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Strong convergence theorems for fixed point problems of a nonexpansive semigroup in a Banach space

Xin Wang, Changsong Hu* and Jinlin Guan

*Correspondence:
huchang1004@yahoo.com.cn
School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China

Abstract
In this paper, we study the implicit and explicit viscosity iteration schemes for a nonexpansive semigroup in a reflexive, strictly convex and uniformly smooth Banach space which satisfies Opial's condition. Our results improve and generalize the corresponding results given by Yao *et al.* (Fixed Point Theory Appl. 2013, doi:10.1186/1687-1812-2013-31) and many others.
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1 Introduction

Let E be a real Banach space, and let K be a nonempty, closed and convex subset of E. A mapping $T : K \to K$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

One parameter family $S := \{T(s) : 0 \le s < \infty\}$ is said to be a nonexpansive semigroup from K into K if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in K$;
(2) $T(s + t) = T(s)T(t)$ for all $s, t \ge 0$;
(3) $\|T(t)x - T(t)y\| \le \|x - y\|$, $\forall x, y \in K$ and $t \ge 0$;
(4) for each $x \in K$, the mapping $T(\cdot)x$ from $[0, \infty)$ into K is continuous.

Let $F(S)$ denote the common fixed point set of the semigroup S, *i.e.*, $F(S) := \{x \in K : T(s)x = x, \forall s > 0\}$. It is known that $F(S)$ is closed and convex.

A continuous operator of the semigroup S is said to be uniformly asymptotically regular (u.a.r.) on K if for all $h \ge 0$ and any bounded subset C of K, $\lim_{s\to\infty} \sup_{x\in C} \|T(h)T(s)x - T(s)x\| = 0$ (see [1]).

Approximation of fixed points of nonexpansive mappings by a sequence of finite means has been considered by many authors (see [2–6]). In 2013, Yao *et al.* [7] introduced two new algorithms for finding a common fixed point of a nonexpansive semigroup $\{T(s)\}_{s\ge0}$ in Hilbert spaces and proved that both approaches converge strongly to a common fixed point of $\{T(s)\}_{s\ge0}$.

Theorem 1.1 [7] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S = \{T(s)\}_{s\geq 0} : C \to C$ be a nonexpansive semigroup with* $\operatorname{Fix}(S) \neq \emptyset$. *Let $\{\gamma_t\}_{0<t<1}$ and $\{\lambda_t\}_{0<t<1}$ be two continuous nets of positive real numbers such that $\gamma_t \in (0,1)$, $\lim_{t\to 0}\gamma_t = 1$ and $\lim_{t\to 0}\lambda_t = +\infty$. Let $\{x_t\}$ be the net defined in the following implicit manner*:

$$x_t = P_C\left[t(\gamma_t x_t) + (1-t)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\,ds\right], \quad \forall t \in (0,1). \tag{1.2}$$

Then, as $t \to 0^+$, the net $\{x_t\}$ strongly converges to $x^* \in$ $\operatorname{Fix}(s)$.

Theorem 1.2 [7] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S = \{T(s)\}_{s\geq 0} : C \to C$ be a nonexpansive semigroup with* $\operatorname{Fix}(S) \neq \emptyset$. *Let $\{x_n\}$ be the sequence generated iteratively by the following explicit algorithm*:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\left[\alpha_n(\gamma_n x_n) + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n\,ds\right], \quad \forall n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0,1]$ *and $\{\lambda_n\}$ is a sequence of positive real numbers. Suppose that the following conditions are satisfied*:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\gamma_n = 1$;
(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;
(iii) $\lim_{n\to\infty}\lambda_n = \infty$ *and* $\lim_{n\to\infty}\frac{\lambda_{n-1}}{\lambda_n} = 1$.

Then the sequence $\{x_n\}$ generated by (1.3) *strongly converges to a point $x^* \in$* $\operatorname{Fix}(s)$.

