Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for mappings satisfying contractive conditions of integral type

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Abstract
Two results involving the existence, uniqueness and iterative approximations of fixed points for two contractive mappings of integral type are proved in complete metric spaces. Two nontrivial examples are included.
MSC: 54H25

Keywords: contractive mappings of integral type; fixed point; complete metric space

1 Introduction

In recent years, there has been increasing interest in the study of fixed points and common fixed points of mappings satisfying contractive conditions of integral type, see, for example, [1–14] and the references cited therein. Branciari [4] introduced first the contractive mapping of integral type as follows:

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le c\int_0^{d(x,y)} \varphi(t)\,dt, \quad \forall x, y \in X,$$

where $c \in (0,1)$ is a constant, $\varphi \in \Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that φ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)\,dt > 0$ for each $\varepsilon > 0\}$ and proved the existence of a fixed point for the mapping in complete metric spaces. Rhoades [10] and Liu *et al.* [8] extended Branciari's result and obtained a few fixed point theorems for the contractive mappings of integral type below:

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le c\int_0^{M(x,y)} \varphi(t)\,dt, \quad \forall x, y \in X$$

and

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le \alpha\bigl(d(x,y)\bigr)\int_0^{d(x,y)} \varphi(t)\,dt, \quad \forall x, y \in X,$$

where $c \in (0,1)$ is a constant, $\varphi \in \Phi$ and $\alpha : \mathbb{R}^+ \to [0,1)$ is a function with $\limsup_{s\to t} \alpha(s) < 1$, $\forall t > 0$. Mongkolkeha and Kumam [9] proved fixed point and common fixed point theorems for ρ-compatible mapping satisfying a generalized weak contraction of integral type in modular spaces. Sintunavarat and Kumam [11, 12] gave common fixed point theorems for single-valued and multi-valued mappings satisfying strict general contractive conditions of integral type.

Inspired and motivated by the results in [1–14], in this paper, we introduce two new classes of contractive mappings of integral type in complete metric spaces and study the existence, uniqueness and iterative approximations of fixed points for the mappings. The results obtained in this paper generalize and improve Theorem 2.1 in [4], Theorem 3.1 in [8] and Theorem 2 in [10]. Two nontrivial examples are constructed.

2 Preliminaries

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$,

$\Phi_1 = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous on $\mathbb{R}^+ \setminus \{0\}$, $\phi(0) = 0$ and $\phi(t) < t$, $\forall t > 0\}$;

$\Phi_2 = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is right upper semi-continuous on $\mathbb{R}^+ \setminus \{0\}$, $\phi(0) = 0$ and $\phi(t) < t$, $\forall t > 0\}$;

$\Phi_3 = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $\phi(0) = 0$, $\phi(t) > 0$, $\forall t > 0$ and $\lim_{n\to\infty} t_n = 0$, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{n\to\infty} \phi(t_n) = 0\}$;

$\Phi_4 = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing, $\phi(0) = 0$, continuous at 0 and $\lim_{n\to\infty} t_n = 0$ for each sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ with $\lim_{n\to\infty} \phi(t_n) = 0\}$;

$\Phi_5 = \{\phi : \phi$ is in Φ_4 and is left continuous on $\mathbb{R}^+ \setminus \{0\}\}$;

(a1) $(\varphi, \phi, \psi) \in \Phi \times \Phi_1 \times \Phi_3$;

(a2) $(\varphi, \phi, \psi) \in \Phi \times \Phi_2 \times \Phi_4$;

(a3) $(\varphi, \phi, \psi) \in \Phi \times \Phi_2 \times \Phi_5$.

Let T be a mapping from a metric space (X, d) into itself, and let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. Put

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\big[d(x,Ty) + d(y,Tx)\big]\right\}, \quad \forall x, y \in X,$$

$\psi(s+)$ and $\psi(s-)$ denote the right and left limits of the function ψ at $s \in \mathbb{R}^+$, respectively.

The following lemmas play important roles in this paper.

Lemma 2.1 [8] *Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\,dt = \int_0^a \varphi(t)\,dt.$$

Lemma 2.2 [8] *Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\,dt = 0$$

if and only if $\lim_{n\to\infty} r_n = 0$.

3 Fixed point theorems and examples

In this section, we prove two fixed point theorems for two classes of contractive mappings of integral type and display two examples as applications of the theorems.

