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A strong convergence theorem for fixed points of generalized asymptotically quasi-ϕ-nonexpansive mappings

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Abstract
The purpose of this paper is to investigate a hybrid projection algorithm for a pair of generalized asymptotically quasi-ϕ-nonexpansive mappings. Strong convergence of the purposed algorithm is obtained in a uniformly smooth and uniformly convex Banach space.
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1 Introduction

The theory of iterative algorithms is a popular research topic of common interest in two areas of nonlinear analysis and optimization. Applications of iterative algorithms are found in a wide range of areas, including economics, image recovery, optimization, signal processing and a lot of real world applications; see [1–22] and the references therein. Many well-known problems can be studied by using algorithms which are iterative in their nature. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set C_m in which the required solution lies. The problem of finding a point in the intersection $\bigcap_{m=1}^{N} C_m$, where $N \geq 1$ is some positive integer, is then of crucial interest, and it cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such a point.

The purpose of this paper is to investigate a hybrid projection algorithm for a pair of generalized asymptotically quasi-ϕ-nonexpansive mappings. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a modified Halpern iterative algorithm is investigated. Strong convergence of the purposed algorithm is obtained in a uniformly convex and uniformly smooth Banach space. Some subresults are also deduced.

2 Preliminaries

Let E be a real Banach space, C be a nonempty subset of E and $T : C \to C$ be a nonlinear mapping. The mapping T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n\to\infty}\left\{\left\|T^{n+1}x - T^n x\right\| : x\in K\right\} = 0.$$

The mapping T is said to be closed if for any sequence $\{x_n\}\subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. A point $x\in C$ is a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T and use $\rightarrow$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x,y\in C.$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}\subset [1,\infty)$ with $k_n\to 1$ as $n\to\infty$ such that

$$\left\|T^n x - T^n y\right\| \le k_n\|x - y\|, \quad \forall x,y\in C, \forall n\ge 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [23] in 1972. In uniformly convex Banach spaces, they proved that if C is nonempty bounded closed and convex, then every asymptotically nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence of iterative algorithms for such a class of mappings.

One of classical iterations is the Halpern iteration [24] which generates a sequence in the following manner:

$$\forall x_1\in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n\ge 1, \tag{2.1}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$ and $u\in C$ is a fixed element.

Since 1967, the Halpern iteration has been studied extensively by many authors; see, for example, [25–31]. It is well known that the following two restrictions

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$

are necessary if the Halpern iterative sequence is strongly convergent for all nonexpansive self-mappings defined on C. To improve the rate of convergence of the Halpern iterative sequence, we cannot rely only on the iteration itself. Hybrid projection methods recently have been applied to solve the problem.

Martinez-Yanes and Xu [27] considered the hybrid projection algorithm for a single nonexpansive mapping in a Hilbert space. Strong convergence theorems are established under condition (C1) only imposed on the control sequence. To be more precise, they proved the following theorem.

Theorem 2.1 *Let H be a real Hilbert space, C be a closed convex subset of H and $T : C\to C$ be a nonexpansive mapping such that $F(T)\neq\emptyset$. Assume that $\{\alpha_n\}\subset (0,1)$ is such that*

$\lim_{n\to\infty}\alpha_n = 0$. *Then the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \quad \forall n \ge 0, \end{cases} \tag{2.2}$$

converges strongly to $P_{F(T)}x_0$.

Recently, some authors considered the problem of extending Theorem MYX to a Banach space. In this paper, we consider, in the framework of Banach spaces, the problem of modifying the Halpern iteration by hybrid projection algorithms such that strong convergence is available under assumption (C1) only. Before proceeding further, we give some definitions and propositions in Banach spaces first.

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit (2.3) is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on the Kadec-Klee property, the readers can refer to [32] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [33] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{2.4}$$

Observe that, in a Hilbert space H, (2.4) is reduced to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x) = \min_{y\in C} \phi(y,x).$$

Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J; see, for example, [32]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of a function ϕ that

$$\bigl(\|x\| - \|y\|\bigr)^2 \le \phi(x,y) \le \bigl(\|y\| + \|x\|\bigr)^2, \quad \forall x,y \in E, \tag{2.5}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle, \quad \forall x,y,z \in E. \tag{2.6}$$

Remark 2.2 If E is a reflexive, strictly convex and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (2.5), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; for more details, see [32] and the references therein.

