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Strong convergence by the shrinking effect of two half-spaces and applications

Muhammad Aqeel Ahmad Khan[1,2*] and Hafiz Fukhar-ud-din[2,3]

*Correspondence:
itsakb@hotmail.com
[1]Department of Mathematics,
Technische Universitat Darmstadt,
Schlossgartenstrasse 7, Darmstadt,
64289, Germany
[2]Department of Mathematics, The
Islamia University of Bahawalpur,
Bahawalpur, 63100, Pakistan
Full list of author information is
available at the end of the article

Abstract

This paper provides a new hybrid-type shrinking projection method for strong convergence results in a Hilbert space. The paper continues - by utilizing the proposed hybrid algorithm - with a strong convergence towards an approximate common element of the set of solutions of a finite family of generalized equilibrium problems and the set of common fixed points of two finite families of k-strict pseudo-contractions in a Hilbert space. Comparatively, our results improve and extend various results announced in the current literature.

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1 Introduction

Let H be a real Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. A map $T : C \to C$ is said to be (i) Lipschitzian if $\|Tx - Ty\| \leq L\|x - y\|$ for some $L > 0$ and for all $x, y \in C$; (ii) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; (iii) k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \text{for all } x, y \in C.$$

Denote by $F(T)$ the set of all fixed points of T.

In 1967, Browder and Petryshyn [1] introduced the class of strict pseudo-contractions as an important generalization of the class of nonexpansive maps. Clearly, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. Moreover, T is said to be pseudo-contraction if $k = 1$ and a strong pseudo-contraction if there exists a positive constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is a pseudo-contraction. Therefore, the class of k-strict pseudo-contractions falls into the one between the classes of nonexpansive maps and pseudo-contractions. We further remark that the class of strong pseudo-contractions is independent of the class of k-strict pseudo-contractions. If T is a k-strict pseudo-contraction, then T satisfies the Lipschitz condition

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\| \quad \text{for all } x, y \in C.$$

Recent developments in fixed point theory reflect that an iterative construction of fixed points of nonlinear maps is an active area of research. Note that the class of k-strict pseudo-contractions is prominent among the classes of nonlinear maps in the current literature. Although strict pseudo-contractions have more powerful applications than nonexpansive maps for solving inverse problems, iterative algorithms for strict pseudo-contractions are far less developed than those for nonexpansive maps.

In fixed point theory, various iterative schemas for computing approximate fixed points of nonlinear maps have been proposed and analyzed. Such iterative schemas can be compared w.r.t. their efficiency and convergence properties. In many situations, strong convergence of an iterative algorithm of a nonlinear map is much more desirable than weak convergence. Obviously, a trivial choice for approximation is the classical Picard algorithm (*i.e.*, $x_{n+1} = T^n(x_0)$). On the other hand, if we take $X = \mathbb{R}$, $C := [0, 1]$, $T(x) := 1 - x$ and $x_0 := 0$, then the Picard algorithm alternates between 0 and 1 and does not converge to the fixed point $\frac{1}{2}$. The classical Mann algorithm, which prevails the Picard algorithm, exhibits weak convergence even in the setting of a Hilbert space. Moreover, Chidume and Mutangadura [2] constructed an example for a Lipschitz pseudo-contraction with a unique fixed point for which the Mann algorithm fails to converge. These facts indicate that the iterative schemas should be modified for the desirable strong convergence properties.

The hybrid projection algorithm in mathematical programming was introduced by Haugazeau [3] in 1968. Later, many researchers studied the hybrid projection method and its various modifications for strong convergence results. In 2000, Solodov and Svaiter [4] established a strong convergence result for finding zeros of maximal monotone operators. They proposed and analyzed the following algorithm:

$$v_k \in Ay_k, \quad v_k + \mu_k(y_k - x_k) = \epsilon_k;$$
$$\|\epsilon_k\| \leq \sigma \max\{\|v_k\|, \mu_k\|y_k - x_k\|\};$$
$$H_k = \{z \in H : \langle z - y_k, v_k \rangle \leq 0\}; \tag{1.1}$$
$$W_k = \{z \in H : \langle x_k - z, x_0 - x_k \rangle \geq 0\};$$
$$x_{k+1} = P_{H_k \cap W_k} x_0.$$

In fact, algorithm (1.1) enforces strong convergence by combining the classical proximal point algorithm with simple projection steps onto intersection of two half-spaces H_k and W_k containing the solution set. Inspired by the seminal work of Solodov and Svaiter [4], Nakajo and Takahashi [5] proposed the following hybrid method for nonexpansive maps in Hilbert spaces:

$$x_0 \in C \quad \text{chosen arbitrarily,}$$
$$y_k = \alpha_k x_k + (1 - \alpha_k) T x_k,$$
$$C_k = \{z \in C : \|y_k - z\| \leq \|x_k - z\|\}, \tag{1.2}$$
$$Q_k = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\},$$
$$x_{k+1} = P_{C_k \cap Q_k} x_0.$$

They showed that algorithm (1.2) converges strongly to $P_{F(T)}x$ under some appropriate conditions. Moreover, algorithm (1.2) is also known as the CQ-method for the Mann al-

gorithm because, at each step, the Mann algorithm is used to construct the sets C_k and Q_k which are in turn used to construct the next iterate of x_{k+1} and hence the name.