Theorem 3.1 *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping satisfying*

$$\int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt \le \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\,dt\right), \quad \forall x,y \in X, \tag{3.1}$$

where φ, ϕ and ψ satisfy (a1) *or* (a2). *Then T has a unique fixed point $a \in X$ and $\lim_{n\to\infty} T^n x_0 = a$ for each $x_0 \in X$.*

Proof Let x_0 be an arbitrary point in X. Put $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$. Assume that $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$. It is easy to see that x_{n_0-1} is a fixed point of T, and there is nothing to prove. Assume that $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$. From (3.1) and one of (a1) and (a2), we obtain that

$$\begin{aligned}\int_0^{\psi(d(x_{n+1},x_n))} \varphi(t)\,dt &= \int_0^{\psi(d(Tx_n,Tx_{n-1}))} \varphi(t)\,dt \le \phi\left(\int_0^{\psi(d(x_n,x_{n-1}))} \varphi(t)\,dt\right) \\ &< \int_0^{\psi(d(x_n,x_{n-1}))} \varphi(t)\,dt, \quad \forall n \in \mathbb{N},\end{aligned} \tag{3.2}$$

which implies that

$$0 < \psi\big(d(x_{n+1},x_n)\big) < \psi\big(d(x_n,x_{n-1})\big), \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Note that (3.3) yields that the sequence $\{\psi(d(x_n,x_{n-1}))\}_{n\in\mathbb{N}}$ is positive and strictly decreasing. Thus, there exists a constant $c \ge 0$ with

$$\lim_{n\to\infty} \psi\big(d(x_n,x_{n-1})\big) = c. \tag{3.4}$$

Suppose that $c > 0$. Taking upper limit in (3.2) and using (3.4), Lemma 2.1 and one of (a1) and (a2), we conclude that

$$\begin{aligned}\int_0^{c} \varphi(t)\,dt &= \limsup_{n\to\infty} \int_0^{\psi(d(x_{n+1},x_n))} \varphi(t)\,dt \le \limsup_{n\to\infty} \phi\left(\int_0^{\psi(d(x_n,x_{n-1}))} \varphi(t)\,dt\right) \\ &\le \phi\left(\int_0^{c} \varphi(t)\,dt\right) < \int_0^{c} \varphi(t)\,dt,\end{aligned}$$

which is absurd, and hence $c = 0$, that is,

$$\lim_{n\to\infty} \psi\big(d(x_n,x_{n-1})\big) = 0,$$

which together with one of (a1) and (a2) guarantees that

$$\lim_{n\to\infty} d(x_n,x_{n-1}) = 0. \tag{3.5}$$

Now, we show that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Suppose that $\{x_n\}_{n\in\mathbb{N}_0}$ is not a Cauchy sequence, which means that there is a constant $\varepsilon > 0$ such that for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ satisfying

$$d(x_{m(k)},x_{n(k)}) > \varepsilon.$$

For each positive integer k, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying the inequality above. It follows that

$$d(x_{m(k)}, x_{n(k)}) > \varepsilon \quad \text{and} \quad d(x_{m(k)-1}, x_{n(k)}) \le \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.6}$$

Note that

$$\begin{aligned} \varepsilon < d(x_{m(k)}, x_{n(k)}) &\le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &\le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) + 2d(x_{n(k)}, x_{n(k)-1}), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{3.7}$$

Letting $k \to \infty$ in (3.7) and using (3.5) and (3.6), we conclude that

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{3.8}$$

In view of (3.1), we deduce that

$$\begin{aligned} \int_0^{\psi(d(x_{m(k)}, x_{n(k)}))} \varphi(t)\,dt &= \int_0^{\psi(d(Tx_{m(k)-1}, Tx_{n(k)-1}))} \varphi(t)\,dt \\ &\le \phi\left(\int_0^{\psi(d(x_{m(k)-1}, x_{n(k)-1}))} \varphi(t)\,dt\right), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Assume that (a1) holds. Taking upper limit in (3.9) and using (3.8) and Lemma 2.1, we get that

$$\begin{aligned} \int_0^{\psi(\varepsilon)} \varphi(t)\,dt &= \limsup_{k\to\infty} \int_0^{\psi(d(x_{m(k)}, x_{n(k)}))} \varphi(t)\,dt \\ &\le \limsup_{k\to\infty} \phi\left(\int_0^{\psi(d(x_{m(k)-1}, x_{n(k)-1}))} \varphi(t)\,dt\right) \\ &\le \phi\left(\int_0^{\psi(\varepsilon)} \varphi(t)\,dt\right) < \int_0^{\psi(\varepsilon)} \varphi(t)\,dt, \end{aligned}$$

which is a contradiction.

