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Strong convergence of viscosity approximation methods for the fixed-point of pseudo-contractive and monotone mappings

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Abstract
In this paper, we introduce a viscosity iterative process, which converges strongly to a common element of the set of fixed points of a pseudo-contractive mapping and the set of solutions of a monotone mapping. We also prove that the common element is the unique solution of certain variational inequality. The strong convergence theorems are obtained under some mild conditions. The results presented in this paper extend and unify most of the results that have been proposed for this class of nonlinear mappings.

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Keywords: pseudo-contractive mappings; monotone mappings; fixed point; variational inequalities; viscosity approximation

1 Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called monotone if and only if

$$\langle x - y, Ax - Ay\rangle \geq 0, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.2}$$

Obviously, the class of monotone mappings includes the class of the α-inverse strongly monotone mappings.

A mapping $T : C \to H$ is called pseudo-contractive if $\forall x, y \in C$, we have

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2. \tag{1.3}$$

A mapping $T : C \to H$ is called κ-strict pseudo-contractive, if there exists a constant $0 \leq \kappa \leq 1$ such that

$$\langle x - y, Tx - Ty\rangle \leq \|x - y\|^2 - \kappa\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C. \tag{1.4}$$

A mapping $T: C \to C$ is called non-expansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.5}$$

Clearly, the class of pseudo-contractive mappings includes the class of strict pseudo-contractive mappings and non-expansive mappings. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

A mapping $f: C \to C$ is called contractive with a contraction coefficient if there exists a constant $\rho \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \tag{1.6}$$

For finding an element of the set of fixed points of the non-expansive mappings, Halpern [1] was the first to study the convergence of the scheme in 1967

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) T(x_n). \tag{1.7}$$

In 2000, Moudafi [2] introduced the viscosity approximation methods and proved the strong convergence of the following iterative algorithm under some suitable conditions

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(x_n). \tag{1.8}$$

Viscosity approximation methods are very important, because they are applied to convex optimization, linear programming, monotone inclusions and elliptic differential equations. In a Hilbert space, many authors have studied the fixed points problems of the fixed points for the non-expansive mappings and monotone mappings by the viscosity approximation methods, and obtained a series of good results, see [3–18].

Suppose that A is a monotone mapping from C into H. The classical variational inequality problem is formulated as finding a point $u \in C$ such that $\langle v - u, Au \rangle \geq 0$, $\forall v \in C$. The set of solutions of variational inequality problems is denoted by $VI(C, A)$.

Takahashi [19, 20] introduced the following scheme and studied the weak and strong convergence theorem of the elements of $F(T) \cap VI(C, A)$, respectively, under different conditions

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_{n+1}) TP_C(x_n - \lambda_n x_n), \tag{1.9}$$

where T is a non-expansive mapping, A is an α-inverse strong monotone operator.

Recently, Zegeye and Shahzad [21] introduced the algorithms and obtained the strong convergence theorems in a Hilbert space, respectively,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n \tag{1.10}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_{r_n} F_{r_n} x_n, \tag{1.11}$$

where T_n are asymptotically non-expansive mappings, and T_{r_n}, F_{r_n} are non-expansive mappings.

Our concern is now the following: Is it possible to construct a new sequence that converges strongly to a common element of the intersection of the set of fixed points of a pseudo-contractive mapping and the solution set of a variational inequality problem for a monotone mapping?