In 2008, Takahashi *et al.* [6] introduced another type of the hybrid method which guarantees strong convergence by the shrinking effect of a sequence of closed convex sets $\{C_{k+1}\}$. More precisely, their algorithm reads as follows:

$$
\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily where } C_0 = C, \\
&y_k = \alpha_k x_k + (1 - \alpha_k) T x_k, \\
&C_{k+1} = \big\{ z \in C_k : \|z - y_k\| \leq \|z - x_k\| \big\}, \\
&x_{k+1} = P_{C_{k+1}} x_0, \quad k \geq 0.
\end{aligned}
\tag{1.3}
$$

Very recently, Dong *et al.* [7] proposed a shrinking projection method similar to (1.3) for nonexpansive maps in a Hilbert space setting. They, in fact, established a strong convergence result by the shrinking effect of one of the half-spaces as defined in (1.2), namely Q_k.

For $i = 1, 2, 3, \ldots, N$, let $A_i : C \to H$ be a finite family of nonlinear maps and $f_i : C \times C \to \mathbb{R}$ (the set of reals) be a finite family of bifunctions. A generalized equilibrium problem is to find the set

$$
\text{GEP}(f_i) = \big\{ x \in C : f_i(x, y) + \langle A_i x, y - x \rangle \geq 0 \big\} \quad \text{for all } y \in C.
\tag{1.4}
$$

Note that if $A_i \equiv 0$ and $f_i(x, y) = f(x, y)$ for all $i \geq 1$, then the problem (1.4) reduces to the classical equilibrium problem EP(f). That is, to find $x \in C$ such that $f(x, y) \geq 0$. Moreover, if $f_i(x, y) \equiv 0$ and $A_i = A$ for all $i \geq 1$, then the problem (1.4) reduces to the classical variational inequality problem VI(C, A). That is, to find $x \in C$ such that $\langle Ax, y - x \rangle \geq 0$.

Equilibrium problems provide a unified approach to address a variety of problems arising in various disciplines of science. The problem (1.4) is very general in the sense that it includes - as special cases - optimization problem, minimax problem, variational inequality problem, Nash equilibrium problem in noncooperative games and others; see, for instance, [8–10]. Combettes and Hirstoaga [9] introduced an iterative method to find an approximate solution of EP(f). Since then, numerous algorithms have been analyzed to find a common element of the set of solutions of EP(f) or GEP(f) and the set of fixed points of a nonlinear map; see, for example, [10–16] and the references therein.

In 2009, Ceng *et al.* [17] introduced an implicit-type algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a strict pseudo-contraction in a real Hilbert space. Recently, Kim *et al.* [18] and Kangtunyakarn [19] approximated a common element of the set of solutions of two generalized equilibrium problems and the set of fixed points of a strict pseudo-contraction using the shrinking projection algorithm as defined in (1.3). Quite recently, Cholamjiak and Suantai [20] established a strong convergence result regarding a system of generalized equilibrium problems and a countable family of strict pseudo-contractions in a real Hilbert space.

Inspired and motivated by the work of Nakajo and Takahashi [5], Takahashi *et al.* [6], Dong *et al.* [7], Ceng *et al.* [17] and Cholamjiak and Suantai [20], we propose a hybrid method based on the shrinking effect of the two half-spaces, namely C_n and Q_n, of the underlying Hilbert space H. The proposed algorithm approximates a common element of the set of solutions of a finite family of generalized equilibrium problems and the set of

common fixed points of two finite families of k-strict pseudo-contractions. Our results refine and improve various results announced in the current literature.

2 Preliminaries

Throughout the paper, we write $x_n \to x$ (resp. $x_n \rightharpoonup x$) to indicate strong convergence (resp. weak convergence). Let H be a real Hilbert space, a map $P_C : H \to C$ defined by

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C$$

is known as a metric projection or a nearest point projection of H onto C. Moreover, P_C is characterized as nonexpansive in Hilbert spaces. We know that for $x \in H$ and $z \in C$, $z = P_C x$ is equivalent to $\langle x - z, z - u \rangle \geq 0$ for all $u \in C$.