Assume that (a2) holds. In view of (3.8), there exists $K \in \mathbb{N}$ satisfying

$$d(x_{m(k)-1}, x_{n(k)-1}) > \frac{\varepsilon}{2}, \quad \forall k \ge K.$$

It follows from the inequality above and $\psi \in \Phi_4$ that

$$\psi\big(d(x_{m(k)-1}, x_{n(k)-1})\big) > \psi\left(\frac{\varepsilon}{2}\right) > 0, \quad \forall k \ge K,$$

which together with (3.9) and $\phi \in \Phi_2$ gives that

$$\begin{aligned} \int_0^{\psi(d(x_{m(k)}, x_{n(k)}))} \varphi(t)\,dt &\le \phi\left(\int_0^{\psi(d(x_{m(k)-1}, x_{n(k)-1}))} \varphi(t)\,dt\right) \\ &< \int_0^{\psi(d(x_{m(k)-1}, x_{n(k)-1}))} \varphi(t)\,dt, \quad \forall k \ge K, \end{aligned}$$

which ensures that

$$\psi\left(d(x_{m(k)},x_{n(k)})\right) < \psi\left(d(x_{m(k)-1},x_{n(k)-1})\right), \quad \forall k \geq K,$$

that is,

$$d(x_{m(k)},x_{n(k)}) < d(x_{m(k)-1},x_{n(k)-1}), \quad \forall k \geq K,$$

which together with (3.6) and (3.8) implies that

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)-1}) = \varepsilon.$$

Taking upper limit in (3.9) and using Lemma 2.1, (a2) and the equations above, we conclude that

$$\begin{aligned}\int_0^{\psi(\varepsilon+)} \varphi(t)\,dt &= \limsup_{k\to\infty} \int_0^{\psi(d(x_{m(k)},x_{n(k)}))} \varphi(t)\,dt \\ &\leq \limsup_{k\to\infty} \phi\left(\int_0^{\psi(d(x_{m(k)-1},x_{n(k)-1}))} \varphi(t)\,dt\right) \\ &\leq \phi\left(\int_0^{\psi(\varepsilon+)} \varphi(t)\,dt\right) < \int_0^{\psi(\varepsilon+)} \varphi(t)\,dt,\end{aligned}$$

which is a contradiction.

Thus, $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since (X,d) is a complete metric space, there exists a point $a \in X$ such that $\lim_{n\to\infty} x_n = a$. By (3.1), Lemma 2.2 and one of $\psi \in \Phi_3$ and $\psi \in \Phi_4$, we arrive at

$$\begin{aligned}0 &\leq \int_0^{\psi(d(x_{n+1},Ta))} \varphi(t)\,dt \leq \phi\left(\int_0^{\psi(d(x_n,a))} \varphi(t)\,dt\right) \\ &\leq \int_0^{\psi(d(x_n,a))} \varphi(t)\,dt \to 0 \quad \text{as } n\to\infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \int_0^{\psi(d(x_{n+1},Ta))} \varphi(t)\,dt = 0,$$

which together with Lemma 2.2 means that

$$\lim_{n\to\infty} \psi\left(d(x_{n+1},Ta)\right) = 0. \tag{3.10}$$

Note that (3.10) and one of $\psi \in \Phi_3$ and $\psi \in \Phi_4$ ensure that $\lim_{n\to\infty} d(x_{n+1},Ta) = 0$. Consequently, we conclude immediately that

$$d(a,Ta) \leq d(a,x_{n+1}) + d(x_{n+1},Ta) \to 0 \quad \text{as } n\to\infty,$$

which gives that $a = Ta$.

Next, we show that a is a unique fixed point T in X. Suppose that T has another fixed point $b \in X \setminus \{a\}$. It follows from (3.1) and one of (a1) and (a2) that

$$\begin{aligned}
0 &< \int_0^{\psi(d(a,b))} \varphi(t)\,dt = \int_0^{\psi(d(Ta,Tb))} \varphi(t)\,dt \\
&\le \phi\left(\int_0^{\psi(d(a,b))} \varphi(t)\,dt\right) < \int_0^{\psi(d(a,b))} \varphi(t)\,dt,
\end{aligned}$$

which is a contradiction. This completes the proof. □

Remark 3.2 Theorem 3.1 generalizes Theorem 2.1 in [4] and Theorem 3.1 in [8]. The example below is an application of Theorem 3.1.