2 Preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space H, a mapping $P_C : H \to C$ is called the metric projection, if $\forall x \in H$, there exists a unique point in C, denote by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

It is well known that P_C is a non-expansive mapping, and $P_C x$ have the property as follows:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall x \in H, y \in C, \tag{2.1}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C. \tag{2.2}$$

Lemma 2.1 [6] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \le (1 - \theta_n)a_n + \sigma_n, \quad n \ge 0,$$

where $\{\theta_n\}$ is a sequence in (0,1) *and $\{\sigma_n\}$ is a real sequence such that*

(i) $\sum_{n=0}^{\infty} \theta_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\sigma_n}{\theta_n} \le 0$ *or* $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 [21] *Let C be a closed convex subset of a Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping, let $T : C \to C$ be a continuous pseudo-contractive mapping, define mappings T_r and F_r as follows*: $x \in H$, $r \in (0,\infty)$

$$T_r(x) = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\bigl\langle y - z, (1 + r)z - x\bigr\rangle \le 0, \forall y \in C \right\},$$

$$F_r(x) = \left\{ z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r and F_r are single-valued*;

(ii) *T_r and F_r are firmly non-expansive mappings, i.e.,* $\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle$, $\|F_r x - F_r y\|^2 \le \langle F_r x - F_r y, x - y\rangle$;

(iii) $F(T_r) = F(T)$, $F(F_r) = VI(C,A)$;

(iv) *$F(T)$ and $VI(C,A)$ are closed convex.*

Lemma 2.3 [22] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequence in a Banach space, and let $\{\beta_n\}$ be a sequence in $[0,1]$, which satisfies the following condition*:

$$0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \quad n \geq 0$$

and

$$\lim_{n\to\infty} \bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Let C be a closed convex subset of a Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping, let $T : C \to C$ be a continuous pseudo-contractive mapping. Then we define the mappings as follows: for $x \in H$, $\tau_n \in (0,\infty)$

$$T_{\tau_n}(x) = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{\tau_n}\bigl\langle y - z, (1+\tau_n)z - x\bigr\rangle \leq 0, \forall y \in C \right\}, \tag{2.3}$$

$$F_{\tau_n}(x) = \left\{ z \in C : \langle y - z, Az\rangle + \frac{1}{\tau_n}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}. \tag{2.4}$$

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping, let $A : C \to H$ be a continuous monotone mapping such that $F = F(T) \cap VI(C,A) \neq \emptyset$, let $f : C \to C$ be a contraction with a contraction coefficient $\rho \in (0,1)$. The mappings T_{τ_n} and F_{τ_n} are defined as (2.3) and (2.4), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$*

$$\begin{cases} y_n = \lambda_n x_n + (1-\lambda_n)F_{\tau_n}x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{\tau_n} y_n, \end{cases} \tag{3.1}$$

where $\lambda_n \in [0,1]$, let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences of nonnegative real numbers in $[0,1]$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 1$;
(iv) $\liminf_{n\to\infty} \tau_n > 0$, $\sum_{n=1}^{\infty} |\tau_{n+1} - \tau_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_F f(\bar{x})$, and also $\bar{x}$ is the unique solution of the variational inequality

$$\bigl\langle f(\bar{x}) - \bar{x}, y - \bar{x}\bigr\rangle \leq 0, \quad \forall y \in F. \tag{3.2}$$

Proof First, we prove that $\{x_n\}$ is bounded. Take $p \in F$, then we have from Lemmas 2.2 that

$$\|y_n - p\| \leq \lambda_n \|x_n - p\| + (1-\lambda_n)\|F_{\tau_n}x_n - F_{\tau_n}p\| \leq \|x_n - p\|. \tag{3.3}$$

For $n \geq 0$, because T_{τ_n} and F_{τ_n} are non-expansive, and f is contractive, we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \big\|\alpha_n\big(f(x_n) - p\big) + \beta_n(x_n - p) + \gamma_n(T_{\tau_n}y_n - p)\big\| \\
&\quad \leq \alpha_n\big\|f(x_n) - f(p)\big\| + \alpha_n\big\|f(p) - p\big\| + \beta_n\|x_n - p\| + \gamma_n\|T_{\tau_n}y_n - p\| \\
&\quad \leq \rho\alpha_n\|x_n - p\| + \alpha_n\big\|f(p) - p\big\| + (1 - \alpha_n)\|x_n - p\| \\
&\quad \leq \big[1 - (1 - \rho)\alpha_n\big]\|x_n - p\| + \alpha_n\big\|f(p) - p\big\| \\
&\quad \leq \max\left\{\left\|x_0 - p, \frac{f(p) - p}{1 - \rho}\right\|\right\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{F_{\tau_n}x_n\}$, $\{T_{\tau_n}y_n\}$ and $\{y_n\}$, $\{f(x_n)\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$.