Let C be a nonempty closed convex subset of E and T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p,Tx) \le \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be relatively asymptotically nonexpansive [34] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p,Tx) \le k_n\phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

The mapping T is said to be quasi-ϕ-nonexpansive [35] if $F(T) \neq \emptyset$ and $\phi(p,Tx) \le \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be asymptotically quasi-ϕ-nonexpansive [36] and [37] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p,Tx) \le k_n\phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

Remark 2.3 The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings which requires the restriction $F(T) = \widetilde{F}(T)$.

Recently, Qin *et al.* [29] further improved the above results by considering the so-called shrinking projection method for a quasi-ϕ-nonexpansive mapping. To be more precise, they proved the following theorem.

Theorem 2.4 *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E, and let $T : C \to C$ be a closed and quasi-ϕ-nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following* *manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J T x_n], \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1. \end{cases} \tag{2.7}$$

Assume that the control sequence satisfies the restriction $\lim_{n\to\infty} \alpha_n = 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x_1$.

Recently, Qin *et al.* [38] introduced a class of generalized asymptotically quasi-ϕ-nonexpansive mappings. Recall that a mapping T is said to be generalized asymptotically quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and there exist a sequence $\{\mu_n\} \subset [1, \infty)$ with $\mu_n \to 1$ as $n \to \infty$ and a sequence $\{\nu_n\} \subset [0, \infty)$ with $\nu_n \to 0$ as $n \to \infty$ such that $\phi(p, Tx) \le \mu_n \phi(p, x) + \nu_n$ for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

In E is a Hilbert space, the mapping T is reduced to a generalized asymptotically quasi-nonexpansive mapping, which was considered by Agarwal *et al.* [39], Shahzad and Zegeye [40] and Lan [41]. Next, we give examples of the mapping.

Let $E = \mathbb{R}^1$ and $C = [0, 1]$. Define the following mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{2}x, & x \in [0, \frac{1}{2}], \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then T is a generalized asymptotically ϕ-nonexpansive mapping with the fixed point set $\{0\}$. We also have the following

$$\phi\bigl(T^n x, T^n y\bigr) = \bigl|T^n x - T^n y\bigr|^2 = \frac{1}{2^{2n}}|x-y|^2 \le |x-y|^2 = \phi(x, y), \quad \forall x, y \in \left[0, \frac{1}{2}\right],$$

$$\phi\bigl(T^n x, T^n y\bigr) = \bigl|T^n x - T^n y\bigr|^2 = 0 \le |x-y|^2 = \phi(x, y), \quad \forall x, y \in \left(\frac{1}{2}, 1\right]$$

and

$$\begin{aligned} \phi\bigl(T^n x, T^n y\bigr) &= \bigl|T^n x - T^n y\bigr|^2 \\ &= \left|\frac{1}{2^n}x - 0\right|^2 \\ &\le \left(\frac{1}{2^n}|x-y| + \frac{1}{2^n}|y|\right)^2 \\ &\le \left(|x-y| + \frac{1}{2^n}\right)^2 \\ &\le |x-y|^2 + \xi_n \\ &= \phi(x, y) + \xi_n, \quad \forall x \in \left[0, \frac{1}{2}\right], \forall y \in \left(\frac{1}{2}, 1\right], \end{aligned}$$

where $\xi_n = \frac{1}{2^{2n}} + \frac{1}{2^{n-1}}$. Hence, we have

$$\phi\left(T^n x, T^n y\right) \le \phi(x,y) + \xi_n, \quad \forall x,y \in [0,1].$$

This shows that T a generalized asymptotically ϕ-nonexpansive mapping instead of an asymptotically ϕ-nonexpansive mapping.

Let $E = l^2$ with the norm $\|\cdot\|$ defined by $\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$ and

$$C = \left\{x = (x_1, x_2, \ldots, x_n, \ldots) | x_1 \le 0, x_i \in \mathbb{R}, i = 2, 3, \ldots\right\}.$$

Define $T : C \to C$ by

$$Tx = (0, 4x_1, 0, \ldots), \quad \forall x \in C.$$

Then T is generalized asymptotically quasi-ϕ-nonexpansive but not asymptotically quasi-ϕ-nonexpansive; for more details, see Lan [41] and the references therein.