Example 3.3 Let $X = [0,1] \cup \{3,5\}$ be endowed with the Euclidean metric $d = |\cdot|$. Define $T : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Tx = \begin{cases} \frac{x}{2}, & \forall x \in [0,1], \\ 0, & x = 3, \\ 1, & x = 5, \end{cases} \qquad \varphi(t) = \begin{cases} 1, & \forall t \in [0,1], \\ e^t, & \forall t \in (1,+\infty), \end{cases}$$

$$\phi(t) = \begin{cases} \frac{2}{3}t, & \forall t \in [0,\frac{3}{4}], \\ 2t-1, & \forall t \in (\frac{3}{4},1), \\ \frac{t^2}{1+t}, & \forall t \in [1,+\infty), \end{cases} \qquad \psi(t) = \begin{cases} \frac{1}{2}t, & \forall t \in [0,1], \\ t^2, & \forall t \in (1,+\infty). \end{cases}$$

It is easy to see that (a2) holds. Put $x, y \in X$ with $x < y$. To verify (3.1), we need to consider four possible cases as follows.

Case 1. Let $x, y \in [0,1]$. It follows that

$$\begin{aligned}
\int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt &= \int_0^{\psi(\frac{1}{2}|x-y|)} \varphi(t)\,dt = \frac{1}{4}|x-y| \le \frac{1}{3}|x-y| \\
&= \phi\left(\int_0^{\frac{1}{2}|x-y|} \varphi(t)\,dt\right) = \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\,dt\right);
\end{aligned}$$

Case 2. Let $x \in [0,1]$ and $y = 3$. Note that $|y-x| \ge 2$ and $e^{|y-x|^2} + 1 - e > e$. It follows that

$$\begin{aligned}
\int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt &= \int_0^{\psi(\frac{x}{2})} \varphi(t)\,dt = \frac{1}{4}x \le \frac{1}{4} < \frac{(e^{|y-x|^2}+1-e)^4}{1+(e^{|y-x|^2}+1-e)^2} \\
&= \phi\left(e^{|y-x|^2}+1-e\right) = \phi\left(\int_0^1 \varphi(t)\,dt + \int_1^{|y-x|^2} \varphi(t)\,dt\right) \\
&= \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\,dt\right);
\end{aligned}$$

Case 3. Let $x \in [0,1]$ and $y = 5$. Notice that $|y-x| \ge 4$. It follows that

$$\begin{aligned}
\int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt &= \int_0^{\psi(|1-\frac{x}{2}|)} \varphi(t)\,dt = \frac{1}{2} - \frac{x}{4} \le \frac{1}{2} \\
&< \frac{(e^{|y-x|^2}+1-e)^4}{1+(e^{|y-x|^2}+1-e)^2} = \phi\left(e^{|y-x|^2}+1-e\right)
\end{aligned}$$

$$\begin{aligned} &= \phi\left(\int_0^1 \varphi(t)\,dt + \int_1^{|y-x|^2} \varphi(t)\,dt\right) \\ &= \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\,dt\right); \end{aligned}$$

Case 4. Let $x = 3$ and $y = 5$. It follows that

$$\begin{aligned} \int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt &= \int_0^{\psi(1)} \varphi(t)\,dt = \int_0^{\frac{1}{2}} \varphi(t)\,dt = \frac{1}{2} < \frac{(e^4+1-e)^2}{1+e^4+1-e} \\ &= \phi\left(e^4+1-e\right) = \phi\left(1 + \int_1^4 \varphi(t)\,dt\right) \\ &= \phi\left(\int_0^1 \varphi(t)\,dt + \int_1^4 \varphi(t)\,dt\right) \\ &= \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\,dt\right). \end{aligned}$$

That is, (3.1) holds. Thus, Theorem 3.1 implies that T has a unique fixed point $0 \in X$ and $\lim_{n\to\infty} T^n x_0 = 0$ for each $x_0 \in X$.

Theorem 3.4 *Let (X,d) be a complete metric space, and let $T: X \to X$ be a mapping satisfying*

$$\int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt \le \phi\left(\int_0^{\psi(M(x,y))} \varphi(t)\,dt\right), \quad \forall x,y \in X, \tag{3.11}$$

where φ, ϕ and ψ satisfy (a1) *or* (a3). *Then T has a unique fixed point $a \in X$ and $\lim_{n\to\infty} T^n x_0 = a$ for each $x_0 \in X$.*

Proof Let x_0 be an arbitrary point in X. Put $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$. Assume that $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$. It is easy to see that x_{n_0-1} is a fixed point of T, and there is nothing to prove. Assume that $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$. From (3.11) and one of (a1) and (a3), we obtain that

$$\begin{aligned} \int_0^{\psi(d(x_{n+1},x_n))} \varphi(t)\,dt &= \int_0^{\psi(d(Tx_n,Tx_{n-1}))} \varphi(t)\,dt \le \phi\left(\int_0^{\psi(M(x_n,x_{n-1}))} \varphi(t)\,dt\right) \\ &< \int_0^{\psi(M(x_n,x_{n-1}))} \varphi(t)\,dt, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} &M(x_n,x_{n-1}) \\ &\quad = \max\left\{d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1}), \frac{1}{2}\left[d(x_n,Tx_{n-1}) + d(x_{n-1},Tx_n)\right]\right\} \\ &\quad = \max\left\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), \frac{1}{2}\left[d(x_n,x_n) + d(x_{n-1},x_{n+1})\right]\right\} \\ &\quad = \max\left\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\right\}, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.13}$$