In this paper, we study the convergence of the following iterative schemes in a reflexive, strictly convex and uniformly smooth Banach space which satisfies Opial's condition:

$$x_t = Q_K\big[t(\gamma_t x_t) + (1-t)T(s_t)x_t\big], \quad \forall t \in (0,1),$$
$$x_{n+1} = (1-\beta_n)x_n + \beta_n Q_K\big[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n\big], \quad \forall n \geq 0.$$

Our work improves and generalizes many others. In particular, our results extend the main results of Yao *et al.* [7].

2 Preliminaries

Let E be a real Banach space and E^* be the dual space of E. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$(x) = \big\{f \in E^* : \langle x,f\rangle = \|x\|^2 = \|f\|^2\big\}. \tag{2.1}$$

By the Hahn-Banach theorem, $J(x)$ is nonempty.

Let $\dim E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x-y}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}. \tag{2.2}$$

E is uniformly convex if $\forall\epsilon \in (0,2]$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x,y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1-\delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$, $\forall \epsilon \in (0,2]$. E is strictly convex if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall \lambda \in (0,1)$.

Let $S(E) = \{x \in E : \|x\| = 1\}$. The space E is said to be smooth if

$$\lim_{t\to 0}\bigl(\|x+ty\| - \|x\|\bigr)/t \tag{2.3}$$

exists for all $x, y \in S(E)$. The norm of E is said to be Fréchet differentiable if for all $x \in S(E)$, the limit (2.3) exists uniformly for all $y \in S(E)$. E is said to have a uniformly Gâteaux differentiable norm if for all $y \in S(E)$, the limit (2.3) is attained uniformly for all $x \in S(E)$. The norm of E is said to be uniformly Fréchet differentiable (or uniformly smooth) if the limit (2.3) is attained uniformly for $x, y \in S(E) \times S(E)$.

It is well known that if E is smooth, then J is single-valued, which is denoted by j. And if E has a uniformly Gâteaux differentiable norm, then J is norm-to-weak* uniformly continuous on each bounded subset of E. The duality mapping J is said to be weakly sequentially continuous if J is single-valued and for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J(x_n) \rightharpoonup^* J(x)$. Gossez and Lami Dozo [8] proved that a space with a weakly continuous duality mappings satisfies Opial's condition. Conversely, if a space satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then it has a weakly continuous duality mapping.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, the mapping $Q : C \to D$ is said to be sunny if

$$Q\bigl(Qx + t(x - Qx)\bigr) = Qx,$$

where $Qx + t(x - Qx) \in C$ for all $x \in C$ and $t \geq 0$.

A mapping $Q : C \to D$ is called a retraction if $Qx = x$ for all $x \in D$.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C into D (see [9, 10]). It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction is coincident with the metric projection from E onto C.

Proposition 2.1 [11] *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C. Let $Q : C \to D$ be a retraction, and let J be the normalized duality mapping on E. Then the following are equivalent*:

(1) *Q is sunny and nonexpansive.*
(2) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$, $\forall x, y \in C$.
(3) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, $\forall x \in C$, $y \in D$.

Proposition 2.2 [12] *Let C be a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space E, and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(S)$ is a sunny nonexpansive retraction of C.*

Lemma 2.3 [13] *Let K be a nonempty closed convex subset of a reflexive Banach space E which satisfies Opial's condition, and suppose that $T : K \to E$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.4 [14] *Let $\{x_n\}$, $\{y_n\}$ be two bounded sequences in a Banach space E and $\beta_n \in (0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.5 [15] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \le (1-\rho_n)a_n + \rho_n\sigma_n, \quad n \ge 0,$$

where $\{\rho_n\}$ and $\{\sigma_n\}$ are sequences of real numbers such that

(i) $0 < \rho_n < 1$;
(ii) $\sum_{n=1}^{\infty} \rho_n = \infty$;
(iii) $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\rho_n\sigma_n|$ *is convergent*.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main result

