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Hybrid projection methods for a bifunction and relatively asymptotically nonexpansive mappings

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Abstract
The purpose of this paper is to investigate a bifunction equilibrium problem and a fixed point problem of relatively asymptotically nonexpansive mappings based on a generalized projection method. A weak convergence theorem for common solutions is established in a uniformly smooth and uniformly convex Banach space.

Keywords: bifunction; equilibrium problem; fixed point; generalized projection; relatively asymptotically nonexpansive mapping

1 Introduction and preliminaries

Let E be a real Banach space, E^* be the dual space of E, and C be a nonempty subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem: Find $\bar{x} \in C$ such that

$$F(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

From now on, we use $EP(F)$ to denote the solution set of equilibrium problem (1.1) and assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;

(A3)

$$\limsup_{t \downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y), \quad \forall x,y,z \in C;$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth iff

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth iff the above limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Recall that E is said to be uniformly convex iff $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. It is well known that E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on the Kadec-Klee property, the readers can refer to [1] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Let $T : C \to C$ be a mapping. From now on, we use $F(T)$ to denote the fixed point set of T. Recall that T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and the weak convergence, respectively.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E.$$

Observe that in a Hilbert space H the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$. As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y\in C} \phi(y,x).$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of a function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|y\| + \|x\|)^2, \quad \forall x,y \in E. \tag{1.2}$$

Remark 1.1 If E is a reflexive, strictly convex, and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$; for more details, see [2] and the references therein.

Recall that a point p in C is said to be an asymptotic fixed point of a mapping T iff C contains a sequence $\{x_n\}$ which converges weakly to p so that $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

Recall that a mapping T is said to be relatively nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

Recall that a mapping T is said to be relatively asymptotically nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi\big(p, T^n x\big) \leq (1 + \mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 1.2 The class of relatively nonexpansive mappings was first considered in Butnariu *et al.* [3]. The class of relatively asymptotically nonexpansive mappings was first considered in Agarwal *et al.* [4] and the references therein.

Recently, many authors investigated fixed point problems of a (relatively) nonexpansive mapping based on hybrid projection methods; for more details, see [5–37] and the references therein. However, most of the results are on strong convergence. In this article, we investigate a bifunction equilibrium problem and a fixed point problem of relatively asymptotically nonexpansive mappings based on a generalized projection method. A weak convergence theorem for common solutions is established in a uniformly smooth and uniformly convex Banach space.

The following lemmas play an important role in this paper.

Lemma 1.3 [37, 38] *Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that $F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0$, $\forall y \in C$. Define a mapping $S_r : E \to C$ by $S_r x = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle, \forall y \in C\}$. Then the following conclusions hold*:

(a) *S_r is single-valued*;

(b) *S_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \leq \langle S_r x - S_r y, Jx - Jy\rangle;$$

(c) *$F(S_r) = EP(F)$ is closed and convex*;

(d) *S_r is relatively nonexpansive*;

(e) *$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q, x)$, $\forall q \in F(S_r)$.*

Lemma 1.4 [4] *Let E be a uniformly smooth and uniformly convex Banach space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a relatively asymptotically nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Lemma 1.5 [2] *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty, closed, and convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 1.6 [2] *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.7 [39] *Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|)$$

for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $t \in [0,1]$.

Lemma 1.8 [40] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then the limit of the sequence $\{a_n\}$ exists. If, in addition, there exists a subsequence $\{\alpha_{n_i}\} \subset \{\alpha_n\}$ such that $\alpha_{n_i} \to 0$, then $\alpha_n \to 0$ as $n \to \infty$.

Lemma 1.9 [41] *Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$.*

2 Main results

Theorem 2.1 *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed and convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T : C \to C$ be a relatively asymptotically nonexpansive mapping with the sequence $\{\mu_{n,1}\}$, and let $S : C \to C$ be a relatively asymptotically nonexpansive mapping with the sequence $\{\mu_{n,2}\}$. Assume that $\Phi := F(T) \cap F(S) \cap EP(F)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} y_0 \in E \quad \text{chosen arbitrarily}, \\ x_n \in C \text{ such that } F(x_n, x) + \frac{1}{r_n}\langle x - x_n, Jx_n - Jy_n\rangle \geq 0, \quad \forall x \in C, \\ y_{n+1} = J^{-1}(\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[0,1]$ and $\{r_n\}$ is a real number sequence in $[r, \infty)$, where $r > 0$ is some real number. Assume that J is weakly sequentially continuous and the following restrictions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=1}^{\infty} \mu_n < \infty$;

(iii) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Phi$, where $\bar{x} = \lim_{n\to\infty} \Pi_\Phi x_n$.