In this paper, motivated by the above results, we investigate a hybrid projection algorithm for a pair of generalized asymptotically quasi-ϕ-nonexpansive mappings. Strong convergence of the purposed algorithm is obtained in a uniformly convex and smooth Banach space. The results presented in this paper mainly improve the corresponding results in Wu and Hao [25], Cho *et al.* [26], Martinez-Yanes and Xu [27], Plubtieng and Ungchittrakool [28], Qin *et al.* [29] and Qin and Su [31].

In order to give our main results, we need the following lemmas.

Lemma 2.5 [33] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.6 [33] *Let E be a reflexive, strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \forall y \in C.$$

Lemma 2.7 [42] *Let E be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g\left(\|x - y\|\right)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.8 [43] *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

3 Main results

Theorem 3.1 *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and generalized asymptotically quasi-ϕ-nonexpansive mapping with a sequence $\{e_n\} \subset [1,\infty)$ such that $e_n \to 1$ as $n \to \infty$ and a sequence $\{\mu_n\} \subset [0,\infty)$, where $\nu_n \to 0$ as $n \to \infty$. Let $S : C \to C$ be a closed and generalized asymptotically quasi-ϕ-nonexpansive mapping with a sequence $\{f_n\} \subset [1,\infty)$ such that $f_n \to 1$ as $n \to \infty$ and a sequence $\{\nu_n\} \subset [0,\infty)$, where $\nu_n \to 0$ as $n \to \infty$. Assume that T and S are asymptotically regular on C and $\mathcal{F} = F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n), \\ y_n = J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)Jz_n), \\ C_{n+1} = \{z \in C_n : \phi(z,y_n) \le \phi(z,x_n) + \alpha_n(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1\rangle) + (k_n - 1)M_n + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $k_n = \max\{e_n, f_n\}$, $\xi_n = \max\{\mu_n, \nu_n\}$, $M_n = \sup\{\phi(z,x_n) : z \in \mathcal{F}\}$ for each $n \ge 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$ such that

(a) $\beta_n + \gamma_n + \delta_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;

(c) $\liminf_{n\to\infty} \gamma_n \delta_n > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$, where $\Pi_{\mathcal{F}}$ is the generalized projection from C onto $\mathcal{F}$.

Proof First, we show that $\mathcal{F}$ is closed and convex. Since T and S are closed, we can easily conclude that $F(T)$ and $F(S)$ are also closed. This proves that $\mathcal{F}$ is closed. Next, we prove the convexity of $\mathcal{F}$. Let $p_1, p_2 \in F(T)$, and $p = tp_1 + (1-t)p_2$, where $t \in (0,1)$. We see that $p = Tp$. Indeed, we see from the definition of T that

$$\phi\bigl(p_1, T^n p\bigr) \le k_n \phi(p_1, p) + \xi_n \tag{3.1}$$

and

$$\phi\bigl(p_2, T^n p\bigr) \le k_n \phi(p_2, p) + \xi_n. \tag{3.2}$$

In view of (2.6), we obtain that

$$\phi\bigl(p_1, T^n p\bigr) = \phi(p_1, p) + \phi\bigl(p, T^n p\bigr) + 2\bigl\langle p_1 - p, Jp - JT^n p\bigr\rangle \tag{3.3}$$

and

$$\phi\bigl(p_1, T^n p\bigr) = \phi(p_1, p) + \phi\bigl(p, T^n p\bigr) + 2\bigl\langle p_1 - p, Jp - JT^n p\bigr\rangle. \tag{3.4}$$

Combining (3.1), (3.2), (3.3) with (3.4) yields that

$$\phi(p, T^n p) \le (k_n - 1)\phi(p_1, p) + 2\langle p - p_1, Jp - JT^n p\rangle + \xi_n \tag{3.5}$$

and

$$\phi(p, T^n p) \le (k_n - 1)\phi(p_2, p) + 2\langle p - p_2, Jp - JT^n p\rangle + \xi_n. \tag{3.6}$$

Multiplying t and $(1 - t)$ on the both sides of (3.5) and (3.6), respectively, yields that

$$\lim_{n\to\infty} \phi(p, T^n p) = 0.$$

By Lemma 2.8, we see that $T^n p \to p$ as $n \to \infty$. Hence $TT^n p = T^{n+1} p \to p$ as $n \to \infty$. In view of the closedness of T, we can obtain that $p \in F(T)$. This shows that $F(T)$ is convex. In the way, we can obtain that $F(S)$ is also convex. This completes the proof that $\mathcal{F}$ is closed and convex.

