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Fixed point theorems for fuzzy contractive mappings in quasi-pseudo-metric spaces

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Abstract
In this paper, we obtain some fixed point theorems for fuzzy mappings in a left K-sequentially complete quasi-pseudo-metric space and in a right K-sequentially complete quasi-pseudo-metric space, respectively. Our analysis is based on the fact that fuzzy fixed point results can be obtained from the fixed point theorem of multivalued mappings with closed values. It is observed that there are many situations in which the mappings are not contractive on the whole space but they may be contractive on its subsets. We feel that this feature of finding the fuzzy fixed points via closed balls was overlooked, and our paper will re-open the research activity into this area.
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1 Introduction

The fixed points of fuzzy mappings were initially studied by Weiss [1] and Butnariu [2]. Then Heilpern [3] initiated the idea of fuzzy contraction mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's [4] fixed point theorem for multivalued mappings. Afterward many authors [5–8] explored the fixed points for generalized fuzzy contractive mappings.

Gregori and Pastor [9] proved a fixed point theorem for fuzzy contraction mappings in left K-sequentially complete quasi-pseudo-metric spaces. Their result is a generalization of the result of Heilpern [3]. In [10] the authors considered a generalized contractive-type condition involving fuzzy mappings in left K-sequentially complete quasi-metric spaces and established the fixed point theorem which is an extension of [11, Theorem 2]. Moreover, the main result of [10] is a quasi-metric version of [11, Theorem 1]. Subsequently, several other authors studied the fixed points of fuzzy contractive mappings in quasi-pseudo-metric space.

In this paper, we establish some local versions of fixed point theorems involving fuzzy contractive mappings in left K-sequentially complete quasi-pseudo-metric spaces and right K-sequentially complete quasi-pseudo-metric spaces, respectively.

2 Preliminaries

Throughout this paper, the letter $\mathbb{N}$ denotes the set of positive integers. If A is a subset of a topological space (X,τ), we will denote by $\mathrm{cl}_\tau A$ the closure of A in (X,τ).

A quasi-pseudo-metric on a nonempty set X is a nonnegative real-valued function d on $X \times X$ such that, for all $x, y, z \in X$,

(i) $d(x,x) = 0$ and

(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

A set X along with a quasi-pseudo-metric d is called a quasi-pseudo-metric space.

Each quasi-pseudo-metric d on X induces a topology $\tau(d)$ which has as a base the family of all d-balls $B_\varepsilon(x)$, where $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$.

If d is a quasi-pseudo-metric on X, then the function d^{-1}, defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$, is also a quasi-pseudo-metric on X. By $d \wedge d^{-1}$ and $d \vee d^{-1}$ we denote $\min\{d, d^{-1}\}$ and $\max\{d, d^{-1}\}$, respectively.

Let d be a quasi-pseudo-metric on X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be

(i) left K-Cauchy [12] if for each $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \mathbb{N}$ with $m \geq n \geq k$.

(ii) right K-Cauchy [12] if for each $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \mathbb{N}$ with $n \geq m \geq k$.

A quasi-pseudo-metric space (X, d) is said to be left (right) K-sequentially complete [12] if each left (right) K-Cauchy sequence in (X, d) converges to some point in X (with respect to the topology $\tau(d)$).

Now let (X, d) be a quasi-pseudo-metric space and let A and B be nonempty subsets of X. Then the Hausdorff distance between subsets A and B is defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(a,B) = \inf\{d(a,x) : x \in B\}$.

Note that $H(A,B) \geq 0$ with $H(A,B) = 0$ iff $\mathrm{cl}\,A = \mathrm{cl}\,B$, $H(A,B) = H(B,A)$ and $H(A,B) \leq H(A,C) + H(C,B)$ for any nonempty subset A, $B$and C of X. Clearly, H is the usual Hausdorff distance if d is a metric on X.

A fuzzy set on X is an element of I^X where $I = [0,1]$. The α-level set of a fuzzy set A, denoted by A_α, is defined by $A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for $\alpha \in (0,1]$, $A_0 = \mathrm{cl}(\{x \in X : A(x) > 0\})$ and $[Tx]_\alpha$ when $A = Tx$ and T is a contraction.