Let $A : C \to H$ be a δ-inverse-strongly monotone map, if there exists $\delta > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \delta \|Ax - Ay\|^2 \quad \text{for all } x, y \in C.$$

If A is a δ-inverse-strongly monotone map of C onto H, then A is $(\frac{1}{\delta})$-Lipschitz continuous. Moreover, for $x, y \in C$ and $r > 0$, we have

$$
\begin{aligned}
\left\| (I - rA)x - (I - rA)y \right\|^2 &= \left\| x - y - r(Ax - Ay) \right\|^2 \\
&= \|x - y\|^2 - 2r \langle x - y, Ax - Ay \rangle + r^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + r(r - 2\delta) \|Ax - Ay\|^2. \tag{2.1}
\end{aligned}
$$

If $r \leq 2\delta$, then $I - rA$ is a nonexpansive map from C onto H.

The following crucial results for a k-strict pseudo-contraction can be found in [21, Proposition 2.1].

Lemma 2.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to H$ is a k-strict pseudo-contraction, then the fixed point set $F(T)$ is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a k-strict pseudo-contraction. Then $(I - T)$ is demiclosed, that is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$.*

The following lemma is well known in the context of a real Hilbert space.

Lemma 2.3 *Let H be a real Hilbert space, then the following identity holds*:

$$\left\| \alpha x + (1 - \alpha) y \right\|^2 = \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (cf. [8] and [9]):
(A1) $f(x, x) = 0$ *for all $x \in C$;*
(A2) f *is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;*
(A3) *for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;*
(A4) $f(x, \cdot)$ *is convex and lower semicontinuous for all $x \in C$.*

A central result in the theory of equilibrium problems - for an approximate solution of EP(f) - is the following result due to [8].

Lemma 2.4 *Let C be a closed convex subset of a real Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \quad \text{for all } y \in C.$$

Moreover, define a map $V_r : H \to C$ by

$$V_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) EP(f) *is closed and convex*;

(2) V_r *is single-valued*;

(3) V_r *is a firmly nonexpansive-type map, i.e.,*

$$\|V_r x - V_r y\|^2 \leq \langle V_r x - V_r y, x - y \rangle \quad \text{for all } x, y \in H;$$

(4) $F(V_r) = \text{EP}(f)$.

3 Main result

Let H be a real Hilbert space and C be its nonempty closed convex subset. Let $T_i(\mathrm{mod}\,N)$, $S_i(\mathrm{mod}\,N) : C \to C$ be two finite families of k-strict pseudo-contractions such that $k = \max\{k_i : 1 \leq i \leq N\}$. Let $f_i(\mathrm{mod}\,N) : C \times C \to \mathbb{R}$ be a finite family of bifunctions and $A_i(\mathrm{mod}\,N) : C \to H$ be a finite family of δ-inverse-strongly monotone maps such that $\delta = \max\{\delta_i : 1 \leq i \leq N\}$.

Throughout this section, we assume that the $\mathrm{mod}\,N$ function takes values in the indexing set $I = \{1, 2, 3, \dots, N\}$ and the set of common fixed points of two finite families of k-strict pseudo-contractions $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ is nonempty, that is, $F = (\bigcap_{i=1}^N F(T_i)) \cap (\bigcap_{i=1}^N F(S_i)) \neq \emptyset$.

Algorithm Our hybrid algorithm reads as follows:

$$
\begin{aligned}
&x_1 \in C_1 = Q_1 = C, \\
&y_{n,i} = \beta_{n,i} z_{n,i} + (1 - \beta_{n,i}) S_i z_{n,i}, \\
&z_{n,i} = \alpha_{n,i} u_{n,i} + (1 - \alpha_{n,i}) T_i u_{n,i}, \\
&f_i(u_{n,i}, y) + \langle A_i x_n, y - u_{n,i} \rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, u_{n,i} - x_n \rangle \geq 0, \quad \forall y \in C, \\
&C_{n+1} = \left\{ z \in C_n : \sup_{i \geq 1} \|y_{n,i} - z\| \leq \|x_n - z\| \right\}, \\
&Q_{n+1} = \left\{ z \in Q_n : \langle z - x_n, x_1 - x_n \rangle \geq 0 \right\}, \\
&x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x_1, \quad n \geq 1.
\end{aligned}
\tag{3.1}
$$

It is worth mentioning that at each iteration, algorithm (3.1) computes a projection onto intersection of two half-spaces. Since these half-spaces are closed and convex, so one can follow Dykstra's algorithm [22] for the computation of such a projection.