Now, we show that C_n is closed and convex for each $n \ge 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some $h \in \mathbb{N}$. For $z \in C_h$, we see that

$$\phi(z, y_h) \le \phi(z, x_h) + \alpha_h\big(\|x_1\|^2 + 2\langle z, Jx_h - Jx_1\rangle\big) + (k_h - 1)M_h + \xi_h$$

is equivalent to

$$2\langle z, Jx_h - Jy_h\rangle + 2\alpha_h\langle z, Jx_1 - Jx_h\rangle \le \|x_h\|^2 - \|y_h\|^2 + \alpha_h\|x_1\|^2 + (k_h - 1)M_h + \xi_h.$$

It is not hard to see that C_{h+1} is closed and convex. Then, for each $n \ge 1$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_1$ is well defined.

Next, we prove that $\mathcal{F} \subset C_n$ for each $n \ge 1$. $\mathcal{F} \subset C_1 = C$ is obvious. Suppose that $\mathcal{F} \subset C_h$ for some $h \in \mathbb{N}$. Then, $\forall w \in \mathcal{F} \subset C_h$, we find from Lemma 2.7 that

$$\begin{aligned}
\phi(w, z_h) &= \phi\big(w, J^{-1}\big(\beta_h Jx_h + \gamma_h JT^h x_h + \delta_h JS^h x_h\big)\big) \\
&= \|w\|^2 - 2\langle w, \beta_h Jx_h + \gamma_h JT^h x_h + \delta_h JS^h x_h\rangle + \big\|\beta_h Jx_h + \gamma_h JT^h x_h + \delta_h JS^h x_h\big\|^2 \\
&\le \|w\|^2 - 2\beta_h\langle w, Jx_h\rangle - 2\gamma_h\langle w, JT^h x_h\rangle - 2\delta_h\langle w, JS^h x_h\rangle \\
&\quad + \beta_h\|x_h\|^2 + \gamma_h\big\|T^h x_h\big\|^2 + \delta_h\big\|S^h x_h\big\|^2 \\
&= \beta_h\phi(w, x_h) + \gamma_h\phi\big(w, T^h x_h\big) + \delta_h\phi\big(w, S^h x_h\big) \\
&\le \beta_h\phi(w, x_h) + \gamma_h k_h\phi(w, x_h) + \gamma_h\xi_n + \delta_h k_h\phi(w, x_h) + \delta_h\xi_h \\
&\le \phi(w, x_h) + (k_h - 1)\phi(w, x_h) + \xi_h.
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(w, y_h) &= \phi\big(w, J^{-1}\big(\alpha_h Jx_1 + (1 - \alpha_h)Jz_h\big)\big) \\
&= \|w\|^2 - 2\langle w, \alpha_h Jx_1 + (1 - \alpha_h)JT^h x_h\rangle + \big\|\alpha_h Jx_1 + (1 - \alpha_h)Jz_h\big\|^2
\end{aligned}$$

$$\begin{aligned}
&\le \|w\|^2 - 2\alpha_h\langle w, Jx_1\rangle - 2(1-\alpha_h)\langle w, Jz_h\rangle + \alpha_h\|x_1\|^2 + (1-\alpha_h)\|z_h\|^2 \\
&= \alpha_h\phi(w,x_1) + (1-\alpha_h)\phi(w,z_h) \\
&\le \alpha_h\phi(w,x_1) + (1-\alpha_h)\phi(w,x_h) + (k_h-1)(1-\alpha_h)\phi(w,x_h) + \xi_h \\
&\le \phi(w,x_h) + \alpha_h\big(\phi(w,x_1) - \phi(w,x_h)\big) + (k_h-1)(1-\alpha_h)\phi(w,x_h) + \xi_h \\
&\le \phi(w,x_h) + \alpha_h\big(\|x_1\|^2 + 2\langle w, Jx_h - Jx_1\rangle\big) + (k_h-1)M_h + \xi_h.
\end{aligned}$$

