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A class of φ -concave operators and applications

Yanbin Sang*

*Correspondence:
syb6662004@163.com
Department of Mathematics, North
University of China, Taiyuan, Shanxi
030051, P.R. China

Abstract

In this paper, by means of the concept of φ -concave operators, which was introduced by Li and Liang (*J. Systems Sci. Math. Sci.* 14(4):355-360, 1994 (in Chinese)), we obtain some new existence and uniqueness theorems of a fixed point of mixed monotone operators with such concavity. Moreover, we apply the main theorem to a class of Hammerstein integral equations.

Keywords: cone and partial order; φ -concave operator; mixed monotone operator; fixed point

1 Introduction

It is well known that concave operators are a class of important operators that are extensively used in nonlinear differential and integral equations (see [1–12]). In [13], Krasnosel'skii introduced in detail many important ideas and results about concave operators. A problem is that there are various concepts of concave operators, such as u_0 -concave operators [5], ordered concave operators [1] and α -concave operators [7], which is somewhat confusing. We should mention that Liang *et al.* [14] proved that both ordered concave operators and α -concave operators are u_0 -concave operators and gave necessary and sufficient conditions on which u_0 -concave operators have a unique fixed point. Furthermore, Zhai *et al.* [11] considered an operator equation with general α -concave or homogeneous operator, their results improved previous results (see Corollary 3.3 in [7]).

We note that Li and Liang [15] introduced the definition of φ -concave operator, which provided a general method to copy with such a class of operators together. Furthermore, [8, 9] extended the concept of φ -concave operator to ϕ -concave($-\psi$) convex operators. In [12], Zhao weakened some conditions and strengthened the conclusions. Mixed monotone operators were introduced by Guo and Lakshmikantham [16] in 1987. Thereafter, many authors have focused on various existence (and uniqueness) theorems of fixed points for mixed monotone operators; for details, see [8, 9, 13, 16–33] and references therein. In [20], Bhaskar and Lakshmikantham established some coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces and discussed the existence and uniqueness of a solution for a periodic boundary value problem. Instead of using a direct proof as in [20], Drici *et al.* [22] employed the notion of reflection operator and investigated fixed point theorems for mixed monotone operators by weakening the requirements in the contractive assumption and strengthening the metric space utilized with a partial order. Sintunavarat and Kumam [34] extended classical coupled fixed point theorems of Bhaskar and Lakshmikantham [20] to the coupled common fixed point theorems

for mappings satisfying a new non-commuting condition. These theorems are generalizations of the results of [20]. Furthermore, Harjani *et al.* [25] improved the main results of [22] using the altering distance functions. Sintunavarat *et al.* [35] established some coupled fixed point theorems for a contraction mapping induced by the cone ball metric in partially ordered spaces and also discussed the condition claim of the uniqueness of a coupled fixed point. Very recently, Chandok *et al.* [21] were concerned with some coupled coincidence point theorems for a pair of mappings having a mixed g -monotone property in partially ordered G -metric spaces. Also, they presented a result on the existence and uniqueness of coupled common fixed points. In [19], the authors introduced the concept of W -compatible mappings. Based on this notion, a tripled coincidence point and a common tripled fixed point for mappings $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ were obtained, where (X, d) is a cone metric space. We should point out that their results do not rely on the assumption of normality condition of the cone.

On the other hand, there is much attention paid to mixed monotone operators with certain concavity and convexity; for example, see [8, 9, 23, 24, 27, 28, 30–33]. In [32], Zhang and Wang modified the methods in [30, 31] to obtain some new existence and uniqueness results of a positive fixed point of mixed monotone operators. Recently, in [29], Zhai and Zhang presented a new fixed point theorem for a class of general mixed monotone operators, which extends the existing corresponding results. Moreover, they investigated some pleasant results of nonlinear eigenvalue problems with mixed monotone properties. Based on them, the local existence-uniqueness of positive solutions for nonlinear boundary value problems which include Neumann boundary value problems, three-point boundary value problems and elliptic boundary value problems for Lane-Emden-Fowler equations was proved.

In this paper, by means of the concept of φ -concave operator, some new theorems of fixed points of mixed monotone operators with such concavity are obtained. Our theorems unify and extend some previous results, then we apply these results in discussing a class of Hammerstein integral equations and give the new condition for determining the unique solution to this equation.