Theorem 3.1 *Let E be a reflexive, strictly convex and uniformly smooth Banach space which satisfies Opial's condition, and let K be a nonempty closed convex subset of E. Let $S = \{T(s) : s \ge 0\} : K \to K$ be a uniformly asymptotically regular nonexpansive semigroup such that $F(S) \neq \emptyset$. Let $\{\gamma_t\}_{0<t<1}$ and $\{s_t\}_{0<t<1}$ be two continuous nets of positive real numbers such that $\gamma_t \in (0,1)$, $\lim_{t\to 0} \gamma_t = 1$ and $\lim_{t\to 0} s_t = +\infty$. Let $\{x_t\}$ be the net defined by*

$$x_t = Q_K\big[t(\gamma_t x_t) + (1-t)T(s_t)x_t\big], \quad \forall t \in (0,1). \tag{3.1}$$

Then, as $t \to 0^+$, the net $\{x_t\}$ converges strongly to a point $x^* \in F(S)$.

Proof Consider a mapping W on K defined by

$$Wx := Q_K\big[t(\gamma_t x) + (1-t)T(s_t)x\big], \quad \forall t \in (0,1).$$

$\forall x, y \in K$, we have

$$\begin{aligned}
\|Wx - Wy\| &\le \big\|t\gamma_t(x-y) + (1-t)\big(T(s_t)x - T(s_t)y\big)\big\| \\
&\le t\gamma_t\|x-y\| + (1-t)\|x-y\| \\
&= \big[1-(1-\gamma_t)t\big]\|x-y\|.
\end{aligned}$$

Hence, W is a contraction. So, it has a unique fixed point, denoted by x_t. That is,

$$x_t = Q_K\big[t(\gamma_t x_t) + (1-t)T(s_t)x_t\big].$$

Therefore, the sequence $\{x_t\}$ defined by (3.1) is well defined.

Let $p \in F(S)$, then

$$\begin{aligned}
\|x_t - p\| &= \big\|Q_K\big[t(\gamma_t x_t) + (1-t)T(s_t)x_t\big] - p\big\| \\
&\le \big\|t\gamma_t(x_t - p) - t(1-\gamma_t)p + (1-t)\big(T(s_t)x_t - p\big)\big\| \\
&\le t\gamma_t\|x_t - p\| + t(1-\gamma_t)\|p\| + (1-t)\|x_t - p\| \\
&= \big[1-(1-\gamma_t)t\big]\|x_t - p\| + t(1-\gamma_t)\|p\|.
\end{aligned}$$

It follows that

$$\|x_t - p\| \le \|p\|.$$

Thus, $\{x_t\}$ is bounded, so is $\{T(s_t)u_n\}$.

Let $R = \|p\|$. It is clear that $\{x_t\} \subset B(p,R)$. Then $B(p,R) \cap K$ is a nonempty bounded closed convex subset of K and $T(s)$-invariant. Since $\{T(s)\}$ is u.a.r. nonexpansive semigroup and $\lim_{t\to 0} s_t = \infty$, then for all $s > 0$,

$$\lim_{t\to 0}\bigl\|T(s)\bigl(T(s_t)x_t\bigr) - T(s_t)x_t\bigr\| \le \lim_{n\to\infty}\sup_{x\in D}\bigl\|T(s)\bigl(T(s_t)x\bigr) - T(s_t)x\bigr\| = 0,$$

where D is any bounded subset of K containing $\{u_n\}$. Since

$$\bigl\|x_t - T(s_t)x_t\bigr\| \le t\bigl\|\gamma_t x_t - T(s_t)x_t\bigr\| \to 0,$$

and

$$\begin{aligned}\bigl\|x_t - T(s)x_t\bigr\| &\le \bigl\|x_t - T(s_t)x_t\bigr\| + \bigl\|T(s_t)x_t - T(s)\bigl(T(s_t)x_t\bigr)\bigr\| + \bigl\|T(s)\bigl(T(s_t)x_t\bigr) - T(s)x_t\bigr\| \\ &\le 2\bigl\|x_t - T(s_t)x_t\bigr\| + \bigl\|T(s_t)x_t - T(s)\bigl(T(s_t)x_t\bigr)\bigr\|.\end{aligned}$$