This shows that $w \in C_{h+1}$. This implies that $\mathcal{F} \subset C_n$. In view of $x_n = \Pi_{C_n}x_1$, we see that

$$\langle x_n - z, Jx_1 - Jx_n\rangle \ge 0, \quad \forall z \in C_n.$$

By $\mathcal{F} \subset C_n$, we find that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \ge 0, \quad \forall w \in \mathcal{F}. \tag{3.7}$$

From Lemma 2.6, we see that

$$\phi(x_n,x_1) = \phi(\Pi_{C_n}x_1, x_1) \le \phi(w,x_1) - \phi(w,x_n) \le \phi(w,x_1)$$

for each $w \in \mathcal{F} \subset C_n$. Therefore, the sequence $\phi(x_n,x_1)$ is bounded. This implies that $\{x_n\}$ is bounded. On the other hand, in view of $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n,x_1) \le \phi(x_{n+1},x_1), \quad \forall n \ge 1.$$

Therefore, $\{\phi(x_n,x_1)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n,x_1)\}$ exists. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m}x_1 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned}
\phi(x_m,x_n) &= \phi(x_m, \Pi_{C_n}x_1) \\
&\le \phi(x_m,x_1) - \phi(\Pi_{C_n}x_1, x_1) \\
&= \phi(x_m,x_1) - \phi(x_n,x_1).
\end{aligned} \tag{3.8}$$

Letting $m,n \to \infty$ in (3.8), we see that $\phi(x_m,x_n) \to 0$. It follows from Lemma 2.8 that $x_m - x_n \to 0$ as $m,n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we can assume that

$$\lim_{n\to\infty} x_n = p \in C. \tag{3.9}$$

Now, we are in a position to show $p \in F(T) \cap F(S)$. By taking $m = n+1$, we obtain that

$$\lim_{n\to\infty} \phi(x_{n+1},x_n) = 0. \tag{3.10}$$

In view of Lemma 2.8, we see that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}$$

Since $x_{n+1} \in C_{n+1}$, we obtain that

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) + \alpha_n\bigl(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1\rangle\bigr) + (k_n - 1)M_n + \xi_n.$$

In view of condition (b), we find from (3.10) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.12}$$

This in turn implies from Lemma 2.8 that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.13}$$

Note that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

Combining (3.11) with (3.13) yields that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.15}$$

On the other hand, we have $Jy_n - Jz_n = \alpha_n(Jx_1 - Jz_n)$. In view of condition (a), we see that

$$\lim_{n\to\infty} \|Jy_n - Jz_n\| = 0. \tag{3.16}$$

Note that

$$\|Jx_n - Jz_n\| \le \|Jx_n - Jy_n\| + \|Jy_n - Jz_n\|.$$

Combining (3.15) with (3.16), we arrive at

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0. \tag{3.17}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.18}$$

Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Let $r = \sup_{n\ge 1}\{\|x_n\|, \|T^n x_n\|, \|S^n x_n\|\}$. From Lemma 2.8, we have

$$\begin{aligned}
&\phi(w, z_n) \\
&\quad = \phi\bigl(w, J^{-1}\bigl(\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\bigr)\bigr)
\end{aligned}$$

$$\begin{aligned}
&= \|w\|^2 - 2\langle w, \beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\rangle + \|\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\|^2 \\
&\le \|w\|^2 - 2\beta_n\langle w, Jx_n\rangle - 2\gamma_n\langle w, JT^n x_n\rangle - 2\delta_n\langle w, JS^n x_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + \gamma_n\|T^n x_n\|^2 + \delta_n\|S^n x_n\|^2 - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|) \\
&\le \beta_n\phi(w,x_n) + \gamma_n\phi(w,T^n x_n) + \delta_n\phi(w,S^n x_n) - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|) \\
&\le \beta_n\phi(w,x_n) + \gamma_n k_n\phi(w,x_n) + \delta_n k_n\phi(w,x_n) - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|) + \xi_n \\
&\le \phi(w,x_n) + (k_n-1)\phi(w,x_n) - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|) + \xi_n.
\end{aligned}$$

It follows that

$$\gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|) \le \phi(w,x_n) - \phi(w,z_n) + (k_n-1)\phi(w,x_n) + \xi_n. \tag{3.19}$$

On the other hand, we have

$$\begin{aligned}
\phi(w,x_n) - \phi(w,z_n) &= \|x_n\|^2 - \|z_n\|^2 - 2\langle w, Jx_n - Jz_n\rangle \\
&\le \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|w\|\|Jx_n - Jz_n\|.
\end{aligned}$$

