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The existence of best proximity points with the weak P-property

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Abstract
We improve some existence theorem of best proximity points with the weak P-property, which has been recently proved by Zhang *et al.*
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1 Introduction

Let (A, B) be a pair of nonempty subsets of a metric space (X, d), and let T be a mapping from A into B. Then $x \in A$ is called a *best proximity point* if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. We have proved many existence theorems of best proximity points. See, for example, [1–6]. Very recently, Caballero *et al.* [7] proved a new type of existence theorem, and Zhang *et al.* [8] generalized the theorem. The theorem proved in [8] is Theorem 8 with an additional assumption of the completeness of B. The essence of the result in [7] becomes very clear in [8], however, we have not learned the essence completely.

Motivated by the fact above, in this paper, we improve the result in [8]. Also, in order to consider the discontinuous case, we give a Kannan version.

2 Preliminaries

In this section, we give some preliminaries.

Definition 1 Let (A, B) be a pair of nonempty subsets of a metric space (X, d), and define A_0 and B_0 by

$$A_0 = \{x \in A : \text{ there exists } u \in B \text{ such that } d(x, u) = d(A, B)\} \tag{1}$$

and

$$B_0 = \{u \in B : \text{ there exists } x \in A \text{ such that } d(x, u) = d(A, B)\}. \tag{2}$$

Then

- (Sankar Raj [9]) (A, B) is said to have the *P-property* if $A_0 \neq \varnothing$ and the following holds:

$$x, y \in A_0, u, v \in B_0, \quad d(x, u) = d(y, v) = d(A, B) \quad \Longrightarrow \quad d(x, y) = d(u, v).$$

- (Zhang *et al.* [8]) (A,B) is said to have the *weak P-property* if $A_0 \neq \varnothing$ and the following holds:

$$x,y \in A_0, u,v \in B_0, \quad d(x,u) = d(y,v) = d(A,B) \quad \Longrightarrow \quad d(x,y) \leq d(u,v).$$

Proposition 2 *Let (A,B) be a pair of nonempty subsets of a metric space (X,d), and define A_0 and B_0 by* (1) *and* (2). *Assume that $A_0 \neq \varnothing$. Then the following are equivalent*:

(i) *(A,B) has the weak P-property.*

(ii) *The conjunction of the following holds*:

(ii-1) *For every $u \in B_0$, there exists a unique $x \in A_0$ with $d(x,u) = d(A,B)$.*

(ii-2) *There exists a nonexpansive mapping Q from B_0 into A_0 such that $d(Qu,u) = d(A,B)$ for every $u \in B_0$.*

Proof We note that $B_0 \neq \varnothing$ because $A_0 \neq \varnothing$. First, we assume (i). Let $x,y \in A_0$ and $u \in B_0$ satisfy $d(x,u) = d(y,u) = d(A,B)$. Then from (i), we have

$$d(x,y) \leq d(u,u) = 0,$$

thus, $x = y$. So (ii-1) holds. We put $Qu = x$. Then from the definition of the weak P-property, we have $d(Qu,Qv) \leq d(u,v)$ for $u,v \in B_0$, that is, Q is nonexpansive. Conversely, we assume (ii). Let $x,y \in A_0$ and $u,v \in B_0$ satisfy $d(x,u) = d(y,v) = d(A,B)$. Then from (ii-1), we have $Qu = x$ and $Qv = y$. Therefore,

$$d(x,y) = d(Qu,Qv) \leq d(u,v)$$

holds. □

Lemma 3 *Let (A,B) be a pair of subsets of a metric space (X,d), and define A_0 and B_0 by* (1) *and* (2). *Assume that $A_0 \neq \varnothing$. Let T be a mapping from A into B, and let Q be a mapping from B_0 into A_0 such that $d(Qu,u) = d(A,B)$ for every $u \in B_0$. Then the following holds*:

$$\{u_n\} \subset B_0, \quad \lim_{n\to\infty} u_n = w, \qquad T\Bigl(\lim_{n\to\infty} Qu_n\Bigr) = w \quad \Longrightarrow \quad w \in B_0. \tag{3}$$