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \lambda_{n+1}\|x_{n+1} - x_n\| + (1 - \lambda_{n+1})\|F_{\tau_{n+1}}x_{n+1} - F_{\tau_n}x_n\| \\
&\quad + |\lambda_{n+1} - \lambda_n|\|x_n - F_{\tau_n}x_n\|.
\end{aligned} \tag{3.4}$$

Let $v_n = F_{\tau_n}x_n$, $v_{n+1} = F_{\tau_{n+1}}x_{n+1}$, by the definition of the mapping F_{τ_n}, we have that

$$\langle y - v_n, Av_n\rangle + \frac{1}{\tau_n}\langle y - v_n, v_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.5}$$

$$\langle y - v_{n+1}, Av_{n+1}\rangle + \frac{1}{\tau_{n+1}}\langle y - v_{n+1}, v_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{3.6}$$

Putting $y := v_{n+1}$ in (3.5), and letting $y := v_n$ in (3.6), we have that

$$\langle v_{n+1} - v_n, Av_n\rangle + \frac{1}{\tau_n}\langle v_{n+1} - v_n, v_n - x_n\rangle \geq 0, \tag{3.7}$$

$$\langle v_n - v_{n+1}, Av_{n+1}\rangle + \frac{1}{\tau_{n+1}}\langle v_n - v_{n+1}, v_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.8}$$

Adding (3.7) and (3.8), we have that

$$\langle v_{n+1} - v_n, Av_n - Av_{n+1}\rangle + \left\langle v_{n+1} - v_n, \frac{v_n - x_n}{\tau_n} - \frac{v_{n+1} - x_{n+1}}{\tau_{n+1}}\right\rangle \geq 0.$$

Since A is a monotone mapping, which implies that

$$\left\langle v_{n+1} - v_n, \frac{v_n - x_n}{\tau_n} - \frac{v_{n+1} - x_{n+1}}{\tau_{n+1}}\right\rangle \geq 0.$$

Therefore, we have that

$$\left\langle v_{n+1} - v_n, v_n - x_n - \frac{\tau_n(v_{n+1} - x_{n+1})}{\tau_{n+1}} + v_{n+1} - v_{n+1}\right\rangle \geq 0,$$

i.e.,

$$\begin{aligned}\|v_{n+1}-v_n\|^2 &\le \left\langle v_{n+1}-v_n, x_{n+1}-x_n+\left(1-\frac{\tau_n}{\tau_{n+1}}\right)(v_{n+1}-x_{n+1})\right\rangle \\ &\le \|v_{n+1}-v_n\|\left\{\|x_{n+1}-x_n\|+\left|1-\frac{\tau_n}{\tau_{n+1}}\right|\|v_{n+1}-x_{n+1}\|\right\}. \end{aligned} \tag{3.9}$$

Without loss of generality, let b be a real number such that $\tau_n > b > 0$, $\forall n \in N$, then we have that

$$\begin{aligned}\|v_{n+1}-v_n\| &\le \|x_{n+1}-x_n\|+\left|1-\frac{\tau_n}{\tau_{n+1}}\right|\|v_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\|+\frac{1}{b}|\tau_{n+1}-\tau_n|K, \end{aligned} \tag{3.10}$$

where $K=\sup\|v_{n+1}-x_{n+1}\|$. Then we have from (3.10) and (3.4) that

$$\begin{aligned}\|y_{n+1}-y_n\| \le{}& \|x_{n+1}-x_n\|+\frac{(1-\lambda_{n+1})|\tau_{n+1}-\tau_n|}{b}K \\ &+|\lambda_{n+1}-\lambda_n|\|x_n-F_{\tau_n}x_n\|. \end{aligned} \tag{3.11}$$