It follows from (3.17) and (3.18) that

$$\phi(w,x_n) - \phi(w,z_n) \to 0 \quad \text{as } n\to\infty. \tag{3.20}$$

In view of the assumption $\liminf_{n\to\infty}\gamma_n\delta_n > 0$, we find from (3.19) that

$$\lim_{n\to\infty} g(\|JT^n x_n - JS^n x_n\|) = 0.$$

It follows from the property of g that

$$\lim_{n\to\infty}\|JT^n x_n - JS^n x_n\| = 0. \tag{3.21}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we arrive at

$$\lim_{n\to\infty}\|T^n x_n - S^n x_n\| = 0. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}
\phi(T^n x_n, S^n x_n) &= \|T^n x_n\|^2 - 2\langle T^n x_n, JS^n x_n\rangle + \|S^n x_n\|^2 \\
&= \|T^n x_n\|^2 - 2\langle T^n x_n, JT^n x_n\rangle + 2\langle T^n x_n, JT^n x_n - JS^n x_n\rangle + \|S^n x_n\|^2 \\
&\le \|S^n x_n\|^2 - \|T^n x_n\|^2 + 2\|T^n x_n\|\|JT^n x_n - JS^n x_n\| \\
&\le (\|S^n x_n\| + \|T^n x_n\|)\|S^n x_n - T^n x_n\| + 2\|T^n x_n\|\|JT^n x_n - JS^n x_n\|.
\end{aligned}$$

From (3.21) and (3.22), we arrive at

$$\lim_{n\to\infty}\phi(T^n x_n, S^n x_n) = 0. \tag{3.23}$$

On the other hand, we have

$$\begin{aligned}
&\phi(T^n x_n, z_n) \\
&\quad = \phi\big(T^n x_n, J^{-1}(\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n)\big) \\
&\quad = \|T^n x_n\|^2 - 2\langle T^n x_n, \beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\rangle \\
&\qquad + \|\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\|^2 \\
&\quad \le \|T^n x_n\|^2 - 2\beta_n\langle T^n x_n, Jx_n\rangle - 2\gamma_n\langle T^n x_n, JT^n x_n\rangle - 2\delta_n\langle T^n x_n, JS^n x_n\rangle \\
&\qquad + \beta_n\|x_n\|^2 + \gamma_n\|T^n x_n\|^2 + \delta_n\|S^n x_n\|^2 \\
&\quad \le \beta_n\phi(T^n x_n, x_n) + \delta_n\phi(T^n x_n, S^n x_n).
\end{aligned}$$

In view of restriction (a), we find (3.23) that

$$\lim_{n\to\infty}\phi(T^n x_n, z_n) = 0. \tag{3.24}$$

It follows from Lemma 2.8 that

$$\lim_{n\to\infty}\|T^n x_n - z_n\| = 0. \tag{3.25}$$

Note that

$$\|T^n x_n - p\| \le \|T^n x_n - z_n\| + \|z_n - x_n\| + \|x_n - p\|.$$

In view of (3.9), (3.18) and (3.25), we find that

$$\lim_{n\to\infty}\|T^n x_n - p\| = 0. \tag{3.26}$$

On the other hand, we have

$$\|T^{n+1}x_n - p\| \le \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - p\|.$$

Since T is asymptotically regular, we obtain that

$$\lim_{n\to\infty}\|T^{n+1}x_n - p\| = 0.$$

That is, $TT^n x_n \to p$ as $n \to \infty$. From the closedness of T, we see that $p \in F(T)$. In the same way, we can also obtain that $p \in F(S)$. This shows that $p \in \mathcal{F}$.

Finally, we show that $p = \Pi_{\mathcal{F}}x_1$. Taking the limit as $n \to \infty$ in (3.7), we obtain that

$$\langle p - w, Jx_1 - Jp\rangle \ge 0, \quad \forall w \in \mathcal{F},$$

and hence $p = \Pi_{\mathcal{F}}x_1$ by Lemma 2.5. This completes the proof. □

Remark 3.2 Theorem 3.1 includes Theorem 2.4 in Section 2 as a special case. The framework of the space can be applicable to L^p, where $p \ge 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth and uniformly convex for every $p \ge 1$.