Suppose that $d(x_{n_0},x_{n_0-1}) \le d(x_{n_0},x_{n_0+1})$ for some $n_0 \in \mathbb{N}$. It follows from (3.12) and (3.13) that

$$\int_0^{\psi(d(x_{n_0+1},x_{n_0}))} \varphi(t)\,dt < \int_0^{\psi(M(x_{n_0},x_{n_0-1}))} \varphi(t)\,dt = \int_0^{\psi(d(x_{n_0+1},x_{n_0}))} \varphi(t)\,dt,$$

which is a contradiction. Consequently, we deduce that

$$d(x_n,x_{n+1}) < d(x_n,x_{n-1}) \quad \text{and} \quad M(x_n,x_{n-1}) = d(x_n,x_{n-1}), \quad \forall n \in \mathbb{N}. \tag{3.14}$$

In view of (3.12), (3.14) and one of (a1) and (a3), we get that

$$\begin{aligned} \int_0^{\psi(d(x_{n+1},x_n))} \varphi(t)\,dt &\le \phi\left(\int_0^{\psi(M(x_n,x_{n-1}))} \varphi(t)\,dt\right) \\ &= \phi\left(\int_0^{\psi(d(x_n,x_{n-1}))} \varphi(t)\,dt\right) \\ &< \int_0^{\psi(d(x_n,x_{n-1}))} \varphi(t)\,dt, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.15}$$

which implies that

$$\psi\big(d(x_{n+1},x_n)\big) < \psi\big(d(x_n,x_{n-1})\big), \quad \forall n \in \mathbb{N}, \tag{3.16}$$

which means that there exists a constant $c \ge 0$ with $\lim_{n\to\infty} \psi(d(x_n,x_{n-1})) = c$.

Now we show that $c = 0$. Otherwise, $c > 0$. Taking upper limit in (3.15) and using Lemma 2.1 and one of (a1) and (a3), we conclude that

$$\begin{aligned} \int_0^c \varphi(t)\,dt &= \limsup_{n\to\infty} \int_0^{\psi(d(x_{n+1},x_n))} \varphi(t)\,dt \le \limsup_{n\to\infty} \phi\left(\int_0^{\psi(d(x_n,x_{n-1}))} \varphi(t)\,dt\right) \\ &\le \phi\left(\int_0^c \varphi(t)\,dt\right) < \int_0^c \varphi(t)\,dt, \end{aligned}$$

which is impossible. Hence $c = 0$, that is,

$$\lim_{n\to\infty} \psi\big(d(x_n,x_{n-1})\big) = 0,$$

which together with one of (a1) and (a3) gives that

$$\lim_{n\to\infty} d(x_n,x_{n-1}) = 0. \tag{3.17}$$

Next, we claim that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Suppose that $\{x_n\}_{n\in\mathbb{N}_0}$ is not a Cauchy sequence, which means that there is a constant $\varepsilon > 0$ such that for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)},x_{n(k)}) > \varepsilon.$$

For each positive integer k, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying the inequality above. Obviously, (3.6)-(3.8) hold. Note that

$$\begin{aligned}
&M(x_{m(k)-1}, x_{n(k)-1}) \\
&\quad = \max\Big\{ d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, Tx_{n(k)-1}), \\
&\quad\quad \frac{1}{2}\big[d(x_{m(k)-1}, Tx_{n(k)-1}) + d(x_{n(k)-1}, Tx_{m(k)-1})\big]\Big\} \\
&\quad = \max\Big\{ d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)}), \\
&\quad\quad \frac{1}{2}\big[d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)})\big]\Big\} \\
&\quad \le \max\Big\{ d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)}), \\
&\quad\quad \frac{1}{2}\big[d(x_{m(k)-1}, x_{n(k)-1}) + 2d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)})\big]\Big\}, \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.18}$$