Thus, for all $s > 0$, we have

$$\lim_{t\to 0}\bigl\|x_t - T(s)x_t\bigr\| = 0. \tag{3.2}$$

Set $y_t = t(\gamma_t x_t) + (1-t)T(s_t)x_t$. Then $x_t = Q_K y_t$. By Proposition 2.1(2), we can get that

$$\begin{aligned}\|x_t - p\|^2 &= \|Q_K y_t - Q_K p\|^2 \\ &\le \bigl\langle y_t - p, j(x_t - p)\bigr\rangle \\ &= t\gamma_t\bigl\langle x_t - p, j(x_t - p)\bigr\rangle - t(1-\gamma_t)\bigl\langle p, j(x_t - p)\bigr\rangle + (1-t)\bigl\langle T(s_t)x_t - p, j(x_t - p)\bigr\rangle \\ &\le \bigl[1 - (1-\gamma_t)t\bigr]\|x_t - p\|^2 - t(1-\gamma_t)\bigl\langle p, j(x_t - p)\bigr\rangle.\end{aligned}$$

Thus

$$\|x_t - p\|^2 \le -\bigl\langle p, j(x_t - p)\bigr\rangle, \quad \forall p \in F(S). \tag{3.3}$$

Since $\{x_t\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup x^*$. From (3.2), we have $x_{t_n} - T(s)x_{t_n} \to 0$ as $n \to \infty$. Since E satisfies Opial's condition, it follows from Lemma 2.3 that $x^* \in F(S)$. From (3.3), we have

$$\|x_{t_n} - p\|^2 \le -\bigl\langle p, j(x_{t_n} - p)\bigr\rangle, \quad \forall p \in F(S). \tag{3.4}$$

In particular, if we substitute x^* for p in (3.4), then we have

$$\bigl\|x_{t_n} - x^*\bigr\|^2 \le -\bigl\langle x^*, j\bigl(x_{t_n} - x^*\bigr)\bigr\rangle. \tag{3.5}$$

Since j is weakly sequentially continuous from E to E^*, it follows from (3.5) that

$$\lim_{n\to\infty}\left\|x_{t_n}-x^*\right\|^2 \le \lim_{n\to\infty} -\left\langle x^*, j\left(x_{t_n}-x^*\right)\right\rangle = 0.$$

Suppose that there exists a subsequence $\{x_{t_m}\}$ of $\{x_t\}$ such that $x_{t_m} \rightharpoonup \widetilde{x}$. Then we have $\widetilde{x} \in F(S)$ and

$$\|x_{t_m}-p\|^2 \le -\langle p, j(x_{t_m}-p)\rangle, \quad \forall p \in F(S). \tag{3.6}$$

Since $x^*, \widetilde{x} \in F(S)$, from (3.4) and (3.6), we have

$$\|x_{t_n}-\widetilde{x}\|^2 \le -\langle \widetilde{x}, j(x_{t_n}-\widetilde{x})\rangle, \tag{3.7}$$

and

$$\left\|x_{t_m}-x^*\right\|^2 \le -\left\langle x^*, j\left(x_{t_m}-x^*\right)\right\rangle. \tag{3.8}$$

Now, in (3.7) and (3.8), taking $n \to \infty$ and $m \to \infty$, respectively. We get

$$\left\|x^*-\widetilde{x}\right\|^2 \le -\left\langle \widetilde{x}, j\left(x^*-\widetilde{x}\right)\right\rangle, \tag{3.9}$$

and

$$\left\|\widetilde{x}-x^*\right\|^2 \le -\left\langle x^*, j\left(\widetilde{x}-x^*\right)\right\rangle. \tag{3.10}$$

Adding up (3.9) and (3.10), we have

$$\left\|x^*-\widetilde{x}\right\|^2 \le 0.$$

We have proved that each cluster point of $\{x_t\}$ (as $t \to 0$) equals x^*. Thus $x_t \to x^*$ as $t \to 0$. □

Remark 3.2 Theorem 3.1 improves and extends Theorem 3.1 of Yao *et al.* [7] in the following aspects.

