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A new general iterative algorithm with Meir-Keeler contractions for variational inequality problems in q-uniformly smooth Banach spaces

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Abstract

In this paper, we generalize the iterative scheme and extend the space studied in (Fixed Point Theory and Applications 2012:46, 2012). Further, we prove some strong convergence theorems of the new iterative scheme for variational inequality problems in q-uniformly smooth Banach spaces under very mild conditions. Our results improve and extend corresponding ones announced by many others.

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1 Introduction

Throughout this paper, we denote by X and X^* a real Banach space and the dual space of X, respectively. Let C be a nonempty closed convex subset of X.

The duality mapping $J : X \to 2^{X^*}$ is defined by $J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2, \|x^*\| = \|x\|\}$, $\forall x \in X$. It is well known that if X is smooth, then J is single-valued, which is denoted by j. Let $q > 1$ be a real number. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\},$$

for all $x \in X$, where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between X and X^*. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. It is well known that if X is smooth, then J_q is single-valued, which is denoted by j_q.

Recall that a mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $W : C \to C$ is said to be nonexpansive if

$$\|W(x) - W(y)\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

A mapping $F : E \to C$ is said to be L-Lipschitzian if there exists a positive constant L such that

$$\left\|F(x) - F(y)\right\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

A mapping $F : E \to C$ is said to be η-strongly accretive if there exist $j_q(x - y) \in J_q(x - y)$ and $\eta > 0$ such that

$$\left\langle Fx - Fy, j_q(x - y)\right\rangle \geq \eta\|x - y\|^q, \quad \forall x, y \in C. \tag{1.4}$$

Without loss of generality, we can assume that $\eta \in (0, 1]$ and $L \in [1, \infty)$.

Recall that if C and D are nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is sunny [1] provided $P(x + t(x - P(x))) = P(x)$ for all $x \in C$ and $t \geq 0$, whenever $x + t(x - P(x)) \in C$. A mapping $P : C \to D$ is called a retraction if $Px = x$ for all $x \in D$. Furthermore, P is a sunny nonexpansive retraction from C onto D if P is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

Let $\{B_n\}$ be a family of mappings from a subset C of a Banach space X into itself with $\bigcap_{n=1}^{\infty} F(B_n) \neq \emptyset$. We say that $\{B_n\}$ satisfies the AKTT-condition if for each bounded subset D of C,

$$\sum_{n=1}^{\infty} \sup_{\omega \in D} \|B_{n+1}\omega - B_n\omega\| < \infty.$$

Proposition 1.1 (Banach [2]) *Let (X, d) be a complete metric space, and let f be a contraction on X, then f has a unique fixed point.*

Proposition 1.2 (Meir and Keeler [3]) *Let (X, d) be a complete metric space, and let ϕ be a Meir-Keeler contraction (MKC, for short) on X, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(\phi(x), \phi(y)) < \varepsilon$ for all $x, y \in X$. Then ϕ has a unique fixed point.*

This proposition is one of generalizations of Proposition 1.1, because the contractions are Meir-Keeler contractions.

Proposition 1.3 [4] *Let C be a closed convex subset of a smooth Banach space X. Let $\widetilde{C}$ be a nonempty subset of C. Let $Q_C : C \to \widetilde{C}$ be a retraction, and let J be the normalized duality mapping on X. Then the following are equivalent:*

(i) *Q_C is sunny and nonexpansive.*
(ii) $\|Q_C x - Q_C y\|^2 \leq \langle x - y, J(Q_C x - Q_C y)\rangle$, $\forall x, y \in C$.
(iii) $\langle x - Q_C x, J(y - Q_C x)\rangle \leq 0$, $\forall x \in C, y \in \widetilde{C}$.

Variational inequality theory has emerged as a great important tool in studying a wide class of unilateral, free, obstacle, moving and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. This field is dynamics and it is experiencing an explosive growth in both theory and applications. Several numerical methods have been developed for solving variational inequalities and related optimization problems; see [4–7] and the references therein.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the classical variational inequality is to find an x^* such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.5}$$

where $F : C \to C$ is a nonlinear mapping. The set of solutions of (1.5) is denoted by $\mathrm{VI}(F, C)$.

In 2008, Yao *et al.* [8] modified Mann's iterative scheme by using the viscosity approximation method which was introduced by Moudafi [1]. More precisely, they introduced and studied the following iterative algorithm:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where T is a nonexpansive mapping of K into itself and f is a contraction on K. They obtained a strong convergence theorem under some mild restrictions on the parameters.

Zhou [9] and Qin *et al.* [10] modified normal Mann's iterative process (1.6) for k-strictly pseudo-contractions to have strong convergence in Hilbert spaces. Qin *et al.* [10] introduced the following iterative algorithm scheme:

$$\begin{cases} x_1 = x \in K, \\ y_n = P_K[\beta_n x_n + (1 - \beta_n) T x_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n A) y_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where T is a k-strictly pseudo-contraction, f is a contraction and A is a strong positive linear bounded operator, P_K is the metric projection. They proved, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.7) converges strongly to a fixed point of the k-strictly pseudo-contraction, which solves some variational inequality.

Very recently, Song *et al.* [11] introduced the following iteration process:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C[\beta_n x_n + (1 - \beta_n) \sum_{i=1}^{\infty} \mu_i^{(n)} T_i x_n], \\ x_{n+1} = \alpha_n \phi(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n F) y_n, \quad n \geq 1, \end{cases} \tag{1.8}$$

where T_i is a k_i-strictly pseudo-contraction, ϕ is an MKC contraction and $F : C \to C$ is an L-Lipschitzian and η-strongly monotone mapping in a Hilbert space, P_C is the metric projection. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_i^n\}$, the sequence $\{x_n\}$ defined by (1.8) converges strongly to a common fixed point of an infinite family of k_i-strictly pseudo-contractions, which solves some variational inequality.

Question 1 Can the space in Song [11] be extended from a Hilbert space to a q-uniformly smooth Banach space?

Question 2 Can the projection P_C in Song [11] be changed to the sunny nonexpansive retraction Q_C and be put to other place of the iteration process?

Question 3 Can we extend the iterative scheme of algorithm (1.8) to a more general iterative scheme?

Question 4 Can we remove the very strict condition $C + C \subset C$ which is necessary in Lemma 3.1 and Theorem 3.2 of Song [11]?

The purpose of this paper is to give affirmative answers to these questions mentioned above. In this paper we study a new general iterative scheme as follows:

$$\begin{cases} x_1 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)\sum_{i=1}^{\infty}\mu_i^{(n)} T_i x_n, \\ x_{n+1} = Q_C[\alpha_n \gamma \phi(x_n) + \gamma_n(\alpha I + (1-\alpha)\sum_{i=1}^{\infty}\mu_i^{(n)} T_i)x_n + ((1-\gamma_n)I - \alpha_n F)y_n], \\ \quad n \geq 1, \end{cases} \tag{1.9}$$

where T_i is a λ_i-strictly pseudo-contraction, ϕ is an MKC contraction, Q_C is the sunny nonexpansive retraction and $F : X \to C$ is an L-Lipschitzian and η-strongly accretive mapping in a q-uniformly smooth Banach space. Under some suitable assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_i^{(n)}\}$, the sequence $\{x_n\}$ defined by (1.9) converges strongly to a common fixed point of an infinite family of λ_i-strictly pseudo-contractions, which solves some variational inequality.

2 Preliminaries

In this section, we first recall some notations. T is said to be a λ-strict pseudo-contraction in the terminology of Browder and Petryshyn [12] if there exists a constant $\lambda \in [0,1)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^q - \lambda\|(I-T)x - (I-T)y\|^q \tag{2.1}$$

for every $x, y \in C$ and for some $j_q(x-y) \in J_q(x-y)$. It is clear that (2.1) is equivalent to the following:

$$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq \lambda\|(I-T)x - (I-T)y\|^q. \tag{2.2}$$

A Banach space X is said to be strictly convex if whenever x and y are not collinear, then $\|x+y\| < \|x\| + \|y\|$.