Definition 2.1 Let (X, d) be a quasi-pseudo-metric space and (V, d_V) be a metric linear space. The families $W^*(X)$ and $W'(X)$ of fuzzy sets on (X, d) and $W(V)$ on (V, d_v) are defined by

$$W^*(X) = \{A \in I^X : A_1 \text{ is nonempty } d\text{-closed and } d^{-1}\text{-compact}\} \quad \text{(see [9])},$$

$$W'(X) = \{A \in I^X : A_1 \text{ is nonempty } d\text{-closed and } d\text{-compact}\} \quad \text{(see [13])},$$

$$W(V) = \left\{A \in I^V : A_\alpha \text{ is compact and convex in } V \text{ for each } \alpha \in [0,1] \text{ and } \sup_{x\in V} A(x) = 1\right\}.$$

Note that for a metric linear space (V, d_V),

$$W(V) \subset W^*(V) = W'(V) = \{A \in I^V : A_1 \textit{ is nonempty and } d\textit{-compact}\} \subset I^V.$$

Definition 2.2 [13] Let (X,d) be a quasi-pseudo-metric space and let $A,B \in W^*(X)$ (or $W'(X)$) and $\alpha \in [0,1]$. Then we define

$$p_\alpha(A,B) = d([A]_\alpha,[B]_\alpha) = \inf\{d(x,y) : x \in [A]_\alpha, y \in [B]_\alpha\},$$
$$D_\alpha(A,B) = H([A]_\alpha,[B]_\alpha),$$

where the Hausdorff metric H is deduced from the quasi-pseudo-metric d on X,

$$p(A,B) = \sup\{p_\alpha(A,B) : \alpha \in (0,1]\}, \qquad D(A,B) = \sup\{D_\alpha(A,B) : \alpha \in (0,1]\}.$$

It is easy to see that p_α is a non-decreasing function of α, and $p_1(A,B) = d([A]_1,[B]_1) = p(A,B)$.

Definition 2.3 Let X be an arbitrary set and Y be any quasi-pseudo-metric space. F is said to be a fuzzy mapping if F is a mapping from X into $W^*(Y)$ (or $W'(Y)$).

Definition 2.4 We say that x is a fixed point of the mapping $F : X \to I^X$ if $\{x\} \subset F(x)$.

Before establishing our main results, we require the following lemmas recorded from ([9, 13]).

Lemma 2.5 *Let (X,d) be a quasi-pseudo-metric space and let $x \in X$ and $A \in W^*(X)$ (or $W'(X)$). Then $\{x\} \subseteq A$ if and only if*

$$d(x,[A]_1) = 0 \qquad (\text{or } d([A]_1,x) = 0).$$

Lemma 2.6 *Let (X,d) be a quasi-pseudo-metric space and let $A \in W^*(X)$ (or $W'(X)$). Then*

$$d(x,[A]_\alpha) \le d(x,y) + d(y,[A]_\alpha) \qquad (\text{or } d([A]_\alpha,x) \le d([A]_\alpha,y) + d(y,x))$$

for any $x,y \in X$ and $\alpha \in (0,1]$.

Lemma 2.7 *Let (X,d) be a quasi-pseudo-metric space and let $\{x_0\} \subseteq A$. Then*

$$d(x_0,[B]_\alpha) \le D_\alpha(A,B) \qquad (\text{or } d([B]_\alpha,x_0) \le D_\alpha(B,A))$$

for each $A,B \in W^*(X)$ (or $W'(X)$) and $\alpha \in (0,1]$.

Lemma 2.8 *Suppose $K \neq \Phi$ is compact in the quasi-pseudo-metric space (X,d^{-1}) (or (X,d)). If $z \in X$, then there exists $k_0 \in K$ such that*

$$d(z,K) = d(z,k_0) \qquad (\text{or } d(K,z) = d(k_0,z)).$$

3 Fixed point theorems for fuzzy contractive maps

In the present section, we prove the local versions of fixed point results for fuzzy contraction mappings in a left (right) K-sequentially complete quasi-pseudo-metric space.

Theorem 3.1 *Let (X,d) be a left K-sequentially complete quasi-pseudo-metric space, $x_0 \in X$, $r > 0$ and $T : X \to W^*(X)$ be a fuzzy mapping. If there exists $k \in (0,1)$ such that*

$$D(Tx,Ty) \leq k\big(d^{-1} \wedge d\big)(x,y) \quad \text{for each } x,y \in \overline{B}_d(x_0,r)$$

and

$$d\big(x_0,[Tx_0]_1\big) < (1-k)r,$$

then there exists $x^* \in \overline{B}_d(x_0,r)$ such that $\{x^*\} \subset Tx^*$.

Proof We apply Lemma 2.8 to the nonempty d^{-1}-compact set $K = [Tx_0]_1$ and x_0 to find $x_1 \in [Tx_0]_1$ such that

$$\begin{aligned} d(x_0,x_1) &= d\big(x_0,[Tx_0]_1\big) \\ &< (1-k)r. \end{aligned}$$

It also implies that $x_1 \in \overline{B}_d(x_0,r)$.