Proof Let $\{u_n\}$ be a sequence in B_0 such that $\{u_n\}$ converges to $w \in X$, and $T(\lim_n Qu_n) = w$. We put $y = \lim_n Qu_n$. Since $Ty = w$, we have $y \in A$ and $w \in B$. Since

$$d(y,w) = \lim_{n\to\infty} d(Qu_n,u_n) = d(A,B),$$

we have $y \in A_0$ and $w \in B_0$. □

Lemma 4 *Let (X,d) be a metric space, let A, A_0, B_0 be nonempty subsets such that A is complete and $A_0 \subset A$. Let T be a mapping from A into X such that $T(A_0) \subset B_0$, and let Q be a nonexpansive mapping from B_0 into A_0. Let $\bar{Q}$ be the mapping whose graph* $\mathrm{Gr}(\bar{Q})$ *is the completion of* $\mathrm{Gr}(Q)$. *Assume* (3). *Then the following hold*:

(i) *$\bar{Q}$ is well-defined and nonexpansive.*

(ii) *$\bar{Q}w = z$ is equivalent to that there exists a sequence $\{u_n\}$ in B_0 such that* $\lim_n u_n = w$ *and* $\lim_n Qu_n = z$.

(iii) *The domain of* $\bar{Q}$ *is* $\bar{B_0}$, *where* $\bar{B_0}$ *is the completion of* B_0.
(iv) *The range of* $\bar{Q}$ *is a subset of* $\bar{A_0}$, *where* $\bar{A_0}$ *is the completion of* A_0.
(v) $T \circ \bar{Q}w = w$ *implies* $T \circ Qw = w$.
(vi) $\bar{Q} \circ Tz = z$ *implies* $Q \circ Tz = z$.
(vii) *The range of* $\bar{Q}$ *is a subset of* A.

Proof We consider that the whole space is the completion of X. Since Q is Lipschitz continuous, $\bar{Q}$ is well-defined. The rest of (i) and (ii)-(iv) are obvious. By using (3), we can easily prove (v) and (vi). From the completeness of A, we obtain (vii). □

3 Fixed point theorems

In this section, we give fixed point theorems, which are used in the proofs of the main results.

Theorem 5 *Let* (X,d) *be a metric space, let* A, A_0, B_0 *be nonempty subsets such that* A *is complete and* $A_0 \subset A$. *Let* T *be a contraction from* A *into* X *such that* $T(A_0) \subset B_0$, *and let* Q *be a nonexpansive mapping from* B_0 *into* A_0. *Assume* (3). *Then* $Q \circ T$ *has a unique fixed point in* A_0.

Proof We consider that the whole space is the completion of X. Define a nonexpansive mapping $\bar{Q}$ as in Lemma 4. Since T is continuous, $T(\bar{A_0})$ is a subset of $\bar{B_0}$. Let S be the restriction of T to $\bar{A_0}$. Then $\bar{Q} \circ S$ is a contraction on $\bar{A_0}$. So the Banach contraction principle yields that there exists a unique fixed point z of $\bar{Q} \circ S$ in $\bar{A_0}$. Since $\bar{Q} \circ Tz = z$, by Lemma 4(vi), z is a fixed point of $Q \circ T$. □

Remark

- If $X = A = A_0 = B_0$ and Q is the identity mapping on B_0, then Theorem 5 becomes the Banach contraction principle [10].
- We can prove Theorem 5 with the mapping $T \circ \bar{Q}$ as in the proof of Theorem 7.

We prove generalizations of Kannan's fixed point theorem [11].

Theorem 6 *Let* (X,d) *be a metric space, let* Y *be a complete subset of* X, *and let* T *be a mapping from* Y *into* X. *Assume that the following hold*:
(i) *There exists* $\alpha \in [0,1/2)$ *such that* $d(Tx,Ty) \le \alpha d(x,Tx) + \alpha d(y,Ty)$ *for all* $x,y \in Y$.
(ii) *There exists a nonempty subset* Z *of* Y *such that* $T(Z) \subset Z$.
Then there exists a unique fixed point z, *and for every* $x \in Z$, $\{T^n x\}$ *converges to* z.