2 Preliminaries

Some definitions, notations and known results are from Refs. [2, 3, 23]. Let E be a real Banach space. A nonempty convex closed set P is called a cone if it satisfies the following conditions:

- (i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
- (ii) $x \in P$ and $-x \in P$ imply $x = \theta$, where θ denotes the zero element of E .

Let E be partially ordered by a cone P of E , *i.e.*, $x \leq y$, if and only if $y - x \in P$ for any $x, y \in E$. Recall that the cone P is said to be solid if the interior P° is nonempty, and P is said to be normal if there exists a positive constant M such that $\theta \leq x \leq y$ ($x, y \in E$) implies $\|x\| \leq M\|y\|$, where M is the normal constant of P . Let $D \subset E$. An operator $A : D \times D \rightarrow E$ is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y , that is, for any $x, y \in D$,

$$x_1, x_2 \in D, \quad x_1 \leq x_2 \quad \implies \quad A(x_1, y) \leq A(x_2, y)$$

and

$$y_1, y_2 \in D, \quad y_1 \leq y_2 \implies A(x, y_1) \geq A(x, y_2).$$

An element $x^* \in D$ is called a fixed point of A if it satisfies $A(x^*, x^*) = x^*$. $A : D \subset E \rightarrow E$ is said to be convex if for $x, y \in D$ with $x \leq y$ and each $t \in [0, 1]$, we have

$$A(tx + (1-t)y) \leq tAx + (1-t)Ay;$$

A is said to be concave if $-A$ is convex.

Let $h > \theta$, write $P_h = \{x \in E \mid \exists \lambda, \mu > 0, \text{ such that } \lambda h \leq x \leq \mu h\}$.

Let $e > \theta$. An operator $A : P \rightarrow P$ is said to be e -concave if it satisfies the following two conditions:

- (i) A is e -positive, i.e., $A(P - \{\theta\}) \subset P_e$;
- (ii) $\forall x \in P_e, \forall 0 < t < 1, \exists \eta = \eta(t, x) > 0$ such that

$$A(tx) \geq (1 + \eta)tAx,$$

where $\eta = \eta(t, x)$ is called the characteristic function of A .

The operator $T : P^\circ \rightarrow P^\circ$ is said to be α -concave ($-\alpha$ -convex) ($0 \leq \alpha < 1$) if

$$T(tx) \geq t^\alpha T(x) \quad (T(tx) \leq t^{-\alpha} T(x)), \quad \forall x \in P^\circ, \forall t \in (0, 1].$$

Let $u_0, v_0 \in E$ with $u_0 \leq v_0$. Write

$$[u_0, v_0] = \{x \in E \mid u_0 \leq x \leq v_0\},$$

where $[u_0, v_0]$ is called an ordering interval.

Definition 2.1 (see [25]) Let $A : P_h \rightarrow P_h$ be a φ -concave operator if there exists a function $\varphi : (0, 1] \times P_h \rightarrow (0, 1]$ such that $t \in (0, 1)$ implies $t < \varphi(t, x)$, and A satisfies the following condition:

$$A(tx) \geq \varphi(t, x)Ax, \quad 0 < t \leq 1, x \in P_h.$$

Definition 2.2 (see [2]) Let S be a bounded set of a real Banach space E . Let

$$\alpha(S) = \inf \left\{ \delta > 0 \mid S = \bigcup_{i=1}^m S_i \text{ with } \text{diam}(S_i) \leq \delta, i = 1, 2, \dots, m \right\}.$$

Clearly, $0 \leq \alpha(S) < \infty$. $\alpha(S)$ is called the Kuratowski measure of noncompactness.

Definition 2.3 (see [36, 37]) Let $D \subset E, A : D \rightarrow E$ be an operator. Then S is said to be a generalized condensing operator if for any $S \subset D, \alpha(S) \neq 0$ implies $\alpha(A(S)) < \alpha(S)$.

Lemma 2.1 (see [2, 37]) Let S, T be bounded subsets of E . Then

- (i) $\alpha(S) = 0$ if and only if S is relatively compact;
- (ii) $S \subset T$ implies $\alpha(S) \leq \alpha(T)$;

- (iii) $\alpha(\bar{S}) = \alpha(S)$;
- (iv) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$;
- (v) $\alpha(\text{co}S) = \alpha(S)$, where $\text{co}S$ denotes the convex hull of S .