Then the modulus of convexity of X is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\|, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0,2]$. X is said to be uniformly convex if $\delta_X(0) = 0$ and $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \leq 2$, and if $\delta_X(\epsilon) \geq c\epsilon^p$ with $p \geq 2$, then X is said to be p-uniformly convex. A Hilbert space H is 2-uniformly convex, while L^p is $\max\{p,2\}$-uniformly convex for every $p > 1$. Let ρ_X :

$[0,\infty) \to [0,\infty)$ be the modulus of smoothness of X defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : x \in S(X), \|y\| \leq t\right\}.$$

A Banach space X is said to be uniformly smooth if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$. A Banach space X is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_X(t) \leq ct^q$ with $q > 1$. A typical example of uniformly smooth Banach spaces is L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$.

The norm of a Banach space X is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.3}$$

exists for all x, y on the unit sphere $S(X) = \{x \in X : \|x\| = 1\}$. If, for each $y \in S(X)$, the limit (2.3) is uniformly attained for $x \in S(X)$, then the norm of X is said to be uniformly Gâteaux differentiable. The norm of X is said to be Fréchet differentiable if, for each $x \in S(X)$, the limit (2.3) is attained uniformly for $y \in S(X)$.

In order to prove our main results, we need the following lemmas.

Lemma 2.1 [13] *Let ϕ be an MKC on a convex subset C of a Banach space X. Then, for each $\varepsilon > 0$, there exists $r \in (0,1)$ such that*

$$\|x-y\| \geq \varepsilon \textit{ implies } \|\phi x - \phi y\| \leq r\|x-y\|, \quad \forall x,y \in C.$$

Lemma 2.2 [14] *Let X be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\big\langle y, J_q(x)\big\rangle + C_q\|y\|^q \quad \textit{for all } x,y \in X.$$

Lemma 2.3 [15] *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the property*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + b_n + \gamma_n c_n, \quad n \geq 0,$$

where $\{\gamma_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy the restrictions:

(i) $\limsup_{n\to\infty} \gamma_n = 0$, $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $b_n \geq 0$, $\sum_{n=1}^{\infty} b_n < \infty$;
(iii) $\limsup_{n\to\infty} c_n \leq 0$. *Then* $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4 [16] *Let C be a nonempty convex subset of a real q-uniformly smooth Banach space X, and let $T : C \to C$ be a λ-strict pseudo-contraction. For $\alpha \in (0,1)$, we define $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then, as $\alpha \in (0,\mu]$, $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q}}\}$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.5 [17] *Let $q > 1$, then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

Lemma 2.6 *Let F be an L-Lipschitzian and η-strongly accretive operator on a nonempty closed convex subset C of a real q-uniformly smooth Banach space X with $0 < \eta q \leq 1$ and $0 < t < (\frac{q\eta}{C_q L^q})^{\frac{1}{q-1}}$. Then $G = (I - tF) : C \to X$ is a contraction with contraction coefficient $\tau_t = 1 - \frac{1}{q}(qt\eta - C_q L^q t^q)$.*

Proof From the definition of η-strongly accretive and L-Lipschitzian operator, we have

$$\begin{aligned}\|Gx - Gy\|^q &= \|x - y - t(Fx - Fy)\|^q \\ &\leq \|x - y\|^q - qt\langle Fx - Fy, j_q(x - y)\rangle + C_q t^q \|Fx - Fy\|^q \\ &\leq \|x - y\|^q - qt\eta\|x - y\|^q + C_q L^q t^q \|x - y\|^q \\ &= \left[1 - \left(qt\eta - C_q L^q t^q\right)\right]\|x - y\|^q.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|Gx - Gy\| &\leq \left[1 - \left(qt\eta - C_q L^q t^q\right)\right]^{\frac{1}{q}}\|x - y\| \\ &\leq \left[1 - \frac{1}{q}\left(qt\eta - C_q L^q t^q\right)\right]\|x - y\|\end{aligned}$$

for all $x, y \in C$. From $0 < \eta q \leq 1$ and $0 < t < (\frac{q\eta}{C_q L^q})^{\frac{1}{q-1}}$, we have $0 < 1 - \frac{1}{q}(qt\eta - C_q L^q t^q) < 1$ and

$$\|Gx - Gy\| \leq \tau_t \|x - y\|,$$

where $\tau_t = 1 - \frac{1}{q}(qt\eta - C_q L^q t^q) \in (0, 1)$. Hence, G is a contraction with contraction coefficient τ_t. This completes the proof. □

Lemma 2.7 ([18], Demiclosedness principle) *Let C be a nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition, and suppose that $T : C \to X$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.8 *Let C be a closed convex subset of a smooth Banach space X. Let $\widetilde{C}$ be a nonempty subset of C. Let $Q_C : C \to \widetilde{C}$ be a retraction, and let j, j_q be the normalized duality mapping and generalized duality mapping on X, respectively. Then the following are equivalent*:

(i) *Q_C is sunny and nonexpansive.*
(ii) *$\|Q_C x - Q_C y\|^2 \leq \langle x - y, j(Q_C x - Q_C y)\rangle$, $\forall x, y \in C$.*
(iii) *$\langle x - Q_C x, j(y - Q_C x)\rangle \leq 0$, $\forall x \in C, y \in \widetilde{C}$.*
(iv) *$\langle x - Q_C x, j_q(y - Q_C x)\rangle \leq 0$, $\forall x \in C, y \in \widetilde{C}$.*

Proof From Proposition 1.3, we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). We need only to prove (iii) $\Leftrightarrow$ (iv).

Indeed, if $y - Q_C x \neq 0$, it follows from the fact $j_q(x) = \|x\|^{q-2} j(x)$ that $\langle x - Q_C x, j(y - Q_C x)\rangle \leq 0 \Leftrightarrow \langle x - Q_C x, j_q(y - Q_C x)\rangle \leq 0$, $\forall x \in C, y \in \widetilde{C}$.

If $y - Q_C x = 0$, then $\langle x - Q_C x, j(y - Q_C x)\rangle = \langle x - Q_C x, j_q(y - Q_C x)\rangle = 0$, $\forall x \in C, y \in \widetilde{C}$. This completes the proof. □

Lemma 2.9 [19] *Suppose that $\{B_n\}$ satisfies the AKTT-condition, then for each bounded subset D of C*:

(i) *$\{B_n\}$ converges strongly to some point in C for each $x \in C$*;

(ii) *Furthermore, if the mapping $B : C \to C$ is defined by $Bx = \lim_{n\to\infty} B_n x$ for all $x \in D$, then $\lim_{n\to\infty} \sup_{\omega\in D} \|B\omega - B_n\omega\| = 0$.*

Lemma 2.10 *Let C be a closed convex subset of a reflexive Banach space X which admits a weakly sequentially continuous duality mapping j_q from X to X^*. Let $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$ and ϕ be an MKC on C. Suppose that $F : C \to X$ is an η-strongly accretive and L-Lipschitzian mapping with coefficient and $\eta > \gamma > 0$. Then the sequence $\{x_t\}$ defined by $x_t = Q_C[t\gamma\phi(x_t) + (1 - tF)Sx_t]$ converges strongly as $t \to 0$ to a fixed point $\widetilde{x}$ of S, which solves the variational inequality*

$$\big\langle (F - \gamma\phi)\widetilde{x}, j_q(\widetilde{x} - z)\big\rangle \leq 0, \quad \forall z \in F(S). \tag{2.4}$$

Proof The definition of $\{x_t\}$ is a good definition. Indeed, from the definition of MKC, we can see that an MKC is also a nonexpansive mapping. Consider a mapping L_t on C defined by

$$L_t x = Q_C\big[t\gamma\phi(x) + (I - tF)Sx\big], \quad x \in C.$$

It is easy to see that L_t is a contraction when $0 < t < (\frac{q(\eta-\gamma)}{C_q L^q})^{\frac{1}{q-1}}$. Indeed, by Lemmas 2.1 and 2.5, we have

$$\begin{aligned}
\|L_t x - L_t y\| &= \big\|Q_C\big[t\gamma\phi(x) + (I - tF)Sx\big] - Q_C\big[t\gamma\phi(y) + (I - tF)Sy\big]\big\| \\
&\leq \big\|t\gamma\phi(x) + (I - tF)Sx - t\gamma\phi(y) - (I - tF)Sy\big\| \\
&\leq t\gamma\big\|\phi(x) - \phi(y)\big\| + \big\|(I - tF)Sx - (I - tF)Sy\big\| \\
&\leq t\gamma\big\|\phi(x) - \phi(y)\big\| + \tau_t\|Sx - Sy\| \\
&\leq t\gamma\|x - y\| + \tau_t\|x - y\| \\
&\leq \theta_t\|x - y\|,
\end{aligned}$$

where $\theta_t = t\gamma + \tau_t \in (0,1)$. Hence L_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = Q_C\big[t\gamma\phi(x_t) + (I - tF)Sx_t\big]. \tag{2.5}$$