We can write

$$kd(x_0,x_1) < k(1-k)r.$$

By Lemma 2.8, choose $x_2 \in [Tx_1]_1$ such that

$$\begin{aligned} d(x_1,x_2) &= d\big(x_1,[Tx_1]_1\big) \\ &\leq D_1(Tx_0,Tx_1) \\ &\leq D(Tx_0,Tx_1) \\ &\leq k\big(d^{-1} \wedge d\big)(x_0,x_1) \\ &\leq kd(x_0,x_1) \\ &< k(1-k)r. \end{aligned}$$

We can show that $x_2 \in \overline{B}_d(x_0,r)$ since

$$\begin{aligned} d(x_0,x_2) &\leq d(x_0,x_1) + d(x_1,x_2) \\ &< (1-k)r + k(1-k)r \\ &< (1-k)\big(1+k+k^2+\cdots\big)r = r. \end{aligned}$$

By Lemma 2.8, choose $x_3 \in [Tx_2]_1$ such that

$$\begin{aligned} d(x_2,x_3) &= d\big(x_2,[Tx_2]_1\big) \\ &\leq D_1(Tx_1,Tx_2) \\ &\leq D(Tx_1,Tx_2) \end{aligned}$$

$$\begin{aligned}
&\leq k\left(d^{-1} \wedge d\right)(x_1, x_2) \\
&\leq kd(x_1, x_2) \\
&\leq k^2 d(x_0, x_1) \\
&< k^2(1-k)r.
\end{aligned}$$

We can show that $x_3 \in \overline{B}_d(x_0, r)$ since

$$\begin{aligned}
d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\
&< (1-k)r + k(1-k)r + k^2(1-k)r \\
&< (1-k)\left(1 + k + k^2 + \cdots\right)r = r.
\end{aligned}$$

We follow the same procedure to obtain $\{x_n\} \subset Tx_{n-1}$ such that

$$d(x_{n-1}, x_n) < k^{n-1} d(x_0, x_1) < k^{n-1}(1-k)r \quad \text{for } n = 3, 4, 5, \ldots.$$

Now, to verify that $\{x_n\}$ is a left K-Cauchy sequence, for $n < m$, we have

$$d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) < \sum_{i=n}^{m-1} k^i d(x_0, x_1) < \left(\frac{k^n}{1-k}\right) d(x_0, x_1).$$

As $k \in (0,1)$ and (X, d) is a left K-sequentially complete quasi-pseudo-metric space, this implies that $\{x_n\}$ is a left K-Cauchy sequence in X. Therefore, there exists $x^* \in \overline{B}_d(x_0, r)$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, from Lemma 2.6 and Lemma 2.7, we get

$$\begin{aligned}
d\left(x^*, \left[Tx^*\right]_1\right) &\leq d\left(x^*, x_n\right) + d\left(x_n, \left[Tx^*\right]_1\right) \\
&\leq d\left(x^*, x_n\right) + D_1\left(Tx_{n-1}, Tx^*\right) \\
&\leq d\left(x^*, x_n\right) + D\left(Tx_{n-1}, Tx^*\right) \\
&\leq d\left(x^*, x_n\right) + k\left(d^{-1} \wedge d\right)\left(x_{n-1}, x^*\right) \\
&\leq d\left(x^*, x_n\right) + kd^{-1}\left(x_{n-1}, x^*\right) \\
&\leq d\left(x^*, x_n\right) + kd\left(x^*, x_{n-1}\right)
\end{aligned}$$

since $d(x^*, x_n)$ and $d(x^*, x_{n-1}) \to 0$ as $n \to \infty$. Thus we have

$$d\left(x^*, \left[Tx^*\right]_1\right) = 0.$$

Lemma 2.5 yields that $\{x^*\} \subset Tx^*$. □

We will furnish the following example in the support of the above result.

Example 3.2 Let $X = \mathbb{R} \cup \{\Upsilon\}$, where $\Upsilon \notin \mathbb{R}$. Define $d : X \times X \longrightarrow [0,\infty)$ by $d(x,y) = |x-y|$, for all $x,y \in \mathbb{R}$, $d(\Upsilon,\Upsilon) = 0$,

$$d(x,\Upsilon) = \begin{Bmatrix} -x+1, & x < -1 \\ 2, & -1 \leq x \leq 1 \\ x+1, & x > 1 \end{Bmatrix}$$

and

$$d(\Upsilon,x) = \begin{Bmatrix} -1-x, & x < -1 \\ 0, & -1 \leq x \leq 1 \\ x-1, & x > 1 \end{Bmatrix},$$