3 Main results

Theorem 3.1 *Let E be a real Banach space and P be a normal cone of E . Let $u_0, v_0 \in E$ with $u_0 \leq v_0$ and $A : [u_0, v_0] \times P \rightarrow P$ be a mixed monotone operator. For fixed $v \in P$, $A(\cdot, v) : [u_0, v_0] \rightarrow P$ is a φ -concave operator. Suppose that*

- (i) *there exists a real positive number r_0 such that $u_0 \geq r_0 v_0$;*
- (ii) *u_0 and v_0 are such that*

$$u_0 \leq A(u_0, v_0), \quad A(v_0, u_0) \leq v_0;$$

- (iii) *there exists an element $w_0 \in [u_0, v_0]$ such that*

$$\varphi(t, x) \geq \varphi(t, w_0), \quad \forall (t, x) \in (0, 1) \times [u_0, v_0],$$

and

$$\lim_{s \rightarrow t^-} \varphi(s, w_0) > t, \quad \forall t \in (0, 1);$$

- (iv) *for fixed $u \in [u_0, v_0]$, $\exists N > 0$ such that*

$$A(u, \cdot) : P \rightarrow P, \quad A(u, v_1) - A(u, v_2) \geq -N(v_1 - v_2), \quad \forall v_1 \geq v_2, v_1, v_2 \in P.$$

Then A has exactly one fixed point x^ in $[u_0, v_0]$.*

Proof We divide the proof into three steps.

Step 1. We prove that for any fixed $u \in [u_0, v_0]$, $A(u, \cdot)$ has exactly one fixed point $T(u) \in [A(u_0, v_0), A(v_0, u_0)]$ such that $A(u, T(u)) = T(u)$. Our proof is the same as Theorem 2.1 in [32]. For completeness, we list it as follows.

For fixed $u \in [u_0, v_0]$, there exists $N > 0$ such that $A(u, v) + Nv$ is increasing in v . Let

$$B(u, v) = \frac{A(u, v) + Nv}{N + 1},$$

then $B(u, v) = v$ is equivalent to $A(u, v) = v$.

Let $x_0 = A(u_0, v_0)$, $y_0 = A(v_0, u_0)$ and $x_{n+1} = B(u, x_n)$, $y_{n+1} = B(u, y_n)$. By $u_0 \leq u \leq v_0$, $x_0 = A(u_0, v_0) \leq v_0$, $u_0 \leq v_0$ and $y_0 = A(v_0, u_0) \geq u_0$, we have

$$A(u, x_0) \geq A(u_0, v_0) = x_0,$$

$$A(u, y_0) \leq A(v_0, u_0) = y_0,$$

thus

$$x_1 = B(u, x_0) = \frac{A(u, x_0) + Nx_0}{N + 1} \geq \frac{x_0 + Nx_0}{N + 1} = x_0,$$

$$y_1 = B(u, y_0) = \frac{A(u, y_0) + Ny_0}{N + 1} \leq \frac{y_0 + Ny_0}{N + 1} = y_0,$$

that is, $x_0 \leq x_1 \leq y_1 \leq y_0$. By induction, we have $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$. Thus we get that

$$\theta \leq y_n - x_n = \frac{A(u, y_{n-1}) + Ny_{n-1} - A(u, x_{n-1}) - Nx_{n-1}}{N + 1} \leq \frac{N}{N + 1}(y_{n-1} - x_{n-1}).$$

Since P is a normal cone and by induction, we get that

$$\|y_n - x_n\| \leq C \left(\frac{N}{N + 1} \right)^n \|y_0 - x_0\|, \tag{3.1}$$

where C is the normal constant of P .

Moreover (for any natural number p),

$$x_{n+p} - x_n \leq y_{n+p} - x_n \leq y_n - x_n, \tag{3.2}$$

$$\|x_{n+p} - x_n\| \leq C \|y_n - x_n\| \leq C^2 \left(\frac{N}{N + 1} \right)^n \|y_0 - x_0\|, \tag{3.3}$$

which implies that x_n is a Cauchy sequence. Noticing E is complete, x_n converges to some element, we denote it by $T(u)$. Equation (3.1) implies that y_n also converges to $T(u)$. By

$$x_n \leq T(u) \leq y_n, \tag{3.4}$$

$$x_{n+1} = B(u, x_n) \leq B(u, T(u)) \leq B(u, y_n) \leq y_{n+1}, \tag{3.5}$$

using the normality of the cone P , we can conclude that x_n, y_n also converge to $B(u, T(u))$. Thus $B(u, T(u)) = T(u)$.

