2) $\alpha = 0$, a contradiction. Hence $u \in F(g) \cap F(f)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors contributed equally in writing this article. They read and approved the final manuscript.

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