Next we show the uniqueness of a solution of the variational inequality (2.4). Suppose that $\widetilde{x} \in F(S)$ and $\hat{x} \in F(S)$ are solutions to (2.4), then, without loss of generality, we may assume that there is a number ε such that $\|\hat{x} - \widetilde{x}\| \geq \varepsilon$. Then, by Lemma 2.1, there is a number $r \in (0,1)$ such that $\|\phi\hat{x} - \phi\widetilde{x}\| \leq r\|\hat{x} - \widetilde{x}\|$. From (2.4) we have

$$\big\langle (F - \gamma\phi)\widetilde{x}, j_q(\widetilde{x} - \hat{x})\big\rangle \leq 0, \tag{2.6}$$

$$\big\langle (F - \gamma\phi)\hat{x}, j_q(\hat{x} - \widetilde{x})\big\rangle \leq 0. \tag{2.7}$$

Adding up (2.6) and (2.7), we obtain

$$\big\langle (F-\gamma\phi)\hat{x}-(F-\gamma\phi)\widetilde{x}, j_q(\hat{x}-\widetilde{x})\big\rangle \le 0. \tag{2.8}$$

Meanwhile, we notice that

$$\begin{aligned}
\big\langle (F-\gamma\phi)\hat{x}-(F-\gamma\phi)\widetilde{x}, j_q(\hat{x}-\widetilde{x})\big\rangle &= \big\langle F\hat{x}-F\widetilde{x}, j_q(\hat{x}-\widetilde{x})\big\rangle - \gamma\big\langle \phi(\hat{x})-\phi(\widetilde{x}), j_q(\hat{x}-\widetilde{x})\big\rangle \\
&\ge \eta\|\hat{x}-\widetilde{x}\|^q - \gamma\|\phi(\hat{x})-\phi(\widetilde{x})\|\|\hat{x}-\widetilde{x}\|^{q-1} \\
&\ge \eta\|\hat{x}-\widetilde{x}\|^q - \gamma r\|\hat{x}-\widetilde{x}\|\|\hat{x}-\widetilde{x}\|^{q-1} \\
&\ge \eta\|\hat{x}-\widetilde{x}\|^q - \gamma r\|\hat{x}-\widetilde{x}\|^q \\
&= (\eta-\gamma r)\|\hat{x}-\widetilde{x}\|^q \\
&\ge (\eta-\gamma r)\varepsilon \\
&> 0.
\end{aligned}$$

Thus $\hat{x}=\widetilde{x}$ and the uniqueness is proved. Below, we use $\widetilde{x}$ to denote the unique solution of (2.3).

First, we prove that $\{x_t\}$ is bounded.

Assume that $0 < t < \frac{\eta-\gamma}{L^2}$ for $\forall z \in F(S)$, fixed ε' for each t.

Case 1. ($\|x_t - z\| < \varepsilon'$). In this case, we can see easily that $\{x_t\}$ is bounded.

Case 2. ($\|x_t - z\| \ge \varepsilon'$). In this case, by Lemma 2.1, there is a number $r' \in (0,1)$ such that $\|\phi(x_t)-\phi(p)\| < r'\|x_t - p\|$, then we have

$$\begin{aligned}
\|x_t - z\| &= \big\|Q_C\big[t\gamma\phi(x_t)+(I-tF)Sx_t\big]-z\big\| \\
&= \big\|t\gamma\phi(x_t)+(I-tF)Sx_t - z\big\| \\
&= \big\|t\big(\gamma\phi(x_t)-Fz\big)+(I-tF)Sx_t-(I-tF)z\big\| \\
&\le t\big\|\gamma\phi(x_t)-p\big\| + \tau_t\|x_t - z\| \\
&\le t\big\|\gamma\phi(x_t)-\gamma\phi(z)\big\| + t\big\|\gamma\phi(z)-Fz\big\| + \tau_t\|x_t - z\| \\
&\le t\gamma r'\|x_t - z\| + t\big\|\phi(z)-Fz\big\| + \tau_t\|x_t - z\|,
\end{aligned}$$

which implies $\|x_t - z\| \le \frac{2\|\gamma\phi(z)-z\|}{\eta-\gamma}$. Thus $\{x_t\}$ is bounded.

Then, we prove that $x_t \to \widetilde{x}$ ($\widetilde{x} \in F(S)$) as $t \to 0$.

Since X is reflexive and $\{x_t\}$ is bounded, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup x^*$. Setting $y_t = t\gamma\phi(x_t)+(I-tF)Sx_t$, we obtain $x_t = Q_C y_t$.

We claim $\|x_{t_n} - x^*\| \to 0$.

It follows from Lemma 2.8 that

$$\big\langle y_t - Q_C y_t, j_q\big(x^* - Q_C y_t\big)\big\rangle \le 0, \tag{2.9}$$

then we have

$$\begin{aligned}
\|x_{t_m} - x^*\|^q &= \big\langle Q_C y_{t_m} - y_{t_m}, j_q\big(x_{t_m}-x^*\big)\big\rangle + \big\langle y_{t_m} - x^*, j_q\big(x_{t_m}-x^*\big)\big\rangle \\
&\le \big\langle y_{t_m} - x^*, j_q\big(x_{t_m}-x^*\big)\big\rangle
\end{aligned}$$

$$\begin{aligned}&= \big\langle (I - t_m F)Sx_{t_m} - (I - t_m F)x^*, j_q\big(x_{t_m} - x^*\big)\big\rangle + t_m\big\langle \gamma\phi x_{t_m} - Fx^*, j_q\big(x_{t_m} - x^*\big)\big\rangle \\ &\le \tau_m \big\|x_{t_m} - x^*\big\|^q + t_m\big\langle \gamma\phi x_{t_m} - Fx^*, j_q\big(x_{t_m} - x^*\big)\big\rangle,\end{aligned}$$

which implies that

$$\big\|x_{t_m} - x^*\big\|^q \le \frac{t_m}{1-\tau_m}\big\langle \gamma\phi x_{t_m} - Fx^*, j_q\big(x_{t_m} - x^*\big)\big\rangle. \tag{2.10}$$

Since $t_m \to 0$ as $m \to \infty$, by (2.10) we obtain that $x_{t_m} \to x^*$. Hence, we have $x_{t_n} \to x^*$.

Now, we prove that x^* solves the variational inequality (2.4).

Since

$$x_t = Q_C y_t = Q_C y_t - y_t + t\gamma\phi(x_t) + (I - tF)Sx_t, \tag{2.11}$$

we get that

$$(F - \gamma\phi)x_t = \frac{1}{t}(Q_C y_t - y_t) - \frac{1}{t}(I - S)x_t + (Fx_t - FSx_t). \tag{2.12}$$

Notice that

$$\begin{aligned}\big\langle (I - S)x_t - (I - S)z, j_q(x_t - z)\big\rangle &= \big\langle x_t - z, j_q(x_t - z)\big\rangle - \big\langle Sx_t - Sz, j_q(x_t - z)\big\rangle \\ &\ge \|x_t - z\|^q - \|Sx_t - Sz\|\|x_t - z\|^{q-1} \\ &\ge \|x_t - z\|^q - \|x_t - z\|^q \\ &= 0.\end{aligned}$$

Then, for $z \in F(S)$,

$$\begin{aligned}\big\langle (F - \gamma\phi)x_t, j_q(x_t - z)\big\rangle &= \frac{1}{t}\big\langle Q_C y_t - y_t, j_q(x_t - z)\big\rangle - \frac{1}{t}\big\langle (I - S)x_t, j_q(x_t - z)\big\rangle \\ &\quad + \big\langle Fx_t - FSx_t, j_q(x_t - z)\big\rangle \\ &= \frac{1}{t}\big\langle Q_C y_t - y_t, j_q(x_t - z)\big\rangle - \frac{1}{t}\big\langle (I - S)x_t - (I - S)z, j_q(x_t - z)\big\rangle \\ &\quad + \big\langle Fx_t - FSx_t, j_q(x_t - z)\big\rangle \\ &\le \big\langle Fx_t - F(Sx_t), j_q(x_t - z)\big\rangle \\ &\le M\|x_t - Sx_t\|,\end{aligned} \tag{2.13}$$

where $M = \sup_{n\ge 0}\{L\|x_t - z\|^{q-1}\} < \infty$. Notice

$$x_t - Sx_t = t\big[\gamma\phi(x_t) - FSx_t\big].$$

Thus, we have

$$x_t - Sx_t \to 0 \quad \text{as } t \to 0.$$

Now replacing t in (2.13) with t_n and letting $n \to \infty$, notice that $(I - S)x_{t_n} \to (I - S)x^* = 0$ for $x^* \in F(S)$, we obtain $\langle (F - \gamma\phi)x^*, j_q(x^* - z)\rangle \le 0$, *i.e.*, $x^* \in F(S)$ is a solution of (2.4).