On the other hand, let $u_n = T_{\tau_n}y_n$, $u_{n+1}=T_{\tau_{n+1}}y_{n+1}$, we have that

$$\langle y-u_n, Tu_n\rangle - \frac{1}{\tau_n}\langle y-u_n,(1+\tau_n)u_n-y_n\rangle \le 0, \quad \forall y\in C, \tag{3.12}$$

$$\langle y-u_{n+1}, Tu_{n+1}\rangle - \frac{1}{\tau_{n+1}}\langle y-u_{n+1},(1+\tau_{n+1})u_{n+1}-y_{n+1}\rangle \le 0, \quad \forall y\in C. \tag{3.13}$$

Let $y := u_{n+1}$ in (3.12), and let $y := u_n$ in (3.13), we have that

$$\langle u_{n+1}-u_n, Tu_n\rangle - \frac{1}{\tau_n}\langle u_{n+1}-u_n,(1+\tau_n)u_n-y_n\rangle \le 0, \tag{3.14}$$

$$\langle u_n-u_{n+1}, Tu_{n+1}\rangle - \frac{1}{\tau_{n+1}}\langle u_n-u_{n+1},(1+\tau_{n+1})u_{n+1}-y_{n+1}\rangle \le 0. \tag{3.15}$$

Adding (3.14) and (3.15), and because T is pseudo-contractive, we have that

$$\left\langle u_{n+1}-u_n, \frac{u_n-y_n}{\tau_n}-\frac{u_{n+1}-y_{n+1}}{\tau_{n+1}}\right\rangle \ge 0.$$

Therefore, we have

$$\left\langle u_{n+1}-u_n, u_n-y_n-\frac{\tau_n(u_{n+1}-y_{n+1})}{\tau_{n+1}}+u_{n+1}-u_{n+1}\right\rangle \ge 0.$$

Hence we have that

$$\|u_{n+1}-u_n\| \le \|y_{n+1}-y_n\|+\frac{1}{b}|\tau_{n+1}-\tau_n|M, \tag{3.16}$$

where $M=\sup\{\|u_n-y_n\| : n\in N\}$.

Let $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$, hence we have that

$$z_{n+1} - z_n = \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(f(x_{n+1}) - f(x_n)\big) + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)f(x_n) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(u_{n+1} - u_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)u_n. \tag{3.17}$$

Hence we have from (3.17), (3.16), (3.11) and condition (iii) that

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\quad\le \frac{(\rho-1)\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\big\{\|f(x_n)\| + \|u_n\|\big\} \\ &\qquad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\frac{|\tau_{n+1} - \tau_n|}{b}\big((1-\lambda_{n+1})K + M\big) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1} - \lambda_n|\|x_n - F_{\tau_n}x_n\|. \end{aligned} \tag{3.18}$$

Notice conditions (ii) and (iv), we have that

$$\limsup_{n\to\infty}\big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) = 0. \tag{3.19}$$

Hence we have from Lemma 2.3 that

$$\limsup_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.20}$$

Therefore, we have that

$$\|x_{n+1} - x_n\| = |1-\beta_n|\|z_n - x_n\| \to 0. \tag{3.21}$$

Hence we have from (3.10) and (3.16) that

$$\|y_{n+1} - y_n\| \to 0, \qquad \|u_{n+1} - u_n\| \to 0, \qquad \|v_{n+1} - v_n\| \to 0. \tag{3.22}$$

Since $x_n = \alpha_{n-1}f(x_{n-1}) + \beta_{n-1}x_{n-1} + \gamma_{n-1}v_{n-1}$, so we have that

$$\begin{aligned} \|x_n - v_n\| &\le \|x_n - v_{n-1}\| + \|v_{n-1} - v_n\| \\ &\le \big[1 - \alpha_{n-1}(1-\rho)\big]\|x_{n-1} - v_{n-1}\| + \|v_{n-1} - v_n\|. \end{aligned}$$

According to Lemma 2.1, we have that

$$\|x_n - v_n\| \to 0.$$

In the same way, we have that

$$\|x_n - y_n\| \to 0, \qquad \|x_n - u_n\| \to 0.$$

Consequently, we have that

$$\|y_n - u_n\| \to 0.$$

Now, we show that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle\leq 0$.