Proof Fix $x \in Z$. Then from the proof in Kannan [11], we obtain that $\{T^n x\}$ converges to a fixed point, and the fixed point is unique. □

Remark If $X = Y = Z$, then Theorem 6 becomes Kannan's fixed point theorem [11].

Using Theorem 6, we obtain the following.

Theorem 7 *Let* (X,d) *be a metric space, let* A, A_0, B_0 *be nonempty subsets such that* A *is complete and* $A_0 \subset A$. *Let* T *be a mapping from* A *into* X *such that* $T(A_0) \subset B_0$, *and let* Q *be a nonexpansive mapping from* B_0 *into* A_0. *Assume that* (3) *and the following hold*:

- *There exist* $\alpha \in [0, 1/2)$ *and* $\mu \in [0, \infty)$ *such that*

$$d(Tx, Ty) \le \alpha\bigl(d(x, Tx) - \mu\bigr) + \alpha\bigl(d(y, Ty) - \mu\bigr)$$

  for $x, y \in A$ *and* $d(Qu, u) \le \mu$ *for all* $u \in B_0$.

Then $T \circ Q$ *has a unique fixed point in* B_0.

Proof We consider that the whole space is the completion of X. Define a nonexpansive mapping $\bar{Q}$ as in Lemma 4. From the continuity of d, $d(\bar{Q}u, u) \le \mu$ for $u \in \bar{B_0}$. For $u, v \in \bar{B_0}$, we have

$$\begin{aligned} & d(T \circ \bar{Q}u, T \circ \bar{Q}v) \\ & \quad \le \alpha\bigl(d(\bar{Q}u, T \circ \bar{Q}u) - \mu\bigr) + \alpha\bigl(d(\bar{Q}v, T \circ \bar{Q}v) - \mu\bigr) \\ & \quad \le \alpha\bigl(d(\bar{Q}u, u) + d(u, T \circ \bar{Q}u) - \mu\bigr) + \alpha\bigl(d(\bar{Q}v, v) + d(v, T \circ \bar{Q}v) - \mu\bigr) \\ & \quad \le \alpha d(u, T \circ \bar{Q}u) + \alpha d(v, T \circ \bar{Q}v). \end{aligned}$$

Hence $T \circ \bar{Q}$ is a Kannan mapping from $\bar{B_0}$ into X. $T \circ \bar{Q}(B_0) = T \circ Q(B_0) \subset B_0$ is obvious. So by Theorem 6, there exists a unique fixed point w of $T \circ \bar{Q}$ in $\bar{B_0}$. By Lemma 4(v), $w \in B_0$ and w is a fixed point of $T \circ Q$. □

Remark

- Since T is not necessarily continuous, the range of $T \circ \bar{Q}$ is not necessarily included by $\bar{B_0}$. Because of the same reason, we cannot prove Theorem 7 with the mapping $\bar{Q} \circ T$.
- It is interesting that we do not need the completeness of any set related to B_0 directly. Of course, we need the completeness of A.

4 Main results

In this section, we give the main results.

Theorem 8 (Zhang *et al.* [8]) *Let* (A, B) *be a pair of subsets of a metric space* (X, d), *and define* A_0 *and* B_0 *by* (1) *and* (2). *Let* T *be a contraction from* A *into* B. *Assume that the following hold*:

(i) (A, B) *has the weak P-property.*
(ii) A *is complete.*
(iii) $T(A_0) \subset B_0$.

Then there exists a unique $z \in A$ *such that* $d(z, Tz) = d(A, B)$.