Hence $\widetilde{x} = x^*$ by uniqueness. Thus, we have shown that every cluster point of $\{x_t\}$ (at $t \to 0$) equals $\widetilde{x}$, therefore $x_t \to \widetilde{x}$ as $t \to 0$. □

Lemma 2.11 *Let X be a q-uniformly smooth Banach space, and let C be a nonempty convex subset of X. Assume that $T_i : C \to C$ is a countable family of λ_i-strict pseudo-contractions for some $0 \le \lambda_i < 1$ and $\inf\{\lambda_i : i \in \mathbb{N}\} > 0$ such that $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\mu_i\}$ is a positive sequence such that $\sum_{i=1}^{\infty} \mu_i = 1$. Then $\sum_{i=1}^{\infty} \mu_i T_i : C \to C$ is a λ-strict pseudo-contraction with $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\}$ and $F(\sum_{i=1}^{\infty} \mu_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof Let $H_n x = \mu_1 T_1 x + \mu_2 T_2 x + \cdots + \mu_n T_n x$, where $\sum_{i=1}^{\infty} \mu_i = 1$. Then $H_n : C \to X$ is a λ-strict pseudo-contraction with $\lambda = \min\{\lambda_i : 1 \le i \le n\}$.

Step 1. We firstly prove the case of $n = 2$.

$$
\begin{aligned}
&\bigl\langle (I - H_2)x - (I - H_2)y, j_q(x - y)\bigr\rangle \\
&\quad = \bigl\langle \mu_1(I - T_1)x + \mu_2(I - T_2)x - \mu_1(I - T_1)y - \mu_2(I - T_2)y, j_q(x - y)\bigr\rangle \\
&\quad = \mu_1\bigl\langle (I - T_1)x - (I - T_1)y, j_q(x - y)\bigr\rangle + \mu_2\bigl\langle (I - T_2)x - (I - T_2)y, j_q(x - y)\bigr\rangle \\
&\quad \ge \mu_1\lambda_1\bigl\|(I - T_1)x - (I - T_1)y\bigr\|^q + \mu_2\lambda_2\bigl\|(I - T_2)x - (I - T_2)y\bigr\|^q \\
&\quad \ge \lambda\bigl[\mu_1\bigl\|(I - T_1)x - (I - T_1)y\bigr\|^q + \mu_2\bigl\|(I - T_2)x - (I - T_2)y\bigr\|^q\bigr] \\
&\quad = \lambda\bigl\|(I - H_2)x - (I - H_2)y\bigr\|^q,
\end{aligned}
$$

where $\lambda = \min\{\lambda_i : i = 1, 2\}$, which shows that $H_2 : C \to C$ is a λ-strict pseudo-contraction.

Using the same means, our proof method can easily carry over to the general finite case.

Step 2. We prove the infinite case. From the definition of λ-strict pseudo-contraction, we have

$$
\bigl\langle (I - T_n)x - (I - T_n)y, j_q(x - y)\bigr\rangle \ge \lambda\bigl\|(I - T_n)x - (I - T_n)y\bigr\|^q, \tag{2.14}
$$

then we obtain

$$
\bigl\|(I - T_n)x - (I - T_n)y\bigr\| \le \left(\frac{1}{\lambda}\right)^{\frac{1}{q-1}} \|x - y\|. \tag{2.15}
$$

Taking $p \in F(T_n)$, it follows from (2.15) that

$$
\bigl\|(I - T_n)x\bigr\| = \bigl\|(I - T_n)x - (I - T_n)p\bigr\| \le \left(\frac{1}{\lambda}\right)^{\frac{1}{q-1}} \|x - p\|. \tag{2.16}
$$

Thus, for $\forall x \in X$, if $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ with $\mu_i > 0$ and $\sum_{i=1}^{\infty} \mu_i = 1$, then $\sum_{i=1}^{\infty} \mu_i T_i$ strongly converges.

Let

$$
Hx = \sum_{i=1}^{\infty} \mu_i T_i x,
$$

then we obtain

$$Hx = \sum_{i=1}^{\infty} \mu_i T_i x = \lim_{n\to\infty} \sum_{i=1}^{n} \mu_i T_i x = \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i T_i x.$$

Therefore

$$\begin{aligned}
&\big\langle (I-H)x-(I-H)y, j_q(x-y)\big\rangle \\
&\quad= \lim_{n\to\infty} \left\langle \left(I - \frac{1}{\sum_{i=1}^{n}\mu_i}\sum_{i=1}^{n}\mu_i T_i\right)x + \left(I - \frac{1}{\sum_{i=1}^{n}\mu_i}\sum_{i=1}^{n}\mu_i T_i\right)y, j_q(x-y)\right\rangle \\
&\quad= \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n}\mu_i}\sum_{i=1}^{n}\mu_i\big\langle (I-T_i)x-(I-T_i)y, j_q(x-y)\big\rangle \\
&\quad\ge \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n}\mu_i}\sum_{i=1}^{n}\mu_i\lambda\big\|(I-T_i)x-(I-T_i)y\big\|^q \\
&\quad\ge \lambda \lim_{n\to\infty} \left\| \left(I - \frac{1}{\sum_{i=1}^{n}\mu_i}\sum_{i=1}^{n}\mu_i T_i\right)x - \left(I - \frac{1}{\sum_{i=1}^{n}\mu_i}\sum_{i=1}^{n}\mu_i T_i\right)y\right\|^q \\
&\quad= \lambda\big\|(I-H)x-(I-H)y\big\|^q.
\end{aligned}$$

Thus, H is a λ-strict pseudo-contraction.

Step 3. We prove $F(\sum_{i=1}^{\infty}\mu_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

Let $x = \sum_{i=1}^{\infty}\mu_i T_i x$, then, for $p \in \bigcap_{i=1}^{\infty} F(T_i)$, we obtain

$$\begin{aligned}
\|x-p\|^q &= \big\langle x-p, j_q(x-p)\big\rangle \\
&= \left\langle \sum_{i=1}^{\infty}\mu_i T_i x - p, j_q(x-p)\right\rangle \\
&= \sum_{i=1}^{\infty}\mu_i\big\langle T_i x - p, j_q(x-p)\big\rangle \\
&\le \|x-p\|^q - \lambda\sum_{i=1}^{\infty}\mu_i\|x-T_i x\|^q,
\end{aligned}$$

where $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\}$. Thus, we obtain $x = T_i x$, it follows that $x \in \bigcap_{i=1}^{\infty} F(T_i)$. □

3 Main results

Lemma 3.1 *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C, and let ϕ be an MKC on C. Let $F : C \to X$ be η-strongly accretive and L-Lipschitzian with $0 < \gamma < \eta$, and let $T_i : C \to C$ be a λ_i-strictly pseudo-contractive non-self-mapping such that $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$. Let $\{x_n\}$ be a sequence of C generated by* (1.9). *We assume that the following parameters are satisfied*:

(i) $0 < \alpha_n < 1$, $\sum_{i=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{i=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$;

(ii) $0 < 1 - (\frac{q\lambda}{C_q})^{\frac{1}{q}} \le \beta_n < 1$, $\sum_{i=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;

(iii) $\sum_{i=1}^{\infty}\mu_i^{(n)} = 1$, $\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}|\mu_i^{(n+1)}-\mu_i^{(n)}| < \infty$;

(iv) $0 < \gamma_n < a < 1$, $\sum_{i=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\alpha_{n+1}\beta_n L + \alpha\gamma_{n+1} + \beta_n > \alpha_{n+1}L + \beta_n\gamma_{n+1}$;
(v) $1 - b \leq \alpha < 1$, $b := \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q}}\}$.