Since sequence $\{x_n\}$ is bounded, then there exists a sub-sequence $\{x_{nk}\}$ of $\{x_n\}$ and $w\in C$ such that $x_{nk}\rightharpoonup w$. Next, we show that $w\in F=F(T)\cap VI(C,A)$.

Since $v_n=F_{\tau_n}x_n$, we have that

$$\langle y-v_{nk},Av_{nk}\rangle+\frac{1}{\tau_{nk}}\langle y-v_{nk},v_{nk}-x_{nk}\rangle\geq 0,\quad\forall y\in C.$$

Let $v_t=tv+(1-t)w$, $t\in(0,1)$, $v\in C$, then we have that

$$\langle v_t-v_{nk},Av_t\rangle\geq\langle v_t-v_{nk},Av_t-Av_{nk}\rangle-\left\langle v_t-v_{nk},\frac{v_{nk}-x_{nk}}{\tau_{nk}}\right\rangle.$$

Since $x_{nk}-v_{nk}\to 0$, and also A is monotone, we have that

$$0\leq\lim_{k\to\infty}\langle v_t-v_{nk},Av_t\rangle=\langle v_t-w,Av_t\rangle.$$

Consequently,

$$\langle v_t-w,Av_t\rangle\geq 0.$$

If $t\to 0$, by the continuity of A, we have $\langle v-w,Aw\rangle\geq 0$, $\forall v\in C$. Thus, $w\in VI(C,A)$.

In addition, since $u_n=F_{\tau_n}y_n$, we have that

$$\langle y-u_{nk},Tu_{nk}\rangle-\frac{1}{\tau_{nk}}\big\langle y-u_{nk},(1+\tau_{nk})u_{nk}-y_{nk}\big\rangle\leq 0,\quad\forall y\in C.$$

Let $v_t=tv+(1-t)w$, $t\in(0,1)$, $v\in C$, then we have that

$$\begin{aligned}\langle u_{nk}-v_t,Tv_t\rangle&\geq\langle v_t-u_{nk},Tu_{nk}-Tv_t\rangle-\left\langle v_t-u_{nk},\frac{1+\tau_{nk}}{\tau_{nk}}u_{nk}-y_{nk}\right\rangle\\&\geq-\langle v_t-u_{nk},v_t\rangle-\frac{1}{\tau_{nk}}\langle v_t-u_{nk},u_{nk}-y_{nk}\rangle.\end{aligned}$$

Since $y_{nk}-u_{nk}\to 0$, if $t\to 0$, by the continuity of T, we have that

$$-\langle v-w,Tw\rangle\geq-\langle v-w,w\rangle,\quad\forall v\in C.$$

Let $v=Tw$, we have $w=Tw$, thus, $w\in F(T)$. Consequently, we conclude that $w\in F=F(T)\cap VI(C,A)$.

Because $\bar{x}=P_Ff(\bar{x})$, we have from (2.1) that

$$\limsup_{n\to\infty}\big\langle f(\bar{x})-\bar{x},x_n-\bar{x}\big\rangle=\big\langle f(\bar{x})-P_Ff(\bar{x}),w-P_Ff(\bar{x})\big\rangle\leq 0.\tag{3.23}$$