Proof By Proposition 2(ii-2), there exists a nonexpansive mapping Q from B_0 into A_0 such that $d(Qu, u) = d(A, B)$ for every $u \in B_0$. Then by Lemma 3, all the assumptions in Theorem 5 hold. So there exists a unique fixed point z of $Q \circ T$ in A_0. This implies that $d(z, Tz) = d(A, B)$. Let $x \in A$ satisfy $d(x, Tx) = d(A, B)$. Then from Proposition 2(ii-1), $x \in A_0$, $Tx \in B_0$ and $Q \circ Tx = x$ hold. Since $Q \circ T$ has a unique fixed point, we obtain $x = z$. Hence z is unique. □

Remark

- If we weaken (i) to the conjunction of $A_0 \neq \varnothing$ and (ii-2) in Proposition 2, we obtain only the existence of best proximity points.
- In [8], we assume the completeness of B.
- Exactly speaking, in [8], we obtained a theorem connected with Geraghty's fixed point theorem [12]. However, in this paper, the difference between the two fixed point theorems is not essential. This means that we can easily modify Theorem 8 to be connected with Geraghty's theorem.

Theorem 9 *Let (A, B) be a pair of subsets of a metric space (X, d), and define A_0 and B_0 by* (1) *and* (2). *Let T be a mapping from A into B. Assume that* (i)-(iii) *in Theorem* 8 *and the following hold*:

(iv) *There exists $\alpha \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq \alpha\big(d(x, Tx) - d(A, B)\big) + \alpha\big(d(y, Ty) - d(A, B)\big)$$

for $x, y \in A$.

Then there exists a unique $z \in A$ such that $d(z, Tz) = d(A, B)$.

Proof By Proposition 2(ii-2), there exists a nonexpansive mapping Q from B_0 into A_0 such that $d(Qu, u) = d(A, B)$ for every $u \in B_0$. Then by Theorem 7, there exists a unique fixed point w of $T \circ Q$ in B_0. This implies that $d(z, Tz) = d(A, B)$, where $z = Qw$. Let $x \in A$ satisfy $d(x, Tx) = d(A, B)$. Then from Proposition 2(ii-1), $x \in A_0$, $Tx \in B_0$ and $Q \circ Tx = x$ hold. Since $T \circ Q \circ Tx = Tx$, we have $Tx = w$, and hence $x = Q \circ Tx = Qw = z$. Therefore, z is unique. □

Remark If we weaken (i) to the conjunction of $A_0 \neq \varnothing$ and (ii-2) in Proposition 2, we obtain only the existence of best proximity points.

5 Additional result

In this section, we give a proposition similar to Proposition 2.

Proposition 10 *Let (A, B) be a pair of nonempty subsets of a metric space (X, d), and define A_0 and B_0 by* (1) *and* (2). *Assume that $A_0 \neq \varnothing$. Then the following are equivalent*:

(i) *(A, B) has the P-property.*

(ii) *The conjunction of the following holds*:

  (ii-1) *For every $u \in B_0$, there exists a unique $x \in A_0$ with $d(x, u) = d(A, B)$.*

  (ii-2) *There exists an isometry Q from B_0 onto A_0 such that $d(Qu, u) = d(A, B)$ for every $u \in B_0$.*

Proof We note $B_0 \neq \varnothing$. First, we assume (i). Let $x, y \in A_0$ and $u \in B_0$ satisfy $d(x, u) = d(y, u) = d(A, B)$. Then from (i), we have $d(x, y) = d(u, u) = 0$, thus, $x = y$. So (ii-1) holds. We put $Qu = x$. Then it is obvious that Q is isometric. For every $x \in A_0$, there exists $u \in B_0$ with $d(x, u) = d(A, B)$. From (ii-1), $Qu = x$ obviously holds, and hence Q is surjective. Conversely, we assume (ii). Let $x, y \in A_0$ and $u, v \in B_0$ satisfy $d(x, u) = d(y, v) = d(A, B)$. Then we have $Qu = x$ and $Qv = y$. Therefore, $d(x, y) = d(Qu, Qv) = d(u, v)$ holds. □

Competing interests
The author declares that he has no competing interests.

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