then (X,d) is a left K-sequentially complete quasi-pseudo-metric space. Now $T : X \longrightarrow W^*(X)$ defined as

$$Tx = \begin{Bmatrix} \chi_{\{-1-x\}}, & x < -1 \\ \chi_{\{\frac{-x}{2}\}}, & -1 \leq x \leq 1 \\ \chi_{\{x-1\}}, & x > 1 \\ \chi_{\{0\}}, & x = \Upsilon \end{Bmatrix}$$

is a fuzzy mapping. For $\alpha \in (0,1]$,

$$\begin{aligned}
[Tx]_\alpha &= \bigl\{t \in X : [Tx](t) \geq \alpha\bigr\} \\
&= \begin{Bmatrix} t \in X : \chi_{\{-1-x\}}(t) \geq \alpha, & x < -1 \\ t \in X : \chi_{\{\frac{-x}{2}\}}(t) \geq \alpha, & -1 \leq x \leq 1 \\ t \in X : \chi_{\{x-1\}}(t) \geq \alpha, & x > 1 \\ t \in X : \chi_{\{0\}}(t) \geq \alpha, & x = \Upsilon \end{Bmatrix} \\
&= \begin{Bmatrix} -1-x, & x < -1 \\ \frac{-x}{2}, & -1 \leq x \leq 1 \\ x-1, & x > 1 \\ 0, & x = \Upsilon \end{Bmatrix}.
\end{aligned}$$

Define $D : W^*(X) \times W^*(X) \longrightarrow [0,\infty)$ by

$$D(Tx,Ty) = \sup_{\alpha \in (0,1]} H\bigl([Tx]_\alpha,[Ty]_\alpha\bigr), \quad \text{for each } x,y \in \overline{B}_d(0,1),$$

where

$$H\bigl([Tx]_\alpha,[Ty]_\alpha\bigr) = \max\Bigl\{\sup_{a \in [Tx]_\alpha} d\bigl(a,[Ty]_\alpha\bigr), \sup_{b \in [Ty]_\alpha} d\bigl(b,[Tx]_\alpha\bigr)\Bigr\}.$$

Now, for $k = \frac{1}{2}$,

$$D(Tx, Ty) \leq k\big(d^{-1} \wedge d\big)(x, y) \quad \text{for each } x, y \in \overline{B}_d(0, 1),$$

and

$$d\big(0, [T0]_1\big) < (1 - k).$$

Then $0 \in \overline{B}_d(0, 1)$ such that $\{0\} \subset T0$.

Note that the fuzzy mapping defined in the above example is not contractive on the whole space; for example, whenever $x = 2$ and $y = \Upsilon$, then

$$D(T2, T\Upsilon) > k\big(d^{-1} \wedge d\big)(2, \Upsilon).$$

When (X, d) is a right K-sequentially complete quasi-pseudo-metric space, using Lemmas 2.5, 2.6, 2.7 and 2.8, for $W'(X)$, we get the following result.

Theorem 3.3 *Let (X, d) be a right K-sequentially complete quasi-pseudo-metric space, $x_0 \in X$, $r > 0$ and $T : X \to W'(X)$ be a fuzzy mapping. If there exists $k \in (0, 1)$ such that*

$$D(Tx, Ty) \leq k\big(d^{-1} \wedge d\big)(x, y) \quad \textit{for each } x, y \in \overline{B}_d(x_0, r),$$

and

$$d\big([Tx_0]_1, x_0\big) < (1 - k)r,$$

then T has a fuzzy fixed point $x^* \in \overline{B}_d(x_0, r)$ such that $\{x^*\} \subset Tx^*$.

The proof of Theorem 3.3 is similar to the proof of Theorem 3.1 and therefore omitted.

Remark 3.4 If (X, d) is a left K-sequentially complete quasi-pseudo-metric space, by imposing the contractive condition on the whole space X in Theorem 3.1, we get the following result of Gregori and Pastor [9].

Corollary 3.5 *Let (X, d) be a left K-sequentially complete quasi-pseudo-metric space and $T : X \to W^*(X)$ be a fuzzy mapping. If there exists $k \in (0, 1)$ such that*

$$D\big(T(x), T(y)\big) \leq k\big(d \wedge d^{-1}\big)(x, y) \quad \textit{for each } x, y \in X,$$

then there exists $x^* \in X$ such that $\{x^*\} \subset T(x^*)$.