Proof Set $\mu_n = \max\{\mu_{n,1}, \mu_{n,2}\}$. Fixing $p \in \Phi$, we find that

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, S_{r_{n+1}} y_{n+1}) \\ &\leq \phi(p, y_{n+1}) \end{aligned}$$

$$\begin{aligned}
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n \rangle \\
&\quad + \|\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n\|^2 \qquad (2.1)\\
&\le \|p\|^2 - 2\alpha_n \langle p, Jx_n\rangle - 2\beta_n \langle p, JT^n x_n\rangle - 2\gamma_n \langle p, JS^n x_n\rangle \\
&\quad + \alpha_n \|x_n\|^2 + \beta_n \|T^n x_n\|^2 + \gamma_n \|S^n x_n\|^2 \\
&= \alpha_n \phi(p, x_n) + \beta_n \phi(p, T^n x_n) + \gamma_n \phi(p, S^n x_n) \\
&\le \phi(p, x_n) + \beta_n \mu_n \phi(p, x_n) + \gamma_n \mu_n \phi(p, x_n) \\
&\le (1 + \mu_n)\phi(p, x_n). \qquad (2.2)
\end{aligned}$$

In view of Lemma 1.8, we obtain that $\lim_{n\to\infty} \phi(p, x_n)$ exits. This implies that the sequence $\{x_n\}$ is bounded. In the light of Lemma 1.7, we find that

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, S_{r_{n+1}} y_{n+1}) \\
&\le \|p\|^2 - 2\langle p, \alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n \rangle \\
&\quad + \|\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n\|^2 \\
&\le \|p\|^2 - 2\alpha_n \langle p, Jx_n\rangle - 2\beta_n \langle p, JT^n x_n\rangle - 2\gamma_n \langle p, JS^n x_n\rangle \\
&\quad + \alpha_n \|x_n\|^2 + \beta_n \|T^n x_n\|^2 + \gamma_n \|S^n x_n\|^2 - \alpha_n \beta_n g(\|JT^n x_n - Jx_n\|) \\
&\le \phi(p, x_n) + \beta_n \mu_n \phi(p, x_n) + \gamma_n \mu_n \phi(p, x_n) - \alpha_n \beta_n g(\|JT^n x_n - Jx_n\|) \\
&\le (1 + \mu_n)\phi(p, x_n) - \alpha_n \beta_n g(\|JT^n x_n - Jx_n\|).
\end{aligned}$$

It follows that

$$\alpha_n \beta_n g(\|JT^n x_n - Jx_n\|) \le (1 + \mu_n)\phi(p, x_n) - \phi(p, x_{n+1}).$$

This finds from the restrictions (ii) and (iii) that

$$\lim_{n\to\infty} g(\|JT^n x_n - Jx_n\|) = 0.$$

This implies that

$$\lim_{n\to\infty} \|JT^n x_n - Jx_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we find that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$$

In the same way, we find that

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0.$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $p \in C$. It follows that $p \in F(T) \cap F(S)$. Next, we prove that $p \in EP(F)$.

Let $r = \sup_{n\geq 1}\{\|x_n\|, \|y_n\|\}$. In view of Lemma 1.9, we find that there exists a continuous, strictly increasing and convex function h with $h(0) = 0$ such that

$$h(x,y) \leq \phi(x,y), \quad \forall x,y \in B_r.$$

It follows from (2.1) that

$$\begin{aligned} h\big(\|x_n - y_n\|\big) &\leq \phi(x_n, y_n) \\ &\leq \phi(p, y_n) - \phi(p, x_n) \\ &\leq \phi(p, x_{n-1}) - \phi(p, x_n) + \mu_{n-1}\phi(p, x_{n-1}). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} h\big(\|x_n - y_n\|\big) = 0.$$

It follows from the property of h that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0.$$

Since $\{r_n\}$ is a real number sequence in $[r,\infty)$, where $r > 0$ is some real number, one finds that

$$\lim_{n\to\infty} \frac{\|Jx_n - Jy_n\|}{r_n} = 0.$$

Notice that $x_n = S_{r_n} y_n$, one sees that

$$F(x_n, x) + \frac{1}{r_n}\langle x - x_n, Jx_n - Jy_n\rangle \geq 0, \quad \forall x \in C.$$

By replacing n by n_i, one finds from (A2) that

$$\begin{aligned} \|x - x_{n_i}\| \frac{\|Jx_{n_i} - Jy_{n_i}\|}{r_{n_i}} &\geq \frac{1}{r_{n_i}}\langle x - x_{n_i}, Jx_{n_i} - Jy_{n_i}\rangle \\ &\geq F(x, x_{n_i}). \end{aligned}$$