If $B(u, x) = x$, then we have for any integer n , $x_n \leq x \leq y_n$, which implies by taking the limit that $x = T(u)$, i.e., $T(u)$ is the unique fixed point of $A(u, \cdot)$ in $[A(u_0, v_0), A(v_0, u_0)]$.

Step 2. We prove that $T(u)$ is increasing in u .

If $u, u' \in [u_0, v_0]$, $u \leq u'$, then we let $x_0 = x'_0 = A(u_0, v_0)$ in Step 1. Since $B(u, v)$ is increasing in both variables, we have

$$x_1 = B(u, x_0) \leq B(u', x'_0) = x'_1.$$

By induction we know $x_n \leq x'_n$. Taking the limit, we have $T(u) \leq T(u')$.

Step 3. We prove that $T(\cdot)$ has a unique fixed point in $[u_0, v_0]$.

Let $u_{n+1} = T(u_n)$, $v_{n+1} = T(v_n)$, we have $u_0 \leq u_1, v_1 \leq v_0$. Thus, by conclusion of Step 2, we get $u_n \leq u_{n+1}, v_{n+1} \leq v_n$, and $u_n \leq v_n$. By condition (i), we have

$$\begin{aligned} u_1 &= A(u_0, T(u_0)) \\ &\geq A(r_0 v_0, T(u_0)) \\ &\geq \varphi(r_0, v_0) A(v_0, T(u_0)) \\ &\geq \varphi(r_0, v_0) A(v_0, T(v_0)) \\ &= \varphi(r_0, v_0) v_1, \end{aligned}$$

that is

$$u_1 \geq \varphi(r_0, v_0) v_1.$$

Let $r_n = \varphi(r_{n-1}, v_{n-1})$, $n = 1, 2, \dots$. By induction, we know

$$u_n \geq r_n v_n, \quad n = 1, 2, \dots$$

Obviously, the sequence $\{r_n\}$ is increasing with $\{r_n\} \subset (0, 1]$. Suppose $r_n \rightarrow r$ ($n \rightarrow \infty$), then $r = 1$. Otherwise, we have $0 < r < 1$. Thus, by condition (iii), we have

$$r_n = \varphi(r_{n-1}, v_{n-1}) \geq \varphi(r_{n-1}, w_0). \tag{3.6}$$

Let $n \rightarrow \infty$ in (3.6), we have

$$r \geq \lim_{s \rightarrow r^-} \varphi(s, w_0) > r,$$

which is a contradiction. Hence, we have $r = 1$.

Therefore, $\forall n, p \geq 1$, we get

$$\theta \leq v_n - u_n \leq v_n - r_n v_n = (1 - r_n)v_n \leq (1 - r_n)v_0,$$

and

$$\theta \leq u_{n+p} - u_n \leq v_{n+p} - u_n \leq v_n - u_n,$$

$$\theta \leq v_n - v_{n+p} \leq v_n - u_{n+p} \leq v_n - u_n.$$

Thus, by the normality of P , it is easy to see that $v_n - u_n \rightarrow 0$ ($n \rightarrow \infty$), and hence $\{u_n\}$, $\{v_n\}$ are Cauchy sequences. Therefore, there exist u^* , v^* such that $u_n \rightarrow u^*$, $v_n \rightarrow v^*$ ($n \rightarrow \infty$) and $u^* = v^*$. Write $x^* = u^* = v^*$.

Now we show that $T(x^*) = x^*$. It is easy to see that

$$T(x^*) \geq T(u_n) = u_{n+1} \rightarrow x^*,$$

and so $T(x^*) \geq x^*$. On the other hand, we have

$$T(x^*) \leq T(v_n) = v_{n+1} \rightarrow x^*,$$

so we obtain $T(x^*) \leq x^*$. Hence $T(x^*) = x^*$.

The proof of the uniqueness is the same as above in Step 1. Finally, from Step 1 we get that x^* is the fixed point of A in $[u_0, v_0]$ such that $T(x^*) = x^*$ and $A(x^*, x^*) = x^*$.