Then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof Let $B_n = \sum_{i=1}^{\infty} \mu_i^{(n)} T_i$, by Lemma 2.11 we obtain that for each $n \geq 0$, B_n is a λ-strict pseudo-contraction on C and $F(B_n) = \bigcap_{i=1}^{\infty} F(T_i)$. Further, we can get that

$$\sum_{n=1}^{\infty} \|B_{n+1}x - B_n x\| = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \bigl|\mu_i^{(n+1)} - \mu_i^{(n)}\bigr| \|T_i x\| < \infty,$$

thus $\{B_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Let $W_n = \alpha I + (1-\alpha)\sum_{i=1}^{\infty} \mu_i^{(n)} T_i$, where $\alpha \in [1-b, 1)$, $b := \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q}}\}$. From Lemma 2.4 and Lemma 2.11 we have that W_n is a nonexpansive mapping and $F(W_n) = \bigcap_{i=1}^{\infty} F(T_i) = \mathfrak{F}$, then the iterative algorithm (1.9) can be rewritten as follows:

$$\begin{cases} x_1 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)B_n x_n, \\ x_{n+1} = Q_C[\alpha_n\gamma\phi(x_n) + \gamma_n W_n x_n + ((1-\gamma_n)I - \alpha_n F)y_n], \quad n \geq 1. \end{cases} \tag{3.1}$$

We divide the rest of the proof into two parts.

Step 1. We will prove that the sequence $\{x_n\}$ is bounded.

Letting

$$L_n x = \beta_n x + (1-\beta_n)B_n x,$$

from Lemma 2.4 and condition (ii), we get that $L_n : C \to C$ is nonexpansive. Taking a point $p \in \bigcap_{i=1}^{\infty} F(T_i)$, we have $L_n p = p$ and $p \in F(W_n)$. Therefore, we obtain

$$\|y_n - p\| = \|L_n x_n - p\| \leq \|x_n - p\|.$$

From the definition of MKC and Lemma 2.1, for any $\varepsilon > 0$, there is a number $r_\varepsilon \in (0,1)$, if $\|x_n - p\| < \varepsilon$, then $\|\phi(x_n) - \phi(p)\| < \varepsilon$; if $\|x_n - p\| \geq \varepsilon$, then $\|\phi(x_n) - \phi(p)\| \leq r_\varepsilon\|x_n - p\|$. It follows from (3.1) and Lemma 2.5 that

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\quad = \bigl\|Q_C\bigl[\alpha_n\gamma\phi(x_n) + \gamma_n W_n x_n + \bigl((1-\gamma_n)I - \alpha_n F\bigr)y_n\bigr] - p\bigr\| \\ &\quad \leq \bigl\|\alpha_n\gamma\phi(x_n) + \gamma_n W_n x_n + \bigl((1-\gamma_n)I - \alpha_n F\bigr)y_n - p\bigr\| \\ &\quad = \bigl\|\alpha_n\bigl(\gamma\phi(x_n) - Fp\bigr) + \gamma_n(W_n x_n - p) + \bigl[(1-\gamma_n)I - \alpha_n F\bigr]y_n - \bigl[(1-\gamma_n)I - \alpha_n F\bigr]p\bigr\| \\ &\quad \leq \left[1 - \gamma_n - \left(\alpha_n\eta - \frac{C_q L^q \alpha_n^q}{q(1-\gamma_n)^{q-1}}\right)\right]\|x_n - p\| + \gamma_n\|x_n - p\| + \alpha_n\bigl\|\gamma\phi(x_n) - Fp\bigr\| \\ &\quad \leq \left[1 - \left(\alpha_n\eta - \frac{C_q L^q \alpha_n^q}{q(1-\gamma_n)^{q-1}}\right)\right]\|x_n - p\| + \alpha_n\gamma\max\bigl\{r\|x_n - p\|, \varepsilon\bigr\} + \alpha_n\bigl\|\gamma\phi(p) - Fp\bigr\| \\ &\quad = \max\left\{\left[1 - \left(\alpha_n\eta - \frac{C_q L^q \alpha_n^q}{q(1-\gamma_n)^{q-1}}\right)\right]\|x_n - p\| + \alpha_n\gamma r\|x_n - p\| + \alpha_n\bigl\|\gamma\phi(p) - Fp\bigr\|,\right. \end{aligned}$$

$$\left[1-\left(\alpha_n\eta-\frac{C_qL^q\alpha_n^q}{q(1-\gamma_n)^{q-1}}\right)\right]\|x_n-p\|+\alpha_n\gamma\varepsilon+\alpha_n\big\|\gamma\phi(p)-Fp\big\|\Bigg\}$$
$$=\max\Bigg\{\left[1-\left(\alpha_n\eta-\frac{C_qL^q\alpha_n^q}{q(1-\gamma_n)^{q-1}}-\alpha_n\gamma r\right)\right]\|x_n-p\|+\alpha_n\big\|\gamma\phi(p)-Fp\big\|,$$
$$\left[1-\left(\alpha_n\eta-\frac{C_qL^q\alpha_n^q}{q(1-\gamma_n)^{q-1}}\right)\right]\|x_n-p\|+\alpha_n\gamma\varepsilon+\alpha_n\big\|\gamma\phi(p)-Fp\big\|\Bigg\}.$$

By induction, we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|\\
&\quad\le\max\Bigg\{\left[1-\alpha_n\left(\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}-\gamma r\right)\right]\|x_n-p\|\\
&\qquad+\alpha_n\left(\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}-\gamma r\right)\frac{\|\gamma\phi(p)-Fp\|}{\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}-\gamma r},\\
&\quad\left[1-\alpha_n\left(\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}\right)\right]\|x_n-p\|\\
&\qquad+\alpha_n\left(\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}\right)\frac{\gamma\varepsilon+\|\gamma\phi(p)-Fp\|}{\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}}\Bigg\}.
\end{aligned}$$

Hence, we obtain

$$\|x_n-p\|\le\max\big\{\|x_0-p\|,M\big\},\quad n\ge 0,$$

where M is a constant such that

$$M=\max\left\{\sup_{n\ge0}\left\{\frac{\|\gamma\phi(p)-Fp\|}{\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}-\gamma r}\right\},\sup\left\{\frac{\gamma\varepsilon+\|\gamma\phi(p)-Fp\|}{\eta-\frac{C_qL^q\alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}}\right\}\right\},$$

which implies that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{L_nx_n\}$.

Step 2. We claim that $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

From (3.1) we have

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\|\\
&\quad=\big\|Q_C\big[\alpha_{n+1}\gamma\phi(x_{n+1})+\gamma_{n+1}W_{n+1}x_{n+1}+\big((1-\gamma_{n+1})I-\alpha_{n+1}F\big)L_{n+1}x_{n+1}\big]\\
&\qquad-Q_C\big[\alpha_n\gamma\phi(x_n)+\gamma_nW_nx_n+\big((1-\gamma_n)I-\alpha_nF\big)L_nx_n\big]\big\|\\
&\quad\le\big\|\alpha_{n+1}\gamma\phi(x_{n+1})+\gamma_{n+1}W_{n+1}x_{n+1}+\big((1-\gamma_{n+1})I-\alpha_{n+1}F\big)L_{n+1}x_{n+1}\\
&\qquad-\alpha_n\gamma\phi(x_n)-\gamma_nW_nx_n-\big((1-\gamma_n)I-\alpha_nF\big)L_nx_n\big\|\\
&\quad=\big\|\big((1-\gamma_{n+1})I-\alpha_{n+1}F\big)L_{n+1}x_{n+1}-\big((1-\gamma_n)I-\alpha_nF\big)L_nx_n\\
&\qquad+\big(\alpha_{n+1}\gamma\phi(x_{n+1})-\alpha_n\gamma\phi(x_n)\big)+(\gamma_{n+1}W_{n+1}x_{n+1}-\gamma_nW_nx_n)\big\|\\
&\quad\le\big\|\big((1-\gamma_{n+1})I-\alpha_{n+1}F\big)L_{n+1}x_{n+1}-\big((1-\gamma_{n+1})I-\alpha_{n+1}F\big)L_{n+1}x_n\big\|\\
&\qquad+\big\|\big((1-\gamma_{n+1})I-\alpha_{n+1}F\big)L_{n+1}x_n-\big((1-\gamma_n)I-\alpha_nF\big)L_nx_n\big\|
\end{aligned}$$