Combining (3.8), (3.17) and (3.18), we infer that

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{3.19}$$

In light of (3.11), we deduce that

$$\begin{aligned}
\int_0^{\psi(d(x_{m(k)}, x_{n(k)}))} \varphi(t)\,dt &= \int_0^{\psi(d(Tx_{m(k)-1}, Tx_{n(k)-1}))} \varphi(t)\,dt \\
&\le \phi\left(\int_0^{\psi(M(x_{m(k)-1}, x_{n(k)-1}))} \varphi(t)\,dt\right), \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.20}$$

Assume that (a1) holds. Taking upper limit in (3.20) and using (3.8), (3.19) and Lemma 2.1, we get that

$$\begin{aligned}
\int_0^{\psi(\varepsilon)} \varphi(t)\,dt &= \limsup_{k\to\infty} \int_0^{\psi(d(x_{m(k)}, x_{n(k)}))} \varphi(t)\,dt \\
&\le \limsup_{k\to\infty} \phi\left(\int_0^{\psi(M(x_{m(k)-1}, x_{n(k)-1}))} \varphi(t)\,dt\right) \\
&\le \phi\left(\int_0^{\psi(\varepsilon)} \varphi(t)\,dt\right) < \int_0^{\psi(\varepsilon)} \varphi(t)\,dt,
\end{aligned}$$

which is a contradiction.

Assume that (a3) holds. Note that (3.19) implies that there exists $K \in \mathbb{N}$ with

$$M(x_{m(k)-1}, x_{n(k)-1}) > \frac{\varepsilon}{2}, \quad \forall k \ge K. \tag{3.21}$$

By virtue of (a3) and (3.21), we deduce that

$$\psi\big(M(x_{m(k)-1}, x_{n(k)-1})\big) > \psi\left(\frac{\varepsilon}{2}\right) > 0, \quad \forall k \ge K. \tag{3.22}$$

In terms of (3.20), (3.22) and (a3), we get that

$$\begin{aligned}\int_0^{\psi(d(x_{m(k)},x_{n(k)}))}\varphi(t)\,dt &\le \phi\left(\int_0^{\psi(M(x_{m(k)-1},x_{n(k)-1}))}\varphi(t)\,dt\right)\\ &< \int_0^{\psi(M(x_{m(k)-1},x_{n(k)-1}))}\varphi(t)\,dt, \quad \forall k\ge K,\end{aligned}$$

which yields that

$$\psi\bigl(d(x_{m(k)},x_{n(k)})\bigr) < \psi\bigl(M(x_{m(k)-1},x_{n(k)-1})\bigr), \quad \forall k\ge K,$$

that is,

$$d(x_{m(k)},x_{n(k)}) < M(x_{m(k)-1},x_{n(k)-1}), \quad \forall k\ge K,$$

which together with (3.6), (3.8) and (3.19) implies that

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \lim_{k\to\infty} M(x_{m(k)-1},x_{n(k)-1}) = \varepsilon. \tag{3.23}$$

Taking upper limit in (3.20) and using (3.23), (a3) and Lemma 2.1, we conclude that

$$\begin{aligned}\int_0^{\psi(\varepsilon+)}\varphi(t)\,dt &= \limsup_{k\to\infty}\int_0^{\psi(d(x_{m(k)},x_{n(k)}))}\varphi(t)\,dt\\ &\le \limsup_{k\to\infty}\phi\left(\int_0^{\psi(M(x_{m(k)-1},x_{n(k)-1}))}\varphi(t)\,dt\right)\\ &\le \phi\left(\int_0^{\psi(\varepsilon+)}\varphi(t)\,dt\right)\\ &< \int_0^{\psi(\varepsilon+)}\varphi(t)\,dt,\end{aligned}$$

which is a contradiction.

Thus, $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. It follows from completeness of (X,d) that there exists a point $a\in X$ with $\lim_{n\to\infty} x_n = a$.

Next, we show that a is a fixed point of T in X. Suppose that $a\neq Ta$. Notice that

$$\begin{aligned}M(x_n,a) &= \max\left\{d(x_n,a), d(x_n,Tx_n), d(a,Ta), \frac{1}{2}\bigl[d(x_n,Ta)+d(a,Tx_n)\bigr]\right\}\\ &= \max\left\{d(x_n,a), d(x_n,x_{n+1}), d(a,Ta), \frac{1}{2}\bigl[d(x_n,Ta)+d(a,x_{n+1})\bigr]\right\}\\ &\to d(a,Ta) \quad \text{as } n\to\infty,\end{aligned}$$