By the construction we can see that it is unique. If \bar{x} satisfies $A(\bar{x}, \bar{x}) = \bar{x}$. By Step 1, $T(\bar{x})$ is the unique fixed point of $A(\bar{x}, \cdot)$, thus $T(\bar{x}) = \bar{x}$. By the uniqueness of the fixed point of $T(\cdot)$, we get $\bar{x} = x^*$. This ends the proof of Theorem 3.1. \square

Remark 3.1 In fact, by Theorem 2.3 in [30] and Theorem 1.1 in [29], we can know that condition $\lim_{s \rightarrow t^-} \varphi(s, w_0) > t$ in Theorem 3.1 can be deleted, while the conclusions remain the same.

Theorem 3.2 *Let E be a real Banach space and P be a cone of E . Let $u_0, v_0 \in E$ with $u_0 \leq v_0$ and $A : [u_0, v_0] \times P \rightarrow P$ be a generalized condensing and mixed monotone operator. For fixed $v \in P$, $A(\cdot, v) : [u_0, v_0] \rightarrow P$ is a φ -concave operator. Suppose that conditions (i)-(iv) of Theorem 3.1 hold. Then A has exactly one fixed point x^* in $[u_0, v_0]$.*

Proof For fixed $u \in [u_0, v_0]$, let $S = \{x_n \mid n = 0, 1, 2, \dots\}$. We show that $S \subset co\{x_0, A(S)\}$. It is easy to see that $x_0 \in co\{x_0, A(S)\}$. If $x_k \in co\{x_0, A(S)\}$ ($k \geq 0$), then for fixed $u \in [u_0, v_0]$,

$$x_{k+1} = \frac{A(u, x_k) + Nx_k}{N+1} = \frac{A(u, x_k)}{N+1} + \left(1 - \frac{1}{N+1}\right)x_k.$$

Noticing that $A(u, x_k), x_k \in co\{x_0, A(S)\}$, we have that $x_{k+1} \in co\{x_0, A(S)\}$. By induction, we have that $S \subset co\{x_0, A(S)\}$. It follows from Lemma 2.1 that

$$\alpha(S) \leq \alpha(co\{x_0, A(S)\}) = \alpha(A(S)) < \alpha(S).$$

This is a contradiction. Consequently, we find that $\alpha(S) = 0$. In view of Lemma 2.1, we have that \bar{S} is a compact set in E . This implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges to x^* in E .

Now, we prove that $\{x_n\}$ itself also converges to x^* . If not, there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$, which converges to another point $x_* \neq x^*$ ($x_* \in E$). Thus, for any fixed $\{x_{n_{i_0}}\}$, and if n_j is large enough, then $x_{n_{i_0}} \leq x_{n_j}$. Let $n_j \rightarrow \infty$, we get that $x_{n_{i_0}} \leq x_*$. Since $x_{n_{i_0}}$ is any fixed element of $\{x_{n_i}\}$, then for any element x_{n_i} of $\{x_{n_i}\}$, we have that $x_{n_i} \leq x_*$. Let $n_i \rightarrow \infty$, we get that $x^* \leq x_*$. Similarly, we can prove that $x_* \leq x^*$. Thus, $x_* = x^*$. This is a contradiction. Therefore, $x_n \rightarrow x^*$. In the same method, we can know that there exists $y^* \in E$ such that $y_n \rightarrow y^*$.

Since $\theta \leq y_n - x_n \leq \left(\frac{N}{N+1}\right)^n (y_0 - x_0)$, $n = 1, 2, 3, \dots$. Let $n \rightarrow \infty$, we have that $y^* = x^*$.