Theorem 3.6 *Let (X, d) be a left K-sequentially complete quasi-pseudo-metric space, $x_0 \in X$, $r > 0$ and $T : X \to W^*(X)$ be a fuzzy mapping. If there exists $k \in (0, \frac{1}{2})$ such that*

$$D_1(Tx, Ty) \leq k \max\big\{\big(d^{-1} \wedge d\big)(x, y), d\big(x, [Tx]_1\big) + d\big(y, [Ty]_1\big)\big\} \quad \textit{for each } x, y \in \overline{B}_d(x_0, r),$$

and

$$d\big(x_0, [Tx_0]_1\big) < (1-k)r,$$

then there exists $x^* \in \overline{B}_d(x_0, r)$ *such that* $\{x^*\} \subset Tx^*$.

Proof We apply Lemma 2.8 to the nonempty d^{-1}-compact set $K = [Tx_0]_1$ and x_0 to find $x_1 \in [Tx_0]_1$ such that

$$\begin{aligned} d(x_0, x_1) &= d\big(x_0, [Tx_0]_1\big) \\ &< (1-k)r. \end{aligned}$$

It also implies that $x_1 \in \overline{B}_d(x_0, r)$.

We can write

$$kd(x_0, x_1) < k(1-k)r. \tag{1}$$

Now choose $x_2 \in X$ such that $x_2 \in [Tx_1]_1$. By Lemma 2.8, we get

$$\begin{aligned} d(x_1, x_2) &= d\big(x_1, [Tx_1]_1\big) \\ &\leq D_1(Tx_0, Tx_1) \\ &\leq k \max\big\{\big(d^{-1} \wedge d\big)(x_0, x_1), d\big(x_0, [Tx_0]_1\big) + d\big(x_1, [Tx_1]_1\big)\big\} \\ &\leq k \max\big\{d(x_0, x_1), d(x_0, x_1) + d(x_1, x_2)\big\}. \end{aligned} \tag{2}$$

Now we consider the following cases.

Case 1: If we consider $d(x_0, x_1)$ as a maximum in above inequality (2) and use inequality (1), we get

$$d(x_1, x_2) \leq kd(x_0, x_1) < k(1-k)r.$$

Case 2: If we consider $d(x_0, x_1) + d(x_1, x_2)$ as a maximum in inequality (2), we have

$$\begin{aligned} d(x_1, x_2) &\leq k\big\{d(x_0, x_1) + d(x_1, x_2)\big\} \\ &\leq \left(\frac{k}{1-k}\right) d(x_0, x_1). \end{aligned}$$

Note that $(\frac{k}{1-k}) < k$, using inequality (1), we get

$$d(x_1, x_2) \leq kd(x_0, x_1) < k(1-k)r.$$

It follows from the above two cases that

$$d(x_1, x_2) < k(1-k)r.$$

We can show that $x_2 \in \overline{B}_d(x_0, r)$ since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &< (1-k)r + k(1-k)r \\ &< (1-k)\bigl(1 + k + k^2 + \cdots\bigr)r = r. \end{aligned}$$

We follow the same procedure to obtain $\{x_n\} \subset Tx_{n-1}$ such that

$$d(x_{n-1}, x_n) < k^{n-1} d(x_0, x_1) < k^{n-1}(1-k)r \quad \text{for } n = 3, 4, 5, \ldots.$$

Now, to verify that $\{x_n\}$ is a left K-Cauchy sequence, for $n < m$, we have

$$d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) < \sum_{i=n}^{m-1} k^i d(x_0, x_1) < \left(\frac{k^n}{1-k}\right) d(x_0, x_1).$$

As $k \in (0, \frac{1}{2})$ and (X, d) is a left K-sequentially complete quasi-pseudo-metric space, this implies that $\{x_n\}$ is a left K-Cauchy sequence in X. Therefore, there exists $x^* \in \overline{B}_d(x_0, r)$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, from Lemma 2.6 and Lemma 2.7, we get

$$\begin{aligned} &d\bigl(x^*, [Tx^*]_1\bigr) \\ &\quad\leq d\bigl(x^*, x_n\bigr) + d\bigl(x_n, [Tx^*]_1\bigr) \\ &\quad\leq d\bigl(x^*, x_n\bigr) + D_1\bigl(Tx_{n-1}, Tx^*\bigr) \\ &\quad\leq d\bigl(x^*, x_n\bigr) + k \max\bigl\{\bigl(d^{-1} \wedge d\bigr)\bigl(x_{n-1}, x^*\bigr), d\bigl(x_{n-1}, [Tx_{n-1}]_1\bigr) + d\bigl(x^*, [Tx^*]_1\bigr)\bigr\} \\ &\quad\leq d\bigl(x^*, x_n\bigr) + k \max\bigl\{d^{-1}\bigl(x_{n-1}, x^*\bigr), d(x_{n-1}, x_{n-2}) + d\bigl(x^*, [Tx^*]_1\bigr)\bigr\} \\ &\quad\leq d\bigl(x^*, x_n\bigr) + k \max\bigl\{d\bigl(x^*, x_{n-1}\bigr), d(x_{n-1}, x_{n-2}) + d\bigl(x^*, [Tx^*]_1\bigr)\bigr\} \end{aligned}$$