(1) From a real Hilbert space to a reflexive, strictly convex and uniformly smooth Banach space which satisfies Opial's condition.
(2) $\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\,ds$ is replaced by $T(s_t)x_t$.

Theorem 3.3 *Let E be a reflexive, strictly convex and uniformly smooth Banach space which satisfies Opial's condition, and let K be a nonempty closed convex subset of E. Let $S = \{T(s) : s \ge 0\} : K \to K$ be a uniformly asymptotically regular nonexpansive semigroup such that $F(S) \ne \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative process*:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n Q_K\left[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n\right], \quad \forall n \ge 0, \tag{3.11}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0,1]$ satisfying the following conditions:

(1) $\lim_{n\to\infty}\gamma_n = 1$, $\sum_{n=1}^{\infty}(1-\gamma_n)\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$.
(2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.
(3) $h, s_n \ge 0$ *such that* $s_{n+1} = h + s_n$ *and* $\lim_{n\to\infty} s_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $x^* \in F(S)$.

Proof Let $p \in F(S)$, we can get

$$\begin{aligned}
\|x_{n+1}-p\| &= \big\|(1-\beta_n)x_n + \beta_n Q_K\big[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n\big] - p\big\| \\
&\le (1-\beta_n)\|x_n-p\| + \beta_n\big\|Q_K\big[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n\big] - p\big\| \\
&\le (1-\beta_n)\|x_n-p\| + \beta_n\big\|\alpha_n\gamma_n(x_n-p) - \alpha_n(1-\gamma_n)p + (1-\alpha_n)\big(T(s_n)x_n - p\big)\big\| \\
&\le (1-\beta_n)\|x_n-p\| + \beta_n\big(\alpha_n\gamma_n\|x_n-p\| - \alpha_n(1-\gamma_n)\|p\| + (1-\alpha_n)\|x_n-p\|\big) \\
&= \big[1-(1-\gamma_n)\alpha_n\beta_n\big]\|x_n-p\| + (1-\gamma_n)\alpha_n\beta_n\|p\| \\
&\le \max\big\{\|x_n-p\|, \|p\|\big\} \\
&\le \max\big\{\|x_0-p\|, \|p\|\big\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded, so is $\{T(s_n)x_n\}$.

Set $y_n = Q_K[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n]$ for all $n \ge 0$. Then $x_{n+1} = (1-\beta_n)x_n + \beta_n y_n$.

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \big\|Q_K\big[\alpha_{n+1}(\gamma_{n+1}x_{n+1}) + (1-\alpha_{n+1})T(s_{n+1})x_{n+1}\big] \\
&\quad - Q_K\big[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n\big]\big\| \\
&\le \big\|\big[\alpha_{n+1}(\gamma_{n+1}x_{n+1}) + (1-\alpha_{n+1})T(s_{n+1})x_{n+1}\big] - \big[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n\big]\big\| \\
&= \big\|\alpha_{n+1}\gamma_{n+1}(x_{n+1}-x_n) + (\alpha_{n+1}\gamma_{n+1} - \alpha_n\gamma_n)x_n + (1-\alpha_{n+1}) \\
&\quad \times \big(T(s_{n+1})x_{n+1} - T(s_{n+1})x_n + T(s_{n+1})x_n - T(s_n)x_n\big) + (\alpha_{n+1}-\alpha_n)T(s_n)x_n\big\| \\
&\le \alpha_{n+1}\gamma_{n+1}\|x_{n+1}-x_n\| + |\alpha_{n+1}\gamma_{n+1} - \alpha_n\gamma_n|\|x_n\| \\
&\quad + (1-\alpha_{n+1})\big(\|x_{n+1}-x_n\| + \big\|T(h)T(s_n)x_n - T(s_n)x_n\big\|\big) \\
&\quad + |\alpha_{n+1}-\alpha_n|\big\|T(s_n)x_n\big\| \\
&= \big[1-(1-\gamma_{n+1})\alpha_{n+1}\big]\|x_{n+1}-x_n\| + |\alpha_{n+1}\gamma_{n+1} - \alpha_n\gamma_n|\|x_n\| \\
&\quad + (1-\alpha_{n+1})\big\|T(h)(s_n)x_n - T(s_n)x_n\big\| + |\alpha_{n+1}-\alpha_n|\big\|T(s_n)x_n\big\|.
\end{aligned}$$