Letting $i \to \infty$ in the above inequality, one obtains from (A4) that

$$F(x,p) \leq 0, \quad \forall x \in C.$$

For $0 < t < 1$ and $y \in C$, define $x_t = tx + (1-t)p$. It follows that $x_t \in C$, which yields that $F(x_t, p) \leq 0$. It follows from (A1) and (A4) that

$$0 = F(x_t, x_t) \leq tF(x_t, x) + (1-t)F(x_x, p) \leq tF(x_t, x).$$

That is,

$$F(x_t, x) \geq 0.$$

Letting $t \downarrow 0$, we obtain from (A3) that $F(p,x) \geq 0$, $\forall x \in C$. This implies that $p \in EP(F)$. This completes the proof that $p \in F(T) \cap F(S) \cap EP(F)$. Define $z_n = \Pi_{F(T)\cap F(S)\cap EP(F)} x_n$. It follows from (2.1) that

$$\phi(z_n, x_{n+1}) \leq (1 + \mu_n)\phi(z_n, x_n). \tag{2.3}$$

This in turn implies from Lemma 1.5 that

$$\begin{aligned}
\phi(z_{n+1}, x_{n+1}) &= \phi(\Pi_{F(T)\cap F(S)\cap EP(F)} x_{n+1}, x_{n+1}) \\
&\leq \phi(z_n, x_{n+1}) - \phi(z_n, \Pi_{F(T)\cap F(S)\cap EP(F)} x_{n+1}) \\
&\leq \phi(z_n, x_{n+1}) - \phi(z_n, z_{n+1}) \\
&\leq \phi(z_n, x_{n+1}).
\end{aligned}$$

It follows from (2.3) that

$$\phi(z_{n+1}, x_{n+1}) \leq (1 + \mu_n)\phi(z_n, x_n).$$

This finds from Lemma 1.8 that the sequence $\{\phi(z_n, x_n)\}$ is a convergence sequence. It follows from (2.1) that

$$\phi(p, x_{n+m}) \leq \phi(p, x_n) + L\left(\sum_{i=1}^{m} \mu_{n+m-i}\right), \tag{2.4}$$

where $L = \sup_{n\geq 1} \phi(p, x_n)$. Since $z_n \in F(T) \cap F(S) \cap EP(F)$, we find that

$$\phi(z_n, x_{n+m}) \leq \phi(z_n, x_n) + M\left(\sum_{i=1}^{m} \mu_{n+m-i}\right),$$

where $M = \sup_{n\geq 1} \phi(z_n, x_n)$. Since $z_{n+m} = \Pi_{F(T)\cap F(S)\cap EP(F)} x_{n+m}$, we find from Lemma 1.5 that

$$\phi(z_n, z_{n+m}) + \phi(z_{n+m}, x_{n+m}) \leq \phi(z_n, x_{n+m}) \leq \phi(z_n, x_n) + M\left(\sum_{i=1}^{m} \mu_{n+m-i}\right).$$

It follows that

$$\phi(z_n, z_{n+m}) \leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + M\left(\sum_{i=1}^{m} \mu_{n+m-i}\right).$$

In view of Lemma 1.9, we find that there exists a continuous, strictly increasing, and convex function g with

$$g\big(\|z_n - z_m\|\big) \leq \phi(z_n, z_m) \leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + M\left(\sum_{i=1}^{m} \mu_{n+m-i}\right).$$

This shows that $\{z_n\}$ is a Cauchy sequence. Since $F(T) \cap F(S) \cap EP(F)$ is closed, one sees that $\{z_n\}$ converges strongly to $z \in F(T) \cap F(S) \cap EP(F)$. Since $p \in F(T) \cap F(S) \cap EP(F)$, we find from Lemma 1.6 that

$$\langle z_{n_k} - p, Jx_{n_k} - Jz_{n_k} \rangle \geq 0.$$

Notice that J is weakly sequentially continuous. Letting $k \to \infty$, we find that $\langle z - p, Jp - Jz\rangle \geq 0$. It follows from the monotonicity of J that $\langle z - p, Jp - Jz\rangle \leq 0$. Since the space is uniformly convex, we find that $z = p$. This completes the proof. □

Remark 2.2 Theorem 2.1 improves Theorem 2.5 in Qin *et al.* [36] on the mappings from the class of relatively nonexpansive mappings to the class of relatively asymptotically nonexpansive mappings.

If $T = S$, then Theorem 2.1 is reduced to the following.

Corollary 2.3 *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed and convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T : C \to C$ be a relatively asymptotically nonexpansive mapping with the sequence $\{\mu_n\}$. Assume that $\Phi := F(T) \cap EP(F)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_0 \in E \quad \text{chosen arbitrarily}, \\ x_n \in C \text{ such that } F(x_n, x) + \frac{1}{r_n}\langle x - x_n, Jx_n - Jy_n\rangle \geq 0, \quad \forall x \in C, \\ y_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and $\{r_n\}$ is a real number sequence in $[r, \infty)$, where $r > 0$ is some real number. Assume that J is weakly sequentially continuous and the following restrictions hold:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$;

(ii) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Phi$, where $\bar{x} = \lim_{n\to\infty} \Pi_\Phi x_n$.

Remark 2.4 Corollary 2.3 is an improvement of Theorem 4.1 in Zembayashi and Takahashi [37] on the nonlinear mapping.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally and read and approved the final manuscript.

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