since $d(x^*, x_n)$ and $d(x^*, x_{n-1})$ and $d(x_{n-1}, x_{n-2}) \to 0$ as $n \to \infty$. Thus we have

$$\begin{aligned} &d\bigl(x^*, [Tx^*]_1\bigr) \leq k \max\bigl\{d\bigl(x^*, [Tx^*]_1\bigr)\bigr\}, \\ &(1-k) d\bigl(x^*, [Tx^*]_1\bigr) \leq 0, \\ &d\bigl(x^*, [Tx^*]_1\bigr) = 0. \end{aligned}$$

Lemma 2.5 yields that $\{x^*\} \subset Tx^*$. □

Example 3.7 Let (X, d) be the left K-sequentially complete quasi-pseudo-metric space of Example 3.2. Now $T : X \longrightarrow W^*(X)$ defined as

$$Tx = \begin{cases} \chi_{\{-1-x\}}, & x < -1 \\ \chi_{\{\frac{-x}{3}\}}, & -1 \leq x \leq 1 \\ \chi_{\{x-1\}}, & x > 1 \\ \chi_{\{0\}}, & x = \Upsilon \end{cases}$$

is a fuzzy mapping. For $\alpha \in (0,1]$,

$$\begin{aligned}[Tx]_\alpha &= \{t \in X : [Tx](t) \geq \alpha\} \\ &= \begin{Bmatrix} t \in X : \chi_{\{-1-x\}}(t) \geq \alpha, & x < -1 \\ t \in X : \chi_{\{\frac{-x}{3}\}}(t) \geq \alpha, & -1 \leq x \leq 1 \\ t \in X : \chi_{\{x-1\}}(t) \geq \alpha, & x > 1 \\ t \in X : \chi_{\{0\}}(t) \geq \alpha, & x = \Upsilon \end{Bmatrix} \\ &= \begin{Bmatrix} -1-x, & x < -1 \\ \frac{-x}{3}, & -1 \leq x \leq 1 \\ x-1, & x > 1 \\ 0, & x = \Upsilon \end{Bmatrix}.\end{aligned}$$

Define $D_1 : W^*(X) \times W^*(X) \longrightarrow [0,\infty)$ by

$$D_1(Tx, Ty) = H([Tx]_1, [Ty]_1) \quad \text{for each } x, y \in \overline{B}_d(0,1),$$

where

$$H([Tx]_\alpha, [Ty]_\alpha) = \max\Big\{\sup_{a \in [Tx]_\alpha} d(a, [Ty]_\alpha), \sup_{b \in [Ty]_\alpha} d(b, [Tx]_\alpha)\Big\}.$$

Now, for $k = \frac{1}{3}$,

$$D_1(Tx, Ty) \leq k \max\{(d^{-1} \wedge d)(x,y), d(x, [Tx]_1) + d(y, [Ty]_1)\} \quad \text{for each } x, y \in \overline{B}_d(0,1),$$

and

$$d(0, [T0]_1) < (1-k).$$

Then $0 \in \overline{B}_d(0,1)$ such that $\{0\} \subset T0$.

If (X,d) is a right K-sequentially complete quasi-pseudo-metric space, using Lemmas 2.5, 2.6, 2.7 and 2.8 for $W'(X)$, we get the following result.

Theorem 3.8 *Let (X,d) be a right K-sequentially complete quasi-pseudo-metric space, $x_0 \in X$, $r > 0$ and $T : X \to W'(X)$ be a fuzzy mapping. If there exists $k \in (0, \frac{1}{2})$ such that*

$$D_1(Tx, Ty) \leq k \max\{(d^{-1} \wedge d)(x,y), d([Tx]_1, x) + d([Ty]_1, y)\} \quad \textit{for each } x, y \in \overline{B}_d(x_0, r),$$

and

$$d([Tx_0]_1, x_0) < (1-k)r,$$

then T has a fuzzy fixed point $x^* \in \overline{B}_d(x_0, r)$ such that $\{x^*\} \subset Tx^*$.

The proof of Theorem 3.8 is similar to the proof of Theorem 3.6 and therefore omitted.