$$\begin{aligned}
&+ \big\|\alpha_{n+1}\gamma\phi(x_{n+1}) - \alpha_n\gamma\phi(x_n)\big\| + \|\gamma_{n+1}W_{n+1}x_{n+1} - \gamma_n W_n x_n\| \\
\le{}& \left(1 - \gamma_{n+1} - \alpha_{n+1}\left(\eta - \frac{C_q L^q \alpha_{n+1}^{q-1}}{q(1-\gamma_n)^{q-1}}\right)\right)\|x_{n+1} - x_n\| \\
&+ (1-\gamma_{n+1})\|L_{n+1}x_n - L_n x_n\| + |\gamma_{n+1} - \gamma_n|\|L_n x_n\| \\
&+ \alpha_{n+1}\|FL_{n+1}x_n - FL_n x_n\| + |\alpha_{n+1} - \alpha_n|\|FL_n x_n\| + \big\|\alpha_{n+1}\gamma\phi(x_{n+1}) - \alpha_{n+1}\gamma\phi(x_n)\big\| \\
&+ \big\|\alpha_{n+1}\gamma\phi(x_n) - \alpha_n\gamma\phi(x_n)\big\| + \|\gamma_{n+1}W_{n+1}x_{n+1} - \gamma_{n+1}W_{n+1}x_n\| \\
&+ \|\gamma_{n+1}W_{n+1}x_n - \gamma_n W_n x_n\| \\
\le{}& \left(1 - \gamma_{n+1} - \alpha_{n+1}\left(\eta - \frac{C_q L^q \alpha_{n+1}^{q-1}}{q(1-\gamma_n)^{q-1}}\right)\right)\|x_{n+1} - x_n\| \\
&+ (1-\gamma_{n+1})\|L_{n+1}x_n - L_n x_n\| + |\gamma_{n+1} - \gamma_n|\|L_n x_n\| \\
&+ \alpha_{n+1}\|FL_{n+1}x_n - FL_n x_n\| + |\alpha_{n+1} - \alpha_n|\|FL_n x_n\| + \alpha_{n+1}\gamma\|x_{n+1} - x_n\| \\
&+ |\alpha_{n+1} - \alpha_n|\big\|\gamma\phi(x_n)\big\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&+ \gamma_{n+1}\|W_{n+1}x_n - W_n x_n\| + |\gamma_{n+1} - \gamma_n|\|W_n x_n\| \\
\le{}& \left(1 - \alpha_{n+1}\left(\eta - \frac{C_q L^q \alpha_{n+1}^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma\right)\right)\|x_{n+1} - x_n\| \\
&+ (1-\gamma_{n+1})\|L_{n+1}x_n - L_n x_n\| + \gamma_{n+1}\|W_{n+1}x_n - W_n x_n\| \\
&+ |\gamma_{n+1} - \gamma_n|\big(\|L_n x_n\| + \|W_n x_n\|\big) + \alpha_{n+1}L\|L_{n+1}x_n - L_n x_n\| \\
&+ |\alpha_{n+1} - \alpha_n|\big(\|FL_n x_n\| + \|\gamma\phi(x_n)\|\big) \\
={}& \left(1 - \alpha_{n+1}\left(\eta - \frac{C_q L^q \alpha_{n+1}^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma\right)\right)\|x_{n+1} - x_n\| \\
&+ |\alpha_{n+1} - \alpha_n|\big(\|FL_n x_n\| + \big\|\gamma\phi(x_n)\big\|\big) \\
&+ |\gamma_{n+1} - \gamma_n|\big(\|L_n x_n\| + \|W_n x_n\|\big) + (1 + \alpha_{n+1}L - \gamma_{n+1})\|L_{n+1}x_n - L_n x_n\| \\
&+ \gamma_{n+1}\|W_{n+1}x_n - W_n x_n\|.
\end{aligned} \tag{3.2}$$

Next, we estimate $\|L_{n+1}x_n - L_n x_n\|$ and $\|W_{n+1}x_n - W_n x_n\|$. Notice that

$$\begin{aligned}
\|L_{n+1}x_n - L_n x_n\| &= \big\|\big[\beta_{n+1}x_n + (1-\beta_{n+1})B_{n+1}x_n\big] - \big[\beta_n x_n + (1-\beta_n)B_n x_n\big]\big\| \\
&\le |\beta_{n+1} - \beta_n|\|x_n - B_{n+1}x_n\| + (1-\beta_n)\|B_{n+1}x_n - B_n x_n\|, \qquad (3.3) \\
\|W_{n+1}x_n - W_n x_n\| &= \big\|\alpha x_n + B_{n+1}x_n - \alpha x_n + (1-\alpha)B_n x_n\big\| \\
&= (1-\alpha)\|B_{n+1}x_n - B_n x_n\|. \qquad (3.4)
\end{aligned}$$

Substituting (3.3) and (3.4) into (3.2) and using condition (iv), we have

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| \le{}& \left(1 - \alpha_{n+1}\left(\eta - \frac{C_q L^q \alpha_{n+1}^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma\right)\right)\|x_{n+1} - x_n\| + M_1\big(|\alpha_{n+1} - \alpha_n| \\
&+ (1 + \alpha_{n+1}L - \gamma_{n+1})|\beta_{n+1} - \beta_n| + |\gamma_{n+1} - \gamma_n|\big) \\
&+ \big[1 - (\alpha_{n+1}\beta_n L + \alpha\gamma_{n+1} + \beta_n - \alpha_{n+1}L - \beta_n\gamma_{n+1})\big]\|B_{n+1}x_n - B_n x_n\|
\end{aligned}$$

$$\leq \left(1-\alpha_{n+1}\left(\eta-\frac{C_q L^q \alpha_{n+1}^{q-1}}{q(1-\gamma_n)^{q-1}}-\gamma\right)\right)\|x_{n+1}-x_n\| + M_1\big(|\alpha_{n+1}-\alpha_n| \\ + (1+\alpha_{n+1}L-\gamma_{n+1})|\beta_{n+1}-\beta_n| + |\gamma_{n+1}-\gamma_n|\big) + \|B_{n+1}x_n - B_n x_n\|, \quad (3.5)$$

where M_1 is an appropriate constant such that

$$M_1 \geq \|x_n - B_{n+1}x_n\| + \|FL_n x_n\| + \|L_n x_n\| + \gamma\big\|\phi(x_n)\big\| + \|W_n x_n\| \quad \text{for all } n.$$

Since $\{B_n\}$ satisfies the AKTT-condition, we get that

$$\sum_{i=1}^{\infty}\|B_{n+1}x_n - B_n x_n\| < \infty.$$

Noticing conditions (i), (iii) and (iv) and applying Lemma 2.3 to (3.5), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \quad (3.6)$$

This completes the proof. □

Lemma 3.2 *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C, and let ϕ be an MKC on C. Let $F: C \to X$ be η-strongly accretive and L-Lipschitzian with $0 < \gamma < \eta$, and let $T_i : C \to C$ be a λ_i-strictly pseudo-contractive non-self-mapping such that $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$. Let $\{x_n\}$ be a sequence of C generated by* (1.9). *We assume that the parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_i^{(n)}\}$ and $\{\gamma_n\}$ satisfy the conditions* (i), (ii), (iii), (iv), (v) *in Lemma* 3.1 *and* (vi) $\lim_{n\to\infty}\beta_n = \alpha$. *Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \mathfrak{F}$, which solves the following variational inequality*:

$$\big\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(p-\widetilde{x})\big\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{\infty} F(T_i).$$

Proof The proof of the lemma will be split into three parts.