So,

$$\begin{aligned}
&\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \\
&\quad \le -(1-\gamma_{n+1})\alpha_{n+1}\|x_{n+1}-x_n\| + |\alpha_{n+1}\gamma_{n+1} - \alpha_n\gamma_n|\|x_n\| \\
&\qquad + (1-\alpha_{n+1})\big\|T(h)(s_n)x_n - T(s_n)x_n\big\| + |\alpha_{n+1}-\alpha_n|\big\|T(s_n)x_n\big\|.
\end{aligned} \tag{3.12}$$

Since $\{T(s) : s \ge 0\}$ is uniformly asymptotically regular and $\lim_{n\to\infty} s_n = \infty$, it follows that

$$\lim_{n\to\infty}\big\|T(h)T(s_n)x_n - T(s_n)x_n\big\| \le \lim_{n\to\infty}\sup_{x\in B}\big\|T(h)T(s_n)x - T(s_n)x\big\| = 0, \tag{3.13}$$

where B is any bounded set containing $\{x_n\}$. Moreover, since $\{x_n\}$, $\{T(s_n)x_n\}$ are bounded, and $\alpha_n \to 0$ as $n \to \infty$, (3.12) implies that

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0.$$

Hence, by Lemma 2.4 we have $\lim_{n\to\infty}\|y_n - x_n\| = 0$ since $x_{n+1} - x_n = \beta_n(y_n - x_n)$. Consequently, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

It follows from (3.11) that

$$\begin{aligned}
\|x_n - T(s_n)x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T(s_n)x_n\| \\
&\le \|x_n - x_{n+1}\| + \big\|(1-\beta_n)\big(x_n - T(s_n)x_n\big) \\
&\quad + \beta_n\big(Q_K\big[\alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n\big] - T(s_n)x_n\big)\big\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T(s_n)x_n\| + \alpha_n\gamma_n\|x_n - T(s_n)x_n\| \\
&\quad + \alpha_n(1-\gamma_n)\|T(s_n)x_n\| \\
&= \|x_n - x_{n+1}\| + (1-\beta_n+\alpha_n\gamma_n)\|x_n - T(s_n)x_n\| + \alpha_n(1-\gamma_n)\|T(s_n)x_n\|.
\end{aligned}$$

So,

$$\|x_n - T(s_n)x_n\| \le \frac{1}{\beta_n - \alpha_n\gamma_n}\left(\|x_n - x_{n+1}\| + \alpha_n(1-\gamma_n)\|T(s_n)x_n\|\right) \to 0. \tag{3.14}$$

Since

$$\begin{aligned}
&\|x_n - T(h)x_n\| \\
&\quad \le \|x_n - T(s_n)x_n\| + \|T(s_n)x_n - T(h)T(s_n)x_n\| + \|T(h)T(s_n)x_n - T(h)x_n\| \\
&\quad \le 2\|x_n - T(s_n)x_n\| + \|T(s_n)x_n - T(h)T(s_n)x_n\|,
\end{aligned}$$

from (3.13) and (3.14), we have

$$\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0. \tag{3.15}$$

Notice that $\{x_n\}$ is bounded. Put $x^* = Q_{F(S)}(0)$. Then there exists a positive number R such that $B(x^*,R)\cap K$ contains $\{x_n\}$. Moreover, $B(x^*,R)\cap K$ is $T(s)$-invariant for all $s \ge 0$ and so, without loss of generality, we can assume that $\{T(s): s \ge 0\}$ is a nonexpansive semigroup on $B(x^*,R)\cap K$. We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle -x^*, j(x_n - x^*)\rangle = \lim_{k\to\infty}\langle x^*, j(x_{n_k} - x^*)\rangle.$$