Theorem 3.9 *Let (X, d) be a left K-sequentially complete quasi-pseudo-metric space, $x_0 \in X$, $r > 0$ and $T : X \to W^*(X)$ be a fuzzy mapping. If there exists $k \in (0, \frac{1}{2})$ such that*

$$D_1(Tx, Ty) \leq k \max\{(d^{-1} \wedge d)(x, y), d(x, [Ty]_1) + d(y, [Tx]_1)\} \quad \text{for each } x, y \in \overline{B}_d(x_0, r),$$

and

$$d(x_0, [Tx_0]_1) < (1 - k)r,$$

then there exists $x^* \in \overline{B}_d(x_0, r)$ such that $\{x^*\} \subset Tx^*$.

Proof We apply Lemma 2.8 to the nonempty d^{-1}-compact set $K = [Tx_0]_1$ and x_0 to find $x_1 \in [Tx_0]_1$ such that

$$\begin{aligned} d(x_0, x_1) &= d(x_0, [Tx_0]_1) \\ &< (1 - k)r. \end{aligned}$$

It also implies that $x_1 \in \overline{B}_d(x_0, r)$.

We can write

$$kd(x_0, x_1) < k(1 - k)r. \tag{3}$$

Now choose $x_2 \in X$ such that $x_2 \in [Tx_1]_1$. By Lemma 2.8, we get

$$\begin{aligned} d(x_1, x_2) &= d(x_1, [Tx_1]_1) \\ &\leq D_1(Tx_0, Tx_1) \\ &\leq k \max\{(d^{-1} \wedge d)(x_0, x_1), d(x_0, [Tx_1]_1) + d(x_1, [Tx_0]_1)\} \\ &\leq k \max\{d(x_0, x_1), d(x_0, x_2) + d(x_1, x_1)\} \\ &\leq k \max\{d(x_0, x_1), d(x_0, x_1) + d(x_1, x_2)\}. \end{aligned} \tag{4}$$

Now we consider the following cases.

Case 1: If we consider $d(x_0, x_1)$ as a maximum in above inequality (4) and use inequality (3), we get

$$d(x_1, x_2) \leq kd(x_0, x_1) < k(1 - k)r.$$

Case 2: If we consider $d(x_0, x_1) + d(x_1, x_2)$ as a maximum in inequality (4), we have

$$\begin{aligned} d(x_1, x_2) &\leq k\{d(x_0, x_1) + d(x_1, x_2)\} \\ &\leq \left(\frac{k}{1 - k}\right) d(x_0, x_1). \end{aligned}$$

Note that $(\frac{k}{1-k}) < k$, using inequality (3), we get

$$d(x_1, x_2) \leq kd(x_0, x_1) < k(1-k)r.$$

It follows from the above two cases that

$$d(x_1, x_2) < k(1-k)r.$$

We can show that $x_2 \in \overline{B}_d(x_0, r)$ since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &< (1-k)r + k(1-k)r \\ &< (1-k)\bigl(1 + k + k^2 + \cdots\bigr)r = r. \end{aligned}$$

We follow the same procedure to obtain $\{x_n\} \subset Tx_{n-1}$ such that

$$d(x_{n-1}, x_n) < k^{n-1}d(x_0, x_1) < k^{n-1}(1-k)r \quad \text{for } n = 3, 4, 5, \ldots.$$

Now, to verify that $\{x_n\}$ is a left K-Cauchy sequence, for $n < m$, we have

$$d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) < \sum_{i=n}^{m-1} k^i d(x_0, x_1) < \left(\frac{k^n}{1-k}\right) d(x_0, x_1).$$

As $k \in (0, \frac{1}{2})$ and (X, d) is a left K-sequentially complete quasi-pseudo-metric space, this implies that $\{x_n\}$ is a left K-Cauchy sequence in X. Therefore, there exists $x^* \in \overline{B}_d(x_0, r)$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, from Lemma 2.6 and Lemma 2.7, we get

$$\begin{aligned} &d\bigl(x^*, [Tx^*]_1\bigr) \\ &\quad \leq d\bigl(x^*, x_n\bigr) + d\bigl(x_n, [Tx^*]_1\bigr) \\ &\quad \leq d\bigl(x^*, x_n\bigr) + D_1\bigl(Tx_{n-1}, Tx^*\bigr) \\ &\quad \leq d\bigl(x^*, x_n\bigr) + k \max\bigl\{\bigl(d^{-1} \wedge d\bigr)\bigl(x_{n-1}, x^*\bigr), d\bigl(x_{n-1}, [Tx^*]_1\bigr) + d\bigl(x^*, [Tx_{n-1}]_1\bigr)\bigr\} \\ &\quad \leq d\bigl(x^*, x_n\bigr) + k \max\bigl\{d^{-1}\bigl(x_{n-1}, x^*\bigr), d\bigl(x_{n-1}, [Tx^*]_1\bigr) + d\bigl(x^*, x_{n-2}\bigr)\bigr\} \\ &\quad \leq d\bigl(x^*, x_n\bigr) + k \max\bigl\{d\bigl(x^*, x_{n-1}\bigr), d\bigl(x_{n-1}, [Tx^*]_1\bigr) + d\bigl(x^*, x_{n-2}\bigr)\bigr\} \end{aligned}$$