Step 1. We will prove that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, where $T : C \to C$ is defined by $Tx = \alpha x + (1-\alpha)Bx$ and $Bx = \lim_{n\to\infty} B_n x$. From (3.1) we have

$$\begin{aligned}
\|L_n x_n - x_{n+1}\| &= \big\|Q_C\big[\alpha_n\gamma\phi(x_n) + \gamma_n W_n x_n + \big((1-\gamma_n)I - \alpha_n F\big)L_n x_n\big] - L_n x_n\big\| \\
&\leq \big\|\alpha_n\gamma\phi(x_n) + \gamma_n W_n x_n + \big((1-\gamma_n)I - \alpha_n F\big)L_n x_n - L_n x_n\big\| \\
&= \big\|\alpha_n\gamma\phi(x_n) - \alpha_n FL_n x_n + \gamma_n W_n x_n - \gamma_n L_n x_n\big\| \\
&\leq \alpha_n\big\|\gamma\phi(x_n) - FL_n x_n\big\| + \gamma_n\|W_n x_n - L_n x_n\| \\
&= \alpha_n\big\|\gamma\phi(x_n) - FL_n x_n\big\| + \gamma_n\big\|\alpha x_n + (1-\alpha)B_n x_n - \beta_n x_n - (1-\beta_n)B_n x_n\big\| \\
&= \alpha_n\big\|\gamma\phi(x_n) - FL_n x_n\big\| + \gamma_n|\beta_n - \alpha|\|x_n - B_n x_n\|.
\end{aligned}$$

Using conditions (i) and (vi), we obtain

$$\lim_{n\to\infty}\|L_n x_n - x_{n+1}\| = 0. \quad (3.7)$$

Since $\{B_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition, from Lemma 2.9 we can obtain $\lim_{n\to\infty}\|B_nx-Bx\|=0$. Furthermore, notice that

$$\begin{aligned}\langle Bx-By,j_q(x-y)\rangle &= \lim_{n\to\infty}\langle B_nx-B_ny,j_q(x-y)\rangle \\ &\le \lim_{n\to\infty}\bigl[\|B_nx-B_ny\|^q-\lambda\|(I-B_n)x-(I-B_n)y\|^q\bigr] \\ &= \|Bx-By\|^q-\lambda\|(I-B)x-(I-B)y\|^q,\end{aligned}$$

therefore, we deduce that $B:C\to C$ is a λ-strict pseudo-contraction. Applying Lemma 2.4, we obtain that T is nonexpansive with $F(T)=F(B)$. Notice that

$$\begin{aligned}&\|Tx_n-x_n\| \\ &\quad\le \|L_nx_n-Tx_n\|+\|L_nx_n-x_{n+1}\|+\|x_{n+1}-x_n\| \\ &\quad= \|\beta_nx_n+(1-\beta_n)B_nx_n-\alpha x_n-(1-\alpha)Bx_n\|+\|L_nx_n-x_{n+1}\|+\|x_{n+1}-x_n\| \\ &\quad= \|(\beta_n-\alpha)(x_n-B_nx_n)+(1-\alpha)(B_nx_n-Bx_n)\|+\|L_nx_n-x_{n+1}\|+\|x_{n+1}-x_n\| \\ &\quad\le |\beta_n-\alpha|\|x_n-B_nx_n\|+(1-\alpha)\|B_nx_n-Bx_n\|+\|L_nx_n-x_{n+1}\|+\|x_{n+1}-x_n\|.\end{aligned}$$

Using (3.6), (3.7) and (vi), we obtain

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0. \tag{3.8}$$

Step 2. We will show that

$$\limsup_{n\to\infty}\langle\gamma\phi(\widetilde{x})-F\widetilde{x},j_q(x_n-\widetilde{x})\rangle\le 0, \tag{3.9}$$

where $\widetilde{x}=\lim_{t\to 0}x_t$ with x_t being the fixed point of the contraction

$$x\mapsto Q_C\bigl[t\gamma\phi(x)+(1-tF)Tx\bigr].$$

From the above, we know that $\widetilde{x}\in\mathfrak{F}=F(T)$, then we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and assume that $x_{n_k}\rightharpoonup\omega$, where $\omega\in F(T)$. Since the Banach space X has a weakly sequentially continuous generalized duality mapping $j_q:X\to X^*$, by using Lemma 2.7, 2.10 and (3.8), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma\phi(\widetilde{x})-F\widetilde{x},j_q(x_n-\widetilde{x})\rangle &= \limsup_{k\to\infty}\langle\gamma\phi(\widetilde{x})-F\widetilde{x},j_q(x_{n_k}-\widetilde{x})\rangle \\ &= \langle\gamma\phi(\widetilde{x})-F\widetilde{x},j_q(\omega-\widetilde{x})\rangle \\ &\le 0.\end{aligned}$$

Step 3. We will prove that $\lim_{n\to\infty}\|x_n-\widetilde{x}\|=0$.

By contradiction, there is a number ε_0 such that

$$\limsup_{n\to\infty}\|x_n-\widetilde{x}\|\ge\varepsilon_0.$$

First, let

$$z_n = \alpha_n \gamma \phi(x_n) + \gamma_n W_n x_n + \big((1-\gamma_n)I - \alpha_n F\big)y_n. \tag{3.10}$$

Now, we will obtain the contradiction from two cases.

Case 1. Fix ε_1 ($\varepsilon_1 < \varepsilon_0$), if for some $n > N \in \mathbb{N}$ such that $\|x_n - \widetilde{x}\| \geq \varepsilon_0 - \varepsilon_1$, and for the other $n > N \in \mathbb{N}$ such that $\|x_n - \widetilde{x}\| < \varepsilon_0 - \varepsilon_1$.

Let

$$M_n = \frac{q\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle}{(\varepsilon_0 - \varepsilon_1)^q}.$$

From (3.9) we know $\limsup_{n\to\infty} M_n \leq 0$, thus there are two numbers h and N. When $n > N$, we have $M_n \leq h$, where $h = \min\{\eta - \frac{C_q L^q \alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}} - \gamma\}$. We extract a number $n_0 > N$ satisfying $\|x_{n_0} - \widetilde{x}\| < \varepsilon_0 - \varepsilon_1$, then from Lemma 2.5 and (3.10), we have

$$\begin{aligned}
&\|x_{n_0+1} - \widetilde{x}\|^q \\
&\quad = \big\langle Q_C z_{n_0} - z_{n_0}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle + \big\langle z_{n_0} - \widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle \\
&\quad \leq \big\langle z_{n_0} - \widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle \\
&\quad = \big\langle \alpha_{n_0}\gamma\phi(x_{n_0}) + \gamma_{n_0} W_{n_0} x_{n_0} + \big((1-\gamma_{n_0})I - \alpha_{n_0}F\big)y_{n_0} - \widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle \\
&\quad = \big\langle \big[(1-\gamma_{n_0})I - \alpha_{n_0}F\big]y_{n_0} - \big[(1-\gamma_{n_0})I - \alpha_{n_0}F\big]\widetilde{x} \\
&\qquad + \alpha_{n_0}\big[\gamma\phi(x_{n_0}) - F\widetilde{x}\big] + \gamma_{n_0}(W_{n_0}x_{n_0} - \widetilde{x}), j_q(x_{n_0+1} - \widetilde{x})\big\rangle \\
&\quad = \big\langle \big[(1-\gamma_{n_0})I - \alpha_{n_0}F\big]y_{n_0} - \big[(1-\gamma_{n_0})I - \alpha_{n_0}F\big]\widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle \\
&\qquad + \big\langle \alpha_{n_0}\big[\gamma\phi(x_{n_0}) - \gamma\phi(\widetilde{x})\big], j_q(x_{n_0+1} - \widetilde{x})\big\rangle \\
&\qquad + \alpha_{n_0}\big\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle + \gamma_{n_0}\big\langle W_{n_0}x_{n_0} - \widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle \\
&\quad \leq \big\|\big[(1-\gamma_{n_0})I - \alpha_{n_0}F\big]y_{n_0} - \big[(1-\gamma_{n_0})I - \alpha_{n_0}F\big]\widetilde{x}\big\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\qquad + \alpha_{n_0}\gamma\|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\qquad + \alpha_{n_0}\big\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle + \gamma_{n_0}\|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\quad \leq \left[1 - \gamma_{n_0} - \alpha_{n_0}\left(\eta - \frac{C_q L^q \alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}}\right)\right] \|y_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\qquad + \alpha_{n_0}\gamma\|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\qquad + \alpha_{n_0}\big\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle + \gamma_{n_0}\|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\quad \leq \left[1 - \gamma_{n_0} - \alpha_{n_0}\left(\eta - \frac{C_q L^q \alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}}\right)\right] \|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\qquad + \alpha_{n_0}\gamma\|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\qquad + \alpha_{n_0}\big\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle + \gamma_{n_0}\|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\quad = \left[1 - \alpha_{n_0}\left(\eta - \frac{C_q L^q \alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}} - \gamma\right)\right] \|x_{n_0} - \widetilde{x}\| \|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\qquad + \alpha_{n_0}\big\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n_0+1} - \widetilde{x})\big\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{q}\left[1-\alpha_{n_0}\left(\eta-\frac{C_qL^q\alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}}-\gamma\right)\right]\|x_{n_0}-\widetilde{x}\|^q \\
&\quad+\frac{q-1}{q}\|x_{n_0+1}-\widetilde{x}\|^q+\alpha_{n_0}\langle\gamma\phi(\widetilde{x})-F\widetilde{x},j_q(x_{n_0+1}-\widetilde{x})\rangle \\
&\leq \frac{1}{q}\left[1-\alpha_{n_0}\left(\eta-\frac{C_qL^q\alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}}-\gamma\right)\right](\varepsilon_0-\varepsilon_1)^q+\frac{q-1}{q}\|x_{n_0+1}-\widetilde{x}\|^q \\
&\quad+\alpha_{n_0}\langle\gamma\phi(\widetilde{x})-F\widetilde{x},j_q(x_{n_0+1}-\widetilde{x})\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n_0+1}-\widetilde{x}\|^q &< \left[1-\alpha_{n_0}\left(\eta-\frac{C_qL^q\alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}}-\gamma\right)\right](\varepsilon_0-\varepsilon_1)^q \\
&\quad+q\alpha_{n_0}\langle\gamma\phi(\widetilde{x})-F\widetilde{x},j_q(x_{n_0+1}-\widetilde{x})\rangle \\
&=\left[1-\alpha_{n_0}\left(\eta-\frac{C_qL^q\alpha_{n_0}^{q-1}}{q(1-\gamma_{n_0})^{q-1}}-\gamma-M_{n_0}\right)\right](\varepsilon_0-\varepsilon_1)^q \\
&\leq(\varepsilon_0-\varepsilon_1)^q.
\end{aligned}$$