We may also assume that $x_{n_k} \rightharpoonup \widetilde{x}$. It follows from Lemma 2.3 and (3.15) that $\widetilde{x} \in F(S)$ and hence

$$\langle -x^*, j(\widetilde{x} - x^*)\rangle \le 0.$$

Since j is weakly sequentially continuous, we have

$$\limsup_{n\to\infty}\langle -x^*, j(x_n - x^*)\rangle = \lim_{k\to\infty}\langle -x^*, j(x_{n_k} - x^*)\rangle = \langle -x^*, j(\widetilde{x} - x^*)\rangle \le 0.$$

Since $\lim_{n\to\infty}\|y_n - x_n\| = 0$, we have $y_n - x^* \to x_n - x^*$, so

$$\limsup_{n\to\infty}\langle -x^*, j(y_n - x^*)\rangle = \limsup_{n\to\infty}\langle -x^*, j(x_n - x^*)\rangle \le 0.$$

Set $u_n = \alpha_n(\gamma_n x_n) + (1-\alpha_n)T(s_n)x_n$. It follows that $y_n = Q_K u_n$ for all $n \ge 0$. By Proposition 2.1(3), we have

$$\langle y_n - u_n, j(y_n - x^*)\rangle \le 0,$$

and so

$$\begin{aligned}
\|y_n - x^*\|^2 &= \langle y_n - x^*, j(y_n - x^*)\rangle \\
&= \langle y_n - u_n, j(y_n - x^*)\rangle + \langle u_n - x^*, j(y_n - x^*)\rangle \\
&\le \langle u_n - x^*, j(y_n - x^*)\rangle \\
&= \alpha_n\gamma_n\langle x_n - x^*, j(y_n - x^*)\rangle - \alpha_n(1-\gamma_n)\langle x^*, j(y_n - x^*)\rangle \\
&\quad + (1-\alpha_n)\langle T(s_n)x_n - x^*, j(y_n - x^*)\rangle \\
&\le \alpha_n\gamma_n\|x_n - x^*\|\|j(y_n - x^*)\| - \alpha_n(1-\gamma_n)\langle x^*, j(y_n - x^*)\rangle \\
&\quad + (1-\alpha_n)\|T(s_n)x_n - x^*\|\|j(y_n - x^*)\| \\
&\le [1-(1-\gamma_n)\alpha_n]\|x_n - x^*\|\|y_n - x^*\| - \alpha_n(1-\gamma_n)\langle x^*, j(y_n - x^*)\rangle \\
&\le \frac{1-(1-\gamma_n)\alpha_n}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|y_n - x^*\|^2 - \alpha_n(1-\gamma_n)\langle x^*, j(y_n - x^*)\rangle,
\end{aligned}$$

that is,

$$\|y_n - x^*\|^2 \le [1-(1-\gamma_n)\alpha_n]\|x_n - x^*\|^2 - 2\alpha_n(1-\gamma_n)\langle x^*, j(y_n - x^*)\rangle.$$

By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|y_n - x^*\|^2 \\
&\le [1-(1-\gamma_n)\alpha_n\beta_n]\|x_n - x^*\|^2 - 2(1-\gamma_n)\alpha_n\beta_n\langle x^*, j(y_n - x^*)\rangle.
\end{aligned}$$

By Lemma 2.5, we conclude that $x_n \to x^*$. □

Remark 3.4 Theorem 3.3 improves and extends Theorem 3.3 of Yao *et al.* [7] in the following aspects.

(1) From a real Hilbert space to a reflexive, strictly convex and uniformly smooth Banach space which satisfies Opial's condition.
(2) $\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n\,ds$ is replaced by $T(s_n)x_n$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper is proposed by XW. All authors read and approved the final manuscript.

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