Next, we show that $x_n\to\bar{x}$. From formula (3.3), we have that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2&=\big\|\alpha_n\big(f(x_n)-\bar{x}\big)+\beta_n(x_n-\bar{x})+\gamma_n(u_n-\bar{x})\big\|^2\\&\leq\alpha_n^2\big\|f(x_n)-\bar{x}\big\|^2+\big\|\beta_n(x_n-\bar{x})+\gamma_n(u_n-\bar{x})\big\|^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\alpha_n\bigl\langle f(x_n) - \bar{x}, \beta_n(x_n - \bar{x}) + \gamma_n(u_n - \bar{x})\bigr\rangle \\
&\leq \alpha_n^2\bigl\|f(x_n) - \bar{x}\bigr\|^2 + (1-\alpha_n)^2\|x_n - \bar{x}\|^2 + 2\alpha_n\beta_n\bigl\langle f(x_n) - \bar{x}, x_n - \bar{x}\bigr\rangle \\
&\quad + 2\alpha_n\gamma_n\bigl\|f(x_n) - \bar{x}\bigr\|\|u_n - \bar{x}\| \\
&\leq \alpha_n^2\bigl\|f(x_n) - \bar{x}\bigr\|^2 + (1-\alpha_n)^2\|x_n - \bar{x}\|^2 + 2\alpha_n\beta_n\bigl\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\bigr\rangle \\
&\quad + 2\alpha_n\beta_n\bigl\langle f(x_n) - f(\bar{x}), x_n - \bar{x}\bigr\rangle + 2\alpha_n\gamma_n\bigl\|f(x_n) - \bar{x}\bigr\|\|u_n - \bar{x}\| \\
&= \bigl[(1-\alpha_n)^2 + 2\rho\alpha_n\beta_n\bigr]\|x_n - \bar{x}\|^2 + \alpha_n^2\bigl\|f(x_n) - \bar{x}\bigr\|^2 + 2\alpha_n\beta_n\bigl\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\bigr\rangle \\
&\quad + 2\alpha_n\gamma_n\bigl\|f(x_n) - \bar{x}\bigr\|\|u_n - \bar{x}\| \\
&= \bigl(1 - 2\alpha_n(1-\rho\beta_n)\bigr)\|x_n - \bar{x}\|^2 + \alpha_n^2\bigl[\bigl\|f(x_n) - \bar{x}\bigr\|^2 + \|x_n - \bar{x}\|^2\bigr] \\
&\quad + 2\alpha_n\gamma_n\bigl\|f(x_n) - \bar{x}\bigr\|\|u_n - \bar{x}\| + 2\alpha_n\beta_n\bigl\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\bigr\rangle.
\end{aligned}
$$

Let $\theta_n = 2\alpha_n(1-\rho\beta_n)$, $\sigma_n = \alpha_n^2[\|f(x_n) - \bar{x}\|^2 + \|x_n - \bar{x}\|^2] + 2\alpha_n\gamma_n\|f(\bar{x}) - \bar{x}\|\|u_n - \bar{x}\| + 2\alpha_n\beta_n\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle$. According to Lemma 2.1 and formula (3.23), we have that $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$, *i.e.*, the sequence $\{x_n\}$ converges strongly to $\bar{x} \in F$.

According to formula (3.23), we conclude that $\bar{x}$ is the solution of the variational inequality (3.2). Now, we show that $\bar{x}$ is the unique solution of the variational inequality (3.2).

Suppose that $\bar{y} \in F$ is another solution of the variational inequality (3.2). Because $\bar{x}$ is the solution of the variational inequality (3.2), *i.e.*, $\langle f(\bar{x}) - \bar{x}, y - \bar{x}\rangle \leq 0$, $\forall y \in F$. Because $\bar{y} \in F$, then we have

$$\bigl\langle f(\bar{x}) - \bar{x}, \bar{y} - \bar{x}\bigr\rangle \leq 0. \tag{3.24}$$

On the other hand, to the solution $\bar{y} \in F$, since $\bar{x} \in F$, so

$$\bigl\langle f(\bar{y}) - \bar{y}, \bar{x} - \bar{y}\bigr\rangle \leq 0. \tag{3.25}$$