Hence, we have

$$\|x_{n_0+1}-\widetilde{x}\|<\varepsilon_0-\varepsilon_1.$$

In the same way, we obtain

$$\|x_n-\widetilde{x}\|<\varepsilon_0-\varepsilon_1,\quad\forall n\geq n_0.$$

It contradicts $\limsup_{n\to\infty}\|x_n-\widetilde{x}\|\geq\varepsilon_0$.

Case 2. Fix ε_1 ($\varepsilon_1<\varepsilon_0$), if $\|x_n-\widetilde{x}\|\geq\varepsilon_0-\varepsilon_1$ for all $n>N\in\mathbb{N}$. In this case, from Lemma 2.1 there exists a number $r\in(0,1)$ such that

$$\|\phi(x_n)-\phi(\widetilde{x})\|\leq r\|x_n-\widetilde{x}\|,\quad n\geq N.$$

From (3.10) we have

$$\begin{aligned}
&\|x_{n+1}-\widetilde{x}\|^q \\
&\quad=\langle Q_Cz_n-z_n,j_q(x_{n+1}-\widetilde{x})\rangle+\langle z_n-\widetilde{x},j_q(x_{n+1}-\widetilde{x})\rangle \\
&\quad\leq\langle z_n-\widetilde{x},j_q(x_{n+1}-\widetilde{x})\rangle \\
&\quad=\langle\alpha_n\gamma\phi(x_n)+\gamma_nW_nx_n+((1-\gamma_n)I-\alpha_nF)y_n-\widetilde{x},j_q(x_{n+1}-\widetilde{x})\rangle \\
&\quad=\langle[(1-\gamma_n)I-\alpha_nF]y_n-[(1-\gamma_n)I-\alpha_nF]\widetilde{x} \\
&\qquad+\alpha_n[\gamma\phi(x_n)-F\widetilde{x}]+\gamma_n(W_nx_n-\widetilde{x}),j_q(x_{n+1}-\widetilde{x})\rangle \\
&\quad=\langle[(1-\gamma_n)I-\alpha_nF]y_n-[(1-\gamma_n)I-\alpha_nF]\widetilde{x},j_q(x_{n+1}-\widetilde{x})\rangle \\
&\qquad+\langle\alpha_n[\gamma\phi(x_n)-\gamma\phi(\widetilde{x})],j_q(x_{n+1}-\widetilde{x})\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle + \gamma_n\langle W_n x_n - \widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle \\
&\leq \big\| \big[(1-\gamma_n)I - \alpha_n F\big]y_n - \big[(1-\gamma_n)I - \alpha_n F\big]\widetilde{x}\big\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_n\gamma r\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_n\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle + \gamma_n\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\leq \left[1 - \gamma_n - \alpha_n\left(\eta - \frac{C_q L^q \alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}\right)\right]\|y_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_n\gamma r\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_n\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle + \gamma_n\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\leq \left[1 - \gamma_n - \alpha_n\left(\eta - \frac{C_q L^q \alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}}\right)\right]\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_n\gamma r\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_n\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle + \gamma_n\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&= \left[1 - \alpha_n\left(\eta - \frac{C_q L^q \alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma r\right)\right]\|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_n\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle \\
&\leq \frac{1}{q}\left[1 - \alpha_n\left(\eta - \frac{C_q L^q \alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma r\right)\right]\|x_n - \widetilde{x}\|^q \\
&\quad + \frac{q-1}{q}\|x_{n+1} - \widetilde{x}\|^q + \alpha_n\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - \widetilde{x}\|^q &< \left[1 - \alpha_n\left(\eta - \frac{C_q L^q \alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma r\right)\right]\|x_n - \widetilde{x}\|^q \\
&\quad + q\alpha_n\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1} - \widetilde{x})\rangle. \qquad (3.11)
\end{aligned}
$$

Put $a_n = \alpha_n(\eta - \frac{C_q L^q \alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma r)$ and $c_n = \frac{q\langle\gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(x_{n+1}-\widetilde{x})\rangle}{\eta - \frac{C_q L^q \alpha_n^{q-1}}{q(1-\gamma_n)^{q-1}} - \gamma r}$. Applying Lemma 2.3 to (3.11), we obtain $x_n \to \widetilde{x}$ as $n \to \infty$, which contradicts $\|x_n - \widetilde{x}\| \geq \varepsilon_0 - \varepsilon_1$. Thus $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. This completes the proof. □

Theorem 3.1 *Let C be a nonempty closed subset of a q-uniformly smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C, and let ϕ be an MKC on C. Let $F : C \to X$ be an η-strongly accretive L-Lipschitzian and linear mapping with $0 < \gamma < \eta$, and let $T_i : C \to C$ be a λ_i-strictly pseudo-contractive non-self-mapping such that $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$. Let $\{x_n\}$ be a sequence of C generated by* (1.9). *We assume that the following parameters are satisfied*:

(i) $0 < \alpha_n < 1$, $\sum_{i=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{i=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $0 < 1 - (\frac{q\lambda}{C_q})^{\frac{1}{q}} \leq \beta_n < 1$, $\sum_{i=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;

(iii) $\sum_{i=1}^{\infty}\mu_i^{(n)} = 1$, $\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}|\mu_i^{(n+1)} - \mu_i^{(n)}| < \infty$;

(iv) $0 < \gamma_n < a < 1$, $\sum_{i=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$, $\alpha_{n+1}\beta_n L + \alpha\gamma_{n+1} + \beta_n > \alpha_{n+1}L + \beta_n\gamma_{n+1}$;

(v) $1 - b \leq \alpha < 1$, $b := \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q}}\}$;

(vi) $\lim_{n\to\infty} \beta_n = \alpha$.

Then $\{x_n\}$ *generated by* (3.1) *converges strongly to* $\widetilde{x} \in \mathfrak{F}$, *which solves the following variational inequality*:

$$\langle \gamma\phi(\widetilde{x}) - F\widetilde{x}, j_q(p - \widetilde{x})\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{\infty} F(T_i).$$

Proof Combining the proof of Lemma 3.1 with Lemma 3.2, we can obtain the conclusion. □

Remark 3.1 Compared with Theorem 3.2 of Song [11], our results are different from those in the following aspects:

(i) Theorem 3.1 improves and extends Theorem 3.2 of Song [11]. Especially, our results extend the above results from a Hilbert space to a more general q-uniformly smooth and uniformly convex Banach space.

(ii) We change the metric projection P_C in Song [11] into the sunny nonexpansive retraction Q_C and put it to the other place of the iteration process so that our iteration process is better defined.

(iii) We generalize the iteration process so that our iteration process is more general.

(iv) We remove the very strict condition $C + C \subset C$ in Lemma 3.1 and Theorem 3.2 of Song [11], and it is worth stressing that the strict condition is also very necessary in Qin *et al.* [10] and Cai *et al.* [20].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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