In the framework of Hilbert spaces, we find the following.

Corollary 3.3 *Let E be a Hilbert space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and generalized asymptotically quasi-nonexpansive mapping with a sequence $\{e_n\} \subset [1,\infty)$ such that $e_n \to 1$ as $n \to \infty$ and a sequence $\{\mu_n\} \subset [0,\infty)$, where $\nu_n \to 0$ as $n \to \infty$. Let $S : C \to C$ be a closed and generalized asymptotically quasi-nonexpansive mapping with a sequence $\{f_n\} \subset [1,\infty)$ such that $f_n \to 1$ as $n \to \infty$ and a sequence $\{\nu_n\} \subset [0,\infty)$, where $\nu_n \to 0$ as $n \to \infty$. Assume that T and S are asymptotically regular on C and $\mathcal{F} = F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ z_n = \beta_n x_n + \gamma_n T^n x_n + \delta_n J S^n x_n, \\ y_n = \alpha_n x_1 + (1 - \alpha_n) z_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \le \|z - x_n\|^2 + \alpha_n(\|x_1\|^2 + 2\langle z, x_n - x_1\rangle) + (k_n - 1)M_n + \xi_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $k_n = \max\{e_n, f_n\}$, $\xi_n = \max\{\mu_n, \nu_n\}$, $M_n = \sup\{\|z - x_n\|^2 : z \in \mathcal{F}\}$ for each $n \ge 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$ such that

(a) $\beta_n + \gamma_n + \delta_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;

(c) $\liminf_{n\to\infty} \gamma_n \delta_n > 0$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_1$, where $P_{\mathcal{F}}$ is the metric projection from C onto $\mathcal{F}$.

For the class of asymptotically quasi-ϕ-nonexpansive mappings, we find from Theorem 3.1 the following.

Corollary 3.4 *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with a sequence $\{e_n\} \subset [1,\infty)$ such that $e_n \to 1$ as $n \to \infty$. Let $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with a sequence $\{f_n\} \subset [1,\infty)$ such that $f_n \to 1$ as $n \to \infty$. Assume that T and S are asymptotically regular on C and $\mathcal{F} = F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\beta_n J x_n + \gamma_n J T^n x_n + \delta_n J S^n x_n), \\ y_n = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J z_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(\|x_1\|^2 + 2\langle z, J x_n - J x_1\rangle) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $k_n = \max\{e_n, f_n\}$, $M_n = \sup\{\phi(z,x_n) : z \in \mathcal{F}\}$ *for each* $n \geq 1$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are real sequences in* $(0,1)$ *such that*

(a) $\beta_n + \gamma_n + \delta_n = 1$;
(b) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;
(c) $\liminf_{n\to\infty} \gamma_n \delta_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_1$, *where* $\Pi_{\mathcal{F}}$ *is the generalized projection from* C *onto* $\mathcal{F}$.

If both T and S are quasi-ϕ-nonexpansive, we find from Theorem 3.1 the following.

Corollary 3.5 *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* C *be a nonempty closed and convex subset of* E. *Let* $T : C \to C$ *be a closed quasi-*ϕ*-nonexpansive mapping, and* $S : C \to C$ *be a closed quasi-*ϕ*-nonexpansive mapping with a nonempty common fixed point set. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\beta_n J x_n + \gamma_n J T x_n + \delta_n J S x_n), \\ y_n = J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J z_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1\rangle)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are real sequences in* $(0,1)$ *such that*

(a) $\beta_n + \gamma_n + \delta_n = 1$;
(b) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;
(c) $\liminf_{n\to\infty} \gamma_n \delta_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_1$, *where* $\Pi_{\mathcal{F}}$ *is the generalized projection from* C *onto* $\mathcal{F}$.

Putting $\beta_n = 0$ and $T = S$, we find from Corollary 3.5 the following.

Corollary 3.6 *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* C *be a nonempty closed and convex subset of* E. *Let* $T : C \to C$ *be a closed quasi-*ϕ*-nonexpansive mapping with a nonempty fixed point set. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J T x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1\rangle)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}}x_1$, *where* $\Pi_{F(T)}$ *is the generalized projection from C onto* $F(T)$.

Remark 3.7 Corollary 3.6 is a Banach version of Theorem 2.1 in Section 2. The sets of Q_n are also relaxed.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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