which guarantees that there exists $K_1\in\mathbb{N}$ satisfying

$$M(x_n,a) = d(a,Ta), \quad \forall n\ge K_1. \tag{3.24}$$

Let (a1) hold. In light of (3.11), (3.24) and Lemma 2.1, we infer that

$$\begin{aligned}\int_0^{\psi(d(a,Ta))}\varphi(t)\,dt &= \limsup_{n\to\infty}\int_0^{\psi(d(x_{n+1},Ta))}\varphi(t)\,dt\\ &= \limsup_{n\to\infty}\int_0^{\psi(d(Tx_n,Ta))}\varphi(t)\,dt\\ &\le \limsup_{n\to\infty}\phi\biggl(\int_0^{\psi(M(x_n,a))}\varphi(t)\,dt\biggr)\\ &= \phi\biggl(\int_0^{\psi(d(a,Ta))}\varphi(t)\,dt\biggr) < \int_0^{\psi(d(a,Ta))}\varphi(t)\,dt,\end{aligned}$$

which is a contradiction.

Let (a3) hold. In view of (3.11) and (3.24), we deduce that

$$\begin{aligned}\int_0^{\psi(d(x_{n+1},Ta))}\varphi(t)\,dt = \int_0^{\psi(d(Tx_n,Ta))}\varphi(t)\,dt &\le \phi\biggl(\int_0^{\psi(M(x_n,a))}\varphi(t)\,dt\biggr)\\ &= \phi\biggl(\int_0^{\psi(d(a,Ta))}\varphi(t)\,dt\biggr) < \int_0^{\psi(d(a,Ta))}\varphi(t)\,dt, \quad \forall n \ge K_1, \end{aligned}\tag{3.25}$$

which yields that

$$\psi\bigl(d(x_{n+1},Ta)\bigr) < \psi\bigl(d(a,Ta)\bigr), \quad \forall n \ge K_1,$$

that is,

$$d(x_{n+1},Ta) < d(a,Ta), \quad \forall n \ge K_1,$$

which together with (3.25) and $(\varphi,\phi,\psi) \in \Phi \times \Phi_2 \times \Phi_5$ means that

$$\begin{aligned}\int_0^{\psi(d(a,Ta))}\varphi(t)\,dt = \int_0^{\psi(d(a,Ta)-)}\varphi(t)\,dt &= \limsup_{n\to\infty}\int_0^{\psi(d(x_{n+1},Ta))}\varphi(t)\,dt\\ &\le \phi\biggl(\int_0^{\psi(d(a,Ta))}\varphi(t)\,dt\biggr) < \int_0^{\psi(d(a,Ta))}\varphi(t)\,dt,\end{aligned}$$

which is a contradiction.

Hence T has a fixed point $a \in X$. Finally, we show that a is a unique fixed point of T in X. Suppose that T has another fixed point $b \in X \setminus \{a\}$. It follows from (3.11) and one of (a1) and (a3) that

$$\begin{aligned}0 &< \int_0^{\psi(d(a,b))}\varphi(t)\,dt = \int_0^{\psi(d(Ta,Tb))}\varphi(t)\,dt\\ &\le \phi\biggl(\int_0^{\psi(d(a,b))}\varphi(t)\,dt\biggr) < \int_0^{\psi(d(a,b))}\varphi(t)\,dt,\end{aligned}$$

which is a contradiction. This completes the proof. □

Remark 3.5 Theorem 3.4 extends Theorem 2 in [10]. The following example is an application of Theorem 3.4.

Example 3.6 Let $X = [0,1] \cup \{2n : n \in \mathbb{N}\}$ be endowed with the Euclidean metric $d = |\cdot|$. Define $T : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Tx = \begin{cases} \frac{x}{2}, & \forall x \in [0,1], \\ \frac{1}{x}, & \forall x \in \{2n : n \in \mathbb{N}\}, \end{cases} \qquad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+,$$

$$\phi(t) = \begin{cases} \frac{t}{2}, & \forall t \in [0,1], \\ \frac{t^2}{1+t}, & \forall t \in (1,+\infty), \end{cases} \qquad \psi(t) = \begin{cases} \frac{t}{2}, & \forall t \in [0,1], \\ \frac{\sqrt{t}}{2}, & \forall t \in (1,+\infty). \end{cases}$$

Clearly, (a1) holds, ϕ and ψ are strictly increasing in $\mathbb{R}^+$. Put $x, y \in X$ with $x < y$. To prove (3.11), we need to consider three possible cases as follows.