In the following, we show that x^* is the fixed point of A . In fact, if $m \geq 1$, for any fixed integer n , $x_n \leq x_{n+m} \leq y_n$. Let $m \rightarrow \infty$, we have that $x_n \leq x^* \leq y_n$. Thus,

$$\begin{aligned} A(u, x^*) &= A(u, x^*) - A(u, x_{n-1}) + A(u, x_{n-1}) \\ &\geq -N(x^* - x_{n-1}) + A(u, x_{n-1}) \\ &= -Nx^* + Nx_{n-1} + A(u, x_{n-1}) \\ &= -Nx^* + (N+1)x_n \rightarrow x^* \quad (n \rightarrow \infty). \end{aligned}$$

On the other hand,

$$\begin{aligned} A(u, x^*) &= A(u, x^*) - A(u, y_{n-1}) + A(u, y_{n-1}) \\ &\leq -N(x^* - y_{n-1}) + A(u, y_{n-1}) \\ &= -Nx^* + Ny_{n-1} + A(u, y_{n-1}) \\ &= -Nx^* + (N+1)y_n \rightarrow x^* \quad (n \rightarrow \infty). \end{aligned}$$

Therefore, for fixed $u \in [u_0, v_0]$, $A(u, x^*) = x^*$. The rest of the proof is the same as that of Theorem 3.1. \square

Corollary 3.1 (see Theorem 2.1 in [32]) *Let P be a normal cone of E , and let $A : P \times P \rightarrow P$ be a mixed monotone operator. Suppose that*

1. *For fixed $v \in P$, $A(\cdot, v) : P \rightarrow P$ is concave; for fixed $u \in P$, $\exists N > 0$ such that $A(u, \cdot) : P \rightarrow P$, $A(u, v_1) - A(u, v_2) \geq -N(v_1 - v_2)$, $\forall v_1 \geq v_2, v_1, v_2 \in P$.*
2. *$\exists \bar{v} > \theta, 0 < c \leq 1$ such that $\theta < A(\bar{v}, \theta) \leq \bar{v}$ and*

$$A(\theta, \bar{v}) \geq cA(\bar{v}, \theta).$$

Then A has exactly one fixed point u^ in $[\theta, \bar{v}]$ and $A(\theta, \bar{v}) \leq u^* \leq A(\bar{v}, \theta)$.*

Proof Set

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \dots,$$

$u_0 = \theta, v_0 = \bar{v}$, it is easy to show that

$$\begin{aligned} u_0 < A(v_0, u_0) \leq v_0, \quad A(u_0, v_0) \geq cA(v_0, u_0), \\ u_1 \geq cv_1, \quad u_1 = A(u_0, v_0) \leq A(u_1, v_1), \\ A(v_1, u_1) \leq v_1 = A(v_0, u_0). \end{aligned}$$

Now let us prove that $A(\cdot, v) : [u_1, v_1] \rightarrow P$ is a φ -concave operator for fixed $v \in [u_1, v_1]$. It suffices to show $A(\cdot, v) : [u_0, v_0] \rightarrow P$ is a φ -concave operator for fixed $v \in [u_0, v_0]$.

For each $u \in [u_0, v_0], t \in (0, 1)$, we see

$$\begin{aligned} A(tu, v) &= A(tu + (1-t)\theta, v) \\ &\geq tA(u, v) + (1-t)A(\theta, v) \\ &\geq tA(u, v) + (1-t)A(\theta, v_0) \\ &\geq tA(u, v) + c(1-t)A(v_0, u_0) \\ &\geq tA(u, v) + c(1-t)A(u, v). \end{aligned}$$

Set $\varphi(t, x) = t + c(1-t)$, then $\varphi : (0, 1] \times [u_0, v_0] \rightarrow (0, 1], \varphi(t, x) > t, \forall t \in (0, 1), \lim_{s \rightarrow t^-} \varphi(s, w_0) = t + c(1-t) > t$. □

So, by Theorem 3.1, we see that A has exactly one fixed point x^* in $[u_1, v_1]$.

Similarly, if P is a solid cone, we have the following corollary.

Corollary 3.2 (see Theorem 2.3 in [32]) *Let P be a normal solid cone of E , and let $A : P^\circ \times P^\circ \rightarrow P^\circ$ be a mixed monotone operator. Suppose that*

1. *For fixed $v \in P^\circ, A(\cdot, v) : P^\circ \rightarrow P^\circ$ is concave; for fixed $u \in P^\circ, \exists N > 0$ such that $A(u, \cdot) : P^\circ \rightarrow P^\circ, A(u, v_1) - A(u, v_2) \geq -N(v_1 - v_2), \forall v_1 \geq v_2, v_1, v_2 \in P^\circ$.*
2. *$\exists u_0 \in P^\circ, v_0 \in P^\circ$ such that $u_0 \leq v_0, u_0 \ll A(u_0, v_0), A(v_0, u_0) \leq v_0$.*

Then A has exactly one fixed point in $[u_0, v_0]$.