Adding (3.24) and (3.25), we have that

$$\bigl\langle \bar{x} - \bar{y} - \bigl(f(\bar{x}) - f(\bar{y})\bigr), \bar{x} - \bar{y}\bigr\rangle \leq 0,$$

i.e.,

$$\bigl\langle \bar{x} - \bar{y} - \bigl(f(\bar{x}) - f(\bar{y})\bigr), \bar{x} - \bar{y}\bigr\rangle \leq \bigl\langle f(\bar{x}) - f(\bar{y}), \bar{x} - \bar{y}\bigr\rangle.$$

Hence

$$\|\bar{x} - \bar{y}\|^2 \leq \bigl\langle f(\bar{x}) - f(\bar{y}), \bar{x} - \bar{y}\bigr\rangle \leq \rho\|\bar{x} - \bar{y}\|^2.$$

Because $\rho \in (0,1)$, hence we conclude that $\bar{x} = \bar{y}$, the uniqueness of the solution is obtained. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping, let $A : C \to H$ be a continuous monotone* *mapping such that $F = F(T) \cap VI(C,A) \neq \emptyset$, let $f : C \to C$ be a contraction with a contraction coefficient $\rho \in (0,1)$. The mappings T_{τ_n} and F_{τ_n} are defined as* (2.3) *and* (2.4)*, respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{\tau_n} F_{\tau_n} x_n, \tag{3.26}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of nonnegative real numbers in $[0,1]$ *and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\liminf_{n\to\infty} \tau_n > 0$, $\sum_{n=1}^{\infty} |\tau_{n+1} - \tau_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_F f(\bar{x})$, and also $\bar{x}$ is the unique solution of the variational inequality

$$\langle f(\bar{x}) - \bar{x}, y - \bar{x} \rangle \leq 0, \quad \forall y \in F. \tag{3.27}$$

Proof Putting $\lambda_n = 0$ in Theorem 3.1, we can obtain the result. □

If in Theorem 3.1 and Theorem 3.2, let $f :\equiv u \in C$ be a constant mapping, we have the following theorems.

Theorem 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping, let $A : C \to H$ be a continuous accretive mapping such that $F = F(T) \cap VI(C,A) \neq \emptyset$. The mappings T_{τ_n} and F_{τ_n} are defined as* (2.3) *and* (2.4)*, respectively. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = \lambda_n x_n + (1 - \lambda_n) F_{\tau_n} x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T_{\tau_n} y_n, \end{cases} \tag{3.28}$$

where $\lambda_n \in [0,1]$, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences of nonnegative real numbers in $[0,1]$ *and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 1$;

(iv) $\liminf_{n\to\infty} \tau_n > 0$, $\sum_{n=1}^{\infty} |\tau_{n+1} - \tau_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_F u$, and also $\bar{x}$ is the unique solution of the variational inequality

$$\langle u - \bar{x}, y - \bar{x} \rangle \leq 0, \quad \forall y \in F. \tag{3.29}$$

Theorem 3.4 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping, let $A : C \to H$ be a continuous monotone mapping such that $F = F(T) \cap VI(C,A) \neq \emptyset$. The mappings T_{τ_n} and F_{τ_n} are defined as* (2.3) *and* (2.4)*, respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T_{\tau_n} F_{\tau_n} x_n, \tag{3.30}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences of nonnegative real numbers in* $[0,1]$ *and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\liminf_{n\to\infty} \tau_n > 0$, $\sum_{n=1}^{\infty} |\tau_{n+1} - \tau_n| < \infty$.

Then the sequence $\{x_n\}$ *converges to* $\bar{x} = P_F u$, *and also* $\bar{x}$ *is the unique solution of the variational inequality*

$$\langle u - \bar{x}, y - \bar{x} \rangle \leq 0, \quad \forall y \in F. \tag{3.31}$$

Competing interests
The author declares that they have no competing interests.

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