Case 1. Let $x, y \in [0,1]$. It follows that

$$M(x,y) = \max\left\{ |x-y|, \left|x - \frac{1}{2}x\right|, \left|y - \frac{1}{2}y\right|, \frac{1}{2}\left(\left|x - \frac{1}{2}y\right| + \left|y - \frac{1}{2}x\right|\right)\right\} \in (0,1]$$

and

$$\begin{aligned} \int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt &= \int_0^{\psi(\frac{1}{2}|x-y|)} \varphi(t)\,dt = \frac{1}{16}(x-y)^2 \le \frac{1}{8}(x-y)^2 \\ &= \phi\left(\frac{1}{4}(x-y)^2\right) = \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\,dt\right) \le \phi\left(\int_0^{\psi(M(x,y))} \varphi(t)\,dt\right). \end{aligned}$$

Case 2. Let $x, y \in \{2n : n \in \mathbb{N}\}$. Notice that

$$M(x,y) = \max\left\{ |x-y|, \left|x - \frac{1}{x}\right|, \left|y - \frac{1}{y}\right|, \frac{1}{2}\left(\left|x - \frac{1}{y}\right| + \left|y - \frac{1}{x}\right|\right)\right\} > 1.$$

Suppose that $M(x,y) \in (1,4]$. It follows that

$$\begin{aligned} \int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt &= \int_0^{\psi(|\frac{1}{x}-\frac{1}{y}|)} \varphi(t)\,dt = \int_0^{\frac{1}{2}|\frac{1}{x}-\frac{1}{y}|} \varphi(t)\,dt = \frac{1}{4}\left(\frac{1}{x} - \frac{1}{y}\right)^2 \\ &< \frac{1}{16} < \frac{1}{8}M(x,y) = \phi\left(\frac{1}{4}M(x,y)\right) = \phi\left(\int_0^{\frac{1}{2}\sqrt{M(x,y)}} \varphi(t)\,dt\right) \\ &= \phi\left(\int_0^{\psi(M(x,y))} \varphi(t)\,dt\right). \end{aligned}$$

Suppose that $M(x,y) > 4$. It follows that

$$\begin{aligned} \int_0^{\psi(d(Tx,Ty))} \varphi(t)\,dt &= \int_0^{\psi(|\frac{1}{x}-\frac{1}{y}|)} \varphi(t)\,dt = \int_0^{\frac{1}{2}|\frac{1}{x}-\frac{1}{y}|} \varphi(t)\,dt = \frac{1}{4}\left(\frac{1}{x} - \frac{1}{y}\right)^2 \\ &< \frac{1}{16} < \frac{M^2(x,y)}{16 + 4M(x,y)} = \frac{\frac{1}{16}M^2(x,y)}{1 + \frac{1}{4}M(x,y)} \\ &= \phi\left(\frac{1}{4}M(x,y)\right) = \phi\left(\int_0^{\frac{1}{2}\sqrt{M(x,y)}} \varphi(t)\,dt\right) \\ &= \phi\left(\int_0^{\psi(M(x,y))} \varphi(t)\,dt\right). \end{aligned}$$

Case 3. Let $x \in [0,1]$ and $y \in \{2n : n \in \mathbb{N}\}$. It follows that

$$M(x,y) = \max\left\{|x-y|, \left|x-\frac{x}{2}\right|, \left|y-\frac{1}{y}\right|, \frac{1}{2}\left(\left|x-\frac{1}{y}\right| + \left|y-\frac{x}{2}\right|\right)\right\} \ge \left|y-\frac{1}{y}\right| > 1$$

and

$$\begin{aligned}\int_0^{\psi(d(Tx,Ty))}\varphi(t)\,dt &= \int_0^{\psi(|\frac{x}{2}-\frac{1}{y}|)}\varphi(t)\,dt = \int_0^{\frac{1}{2}|\frac{x}{2}-\frac{1}{y}|}\varphi(t)\,dt \\ &= \frac{1}{4}\left(\frac{x}{2}-\frac{1}{y}\right)^2 \le \frac{1}{16} < \max\left\{\frac{1}{8}M(x,y), \frac{M^2(x,y)}{16+4M(x,y)}\right\} \\ &= \phi\left(\frac{1}{4}M(x,y)\right) = \phi\left(\int_0^{\psi(M(x,y))}\varphi(t)\,dt\right),\end{aligned}$$

that is, (3.11) holds. Thus, Theorem 3.4 implies that T has a unique fixed point $0 \in X$ and $\lim_{n\to\infty} T^n x_0 = 0$ for each $x_0 \in X$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the referees for useful comments and suggestions. This research was supported by the Science Research Foundation of Educational Department of Liaoning Province (L2012380).

Received: 19 June 2013 Accepted: 28 August 2013 Published: 07 Nov 2013

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10.1186/1687-1812-2013-267
Cite this article as: Liu et al.: Fixed point theorems for mappings satisfying contractive conditions of integral type. *Fixed Point Theory and Applications* 2013, 2013:267