since $d(x^*, x_n)$ and $d(x^*, x_{n-1})$ and $d(x^*, x_{n-2}) \to 0$ as $n \to \infty$. Thus we have

$$\begin{aligned} &d\bigl(x^*, [Tx^*]_1\bigr) \leq k \max\bigl\{d\bigl(x^*, [Tx^*]_1\bigr)\bigr\}, \\ &(1-k)d\bigl(x^*, [Tx^*]_1\bigr) \leq 0, \\ &d\bigl(x^*, [Tx^*]_1\bigr) = 0. \end{aligned}$$

Lemma 2.5 yields that $\{x^*\} \subset Tx^*$. □

Example 3.10 Let (X,d) be the left K-sequentially complete quasi-pseudo-metric space of Example 3.2. Now $T : X \longrightarrow W^*(X)$ defined as

$$Tx = \begin{Bmatrix} \chi_{\{-1-x\}}, & x < -1 \\ \chi_{\{\frac{-x}{4}\}}, & -1 \le x \le 1 \\ \chi_{\{x-1\}}, & x > 1 \\ \chi_{\{0\}}, & x = \Upsilon \end{Bmatrix}$$

is a fuzzy mapping. For $\alpha \in (0,1]$,

$$\begin{aligned} [Tx]_\alpha &= \bigl\{t \in X : [Tx](t) \ge \alpha\bigr\} \\ &= \begin{Bmatrix} t \in X : \chi_{\{-1-x\}}(t) \ge \alpha, & x < -1 \\ t \in X : \chi_{\{\frac{-x}{4}\}}(t) \ge \alpha, & -1 \le x \le 1 \\ t \in X : \chi_{\{x-1\}}(t) \ge \alpha, & x > 1 \\ t \in X : \chi_{\{0\}}(t) \ge \alpha, & x = \Upsilon \end{Bmatrix} \\ &= \begin{Bmatrix} -1-x, & x < -1 \\ \frac{-x}{4}, & -1 \le x \le 1 \\ x-1, & x > 1 \\ 0, & x = \Upsilon \end{Bmatrix}. \end{aligned}$$

Now, for $k = \frac{1}{4}$,

$$D_1(Tx, Ty) \le k \max\bigl\{\bigl(d^{-1} \wedge d\bigr)(x,y), d\bigl(x, [Ty]_1\bigr) + d\bigl(y, [Tx]_1\bigr)\bigr\} \quad \text{for each } x, y \in \overline{B}_d(0,1),$$

and

$$d\bigl(0, [T0]_1\bigr) < (1-k).$$

Then $0 \in \overline{B}_d(0,1)$ such that $\{0\} \subset T0$.

If (X,d) is a right K-sequentially complete quasi-pseudo-metric space, using Lemmas 2.5, 2.6, 2.7 and 2.8 for $W'(X)$, we get the following result.

Theorem 3.11 *Let (X,d) be a right K-sequentially complete quasi-pseudo-metric space, $x_0 \in X$, $r > 0$ and $T : X \to W'(X)$ be a fuzzy mapping. If there exists $k \in (0, \frac{1}{2})$ such that*

$$D_1(Tx, Ty) \le k \max\bigl\{\bigl(d^{-1} \wedge d\bigr)(x,y), d\bigl([Ty]_1, x\bigr) + d\bigl([Tx]_1, y\bigr)\bigr\} \quad \text{for each } x, y \in \overline{B}_d(x_0, r),$$

and

$$d\bigl([Tx_0]_1, x_0\bigr) < (1-k)r,$$

then T has a fuzzy fixed point $x^* \in \overline{B}_d(x_0, r)$ such that $\{x^*\} \subset Tx^*$.

The proof of Theorem 3.11 is similar to the proof of Theorem 3.9 and therefore omitted.

4 Conclusion

From the application point of view, it often happens that a mapping T is a fuzzy contraction on a subset Y of X but not on the entire space X. However, if Y is closed, then it is complete, so that T has a fuzzy fixed x in Y, provided we impose a restriction on the choice of x_o, so that the sequence x_m remains in the closed subset Y. In this paper, we used this method to find fixed points of fuzzy mappings on a left (right) K-sequentially complete quasi-pseudo-metric space X.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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