Proof It is easy to show that there exists a real number $r_0 > 0$ such that $u_0 \geq r_0 v_0$ since $u_0, v_0 \in P^\circ$, which completes the proof of Corollary 3.2 by means of Corollary 3.1. □

Corollary 3.3 (see Remark 2.4 in [32]) *Let P be a normal solid cone of E , and let $A : P^\circ \times P \rightarrow P^\circ$ be a mixed monotone operator. Suppose that*

1. *For fixed $v \in P$, $A(\cdot, v) : P^\circ \rightarrow P^\circ$ is α -concave; for fixed $u \in P$, $\exists N > 0$ such that $A(u, \cdot) : P \rightarrow P$, $A(u, v_1) - A(u, v_2) \geq -N(v_1 - v_2)$, $\forall v_1 \geq v_2, v_1, v_2 \in P$.*
2. *$\exists u_0 \in P^\circ, v_0 \in P$ such that $u_0 \leq v_0, u_0 \ll A(u_0, v_0), A(v_0, u_0) \leq v_0$.*

Then A has exactly one fixed point in $[u_0, v_0]$.

Proof Take $\varphi(t, x) = t^\alpha, \forall t \in (0, 1)$. Thus, by Corollary 3.2, we easily see that the conclusions of Corollary 3.3 hold. □

Corollary 3.4 (see Theorem 2.2 in [11] or Theorem 2.1 in [10]) *Assume that the operator A satisfies the following conditions:*

- (H₁) *$A : P_h \rightarrow P_h$ is increasing in P_h ;*
 (H₂) *for $\forall x \in P_h$ and $t \in (0, 1)$, there exists $\alpha(t) \in (0, 1)$ such that*

$$A(tx) \geq t^{\alpha(t)}Ax.$$

Then A has a unique solution in P_h .

Proof If $A(u, v)$ is independent of v , then take $\varphi(t, x) = t^{\alpha(t)}, \forall t \in (0, 1)$. It is easy to check that conditions (i), (iii) in Theorem 3.1 hold. By Lemma 2.1 in [8], assume that the operator A satisfies conditions (H₁) and (H₂), we can know that there are $u_0, v_0 \in P_h$ such that $u_0 < v_0, u_0 \leq Au_0 \leq Av_0 \leq v_0$. Thus, condition (ii) in Theorem 3.1 holds. Therefore, it follows from Theorem 3.1 that the conclusions of Corollary 3.4 hold. □

Corollary 3.5 (see Theorem 2.6 in [10]) *Assume that the operator A satisfies the following conditions:*

- (H₃) *$A : P_h \rightarrow P_h$ is increasing in P_h ;*
 (H₄)

$$A(tx) \geq t^{\alpha(t,x)}Ax, \quad \forall x \in P_h, t \in (0, 1),$$

where $\alpha : (0, 1) \times P_h \rightarrow (0, 1)$ is increasing in x for fixed $t \in (0, 1)$;

- (H₅) *there exists $t_0 \in (0, 1)$ such that*

$$t_0h \leq Ah \leq \frac{1}{t_0^{1-\alpha(t_0, \frac{1}{t_0}h)}}h.$$

Then A has a unique solution in P_h .

Proof By Corollary 3.4, we only need to check condition (ii) in Theorem 3.1. By the proof of Theorem 2.6 in [10], choose $k \in R$ such that $k > \frac{1}{1-\alpha(t_0)}$. Put $u_0 = t_0^k h, v_0 = \frac{1}{t_0} h$, set

$$u_n = Au_{n-1}, \quad v_n = Av_{n-1}, \quad n = 1, 2, \dots,$$

we have $u_0, v_0 \in P_h, u_0 < v_0$, and $u_0 \leq Au_0 \leq Av_0 \leq v_0$. □

Furthermore, we can easily obtain the following new result.

Theorem 3.3 Let P be a normal cone of the real Banach space E , $e > \theta$ and $u_0, v_0 \in P$ with $u_0 \leq v_0$, and let $A : P \times P \rightarrow P$ be a mixed monotone operator. Suppose that

- (i) there exists a real positive number r_0 such that $u_0 \geq r_0 v_0$;
- (ii) $u_0 \leq A(u_0, v_0), A(v_0, u_0) \leq v_0$;
- (iii) for fixed v , $A(\cdot, v) : P \rightarrow P$ is e -concave with its characteristic function, $\eta(t, x)$ is monotone in x and continuous in t from left;
- (iv) for fixed $u \in P$, $\exists N > 0$ such that

$$A(u, \cdot) : P \rightarrow P, \quad A(u, v_1) - A(u, v_2) \geq -N(v_1 - v_2), \quad \forall v_1 \geq v_2, v_1, v_2 \in P.$$

Then A has exactly one fixed point x^* in $[u_0, v_0]$.

Corollary 3.6 (see Theorem 2.5 in [11]) Assume that the operator A satisfies the following conditions:

- (H₆) $A : P_h \rightarrow P_h$ is increasing in P_h ;
- (H₇) for each $x \in P_h$ and $t \in (0, 1)$, there exists $\eta(t) \in (0, 1)$ such that

$$A(tx) \geq t(1 + \eta(t))Ax.$$

Then A has a unique solution in P_h .

Proof Set $\eta(t, x) = \eta(t)$, by Corollary 3.4 and Theorem 3.3, we can know that the conclusions of Corollary 3.6 hold. \square

4 Application

In this section, we present an example to explain our results.

Example 4.1 Consider the following nonlinear integral equation:

$$x(t) = (Ax)(t) = \int_{R^N} K(t, s) [x^{\frac{1}{2}}(s) + x(s) + x^{-\frac{1}{3}}(s)] ds. \quad (4.1)$$

Conclusion 4.1 Suppose that $K : R^N \times R^N \rightarrow R^1$ is nonnegative and continuous with

$$\frac{1}{111} \leq \int_{R^N} K(t, s) ds \leq \frac{1}{6}. \quad (4.2)$$

Then Eq. (4.1) has a unique positive solution $x^*(t)$ satisfying $10^{-2} \leq x^*(t) \leq 1$.

Proof We use Theorem 3.3 to prove Conclusion 4.1. Let $E = C_B(R^N)$ denote the set of all bounded continuous functions on R^N ; we define $\|x\| = \sup_{t \in R^N} |x(t)|$, and then E is a real Banach space. Let $P = C_B^+(R^N)$ denote the set of all nonnegative functions of $C_B(R^N)$. Then P is a normal cone of E . Obviously, Eq. (4.1) can be written in the form $x = A(x, x)$, where

$$\begin{aligned} A(x, y) &= A_1(x) + A_2(y), \\ A_1(x) &= \int_{R^N} K(t, s) [x^{\frac{1}{2}}(s) + x(s)] ds, \\ A_2(y) &= \int_{R^N} K(t, s) y^{-\frac{1}{3}}(s) ds. \end{aligned}$$

Now, let us show that the operator A satisfies all the conditions in Theorem 3.3. In fact, set $u_0 = 10^{-2}$, $v_0 = 1$. It is easy to see that $A : P \times P \rightarrow P$ is a mixed monotone operator. It is obvious that $u_0, v_0 \in P$, $u_0 < v_0$ and there exists a real number $\epsilon_0 > 0$ such that $u_0 \geq \epsilon_0 v_0$. By (4.2) we can easily get

$$A(u_0, v_0) = \int_{R^N} K(t, s)(10^{-1} + 10^{-2} + 1) ds \geq 10^{-2} = u_0,$$

and

$$A(v_0, u_0) = \int_{R^N} K(t, s)(1 + 1 + 10^{\frac{2}{3}}) ds \leq 1 = v_0.$$

For fixed y , $\forall t \in (0, 1)$, $\exists \eta = \eta(t, x) = \frac{(\sqrt{t}-t)\sqrt{x}}{t(\sqrt{x}+x)} > 0$ such that

$$A(tx, y) \geq (1 + \eta)tA(x, y),$$

where $\eta(t, x)$ is decreasing in x and continuous in t from left.

For fixed x , $\exists N = \frac{500}{9}$ such that

$$A(x, v_1) - A(x, v_2) \geq -\frac{500}{9}(v_1 - v_2), \quad \forall v_1 \geq v_2, v_1, v_2 \in [10^{-2}, 1].$$

Therefore, we see Conclusion 4.1 holds by means of Theorem 3.3. □

Competing interests

The author declares that they have no competing interests.

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