The main result of this section is as follows.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i(\mathrm{mod}\,N), S_i(\mathrm{mod}\,N) : C \to C$ be two finite families of k-strict pseudo-contractions. Let $f_i(\mathrm{mod}\,N) : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying* (A1)-(A4) *and let $A_i(\mathrm{mod}\,N) : C \to H$ be a finite family of δ-inverse-strongly monotone maps. Let $\{r_{n,i}\} \subset (0, \infty)$ and $\{\alpha_{n,i}\}, \{\beta_{n,i}\}$ be two control sequences such that*

(C1) $0 \le k < a \le \alpha_{n,i}, \beta_{n,i} \le b < 1$;

(C2) $0 < r < r_{n,i} < s < 2\delta$ *for all $i \ge 1$.*

Assume that $\mathcal{F} := [\bigcap_{i=1}^{N} F(T_i)] \cap [\bigcap_{i=1}^{N} F(S_i)] \cap [\bigcap_{i=1}^{N} \mathrm{GEP}(f_i, A_i)] \ne \emptyset$, then the sequence $\{x_n\}$ generated by (3.1) *converges strongly to $x = P_{\mathcal{F}} x_1$, where $P_{\mathcal{F}}$ is the metric projection of H onto $\mathcal{F}$.*

Proof It follows from Lemma 2.4 that $u_{n,i}$ can be written as $u_{n,i} = V_{r_{n,i}}(x_n - r_{n,i} A_i x_n)$ for all $n \ge 1$. Moreover, it follows from the nonexpansiveness of $I - r_{n,i} A_i$ that

$$\|u_{n,i} - p\| \le \|x_n - p\|,$$

where $p = \bigcap_{i=1}^{N} T_i p = \bigcap_{i=1}^{N} S_i p = \bigcap_{i=1}^{N} V_{r_{n,i}}(p - r_{n,i} A_i p)$.

We proceed to show that algorithm (3.1) is well defined. It is obvious from the definitions of respective sets that C_{n+1} is closed and Q_{n+1} is closed and convex. Now, we show that C_{n+1} is convex. Since $C_1 = C$ is convex, we assume that C_k is convex for some $k \ge 2$. For any $z \in C_k$, the inequality $\|y_{n,i} - z\| \le \|x_n - z\|$ is equivalent to

$$\|y_{n,i}\|^2 - \|x_n\|^2 - 2\langle z, y_{n,i} - x_n \rangle \ge 0.$$

Hence, C_{k+1} is convex, and consequently C_{n+1} is convex for each $n \ge 1$.

Next, we show by induction that $\mathcal{F} \subset C_{n+1} \cap Q_{n+1}$ for all $n \ge 1$. Obviously, $\mathcal{F} \subset C_1 \cap Q_1 = C$. Let $p \in \mathcal{F}$. From (2.1), (3.1) and Lemma 2.3, we have the following estimate:

$$
\begin{aligned}
\|z_{n,i} - p\|^2 &= \left\| \alpha_{n,i}(u_{n,i} - p) + (1 - \alpha_{n,i})(T_i u_{n,i} - p) \right\|^2 \\
&= \alpha_{n,i} \|u_{n,i} - p\|^2 + (1 - \alpha_{n,i}) \|T_i u_{n,i} - p\|^2 \\
&\quad - \alpha_{n,i}(1 - \alpha_{n,i}) \|u_{n,i} - T_i u_{n,i}\|^2 \\
&\le \alpha_{n,i} \|u_{n,i} - p\|^2 + (1 - \alpha_{n,i})\left(\|u_{n,i} - p\|^2 + k\|u_{n,i} - T_i u_{n,i}\|^2 \right) \\
&\quad - \alpha_{n,i}(1 - \alpha_{n,i}) \|u_{n,i} - T_i u_{n,i}\|^2 \\
&= \|u_{n,i} - p\|^2 - (1 - \alpha_{n,i})(\alpha_{n,i} - k) \|u_{n,i} - T_i u_{n,i}\|^2.
\end{aligned}
\tag{3.2}
$$

Since $\alpha_{n,i} - k > 0$ (by (C1)), therefore the above estimate (3.2) yields

$$\|z_{n,i} - p\|^2 \le \|u_{n,i} - p\|^2 \le \|x_n - p\|^2. \tag{3.3}$$

Moreover,

$$
\begin{aligned}
\|y_{n,i} - p\|^2 &= \big\| \beta_{n,i}(z_{n,i} - p) + (1 - \beta_{n,i})(S_i z_{n,i} - p) \big\|^2 \\
&= \beta_{n,i} \|z_{n,i} - p\|^2 + (1 - \beta_{n,i}) \|S_i z_{n,i} - p\|^2 \\
&\quad - \beta_{n,i}(1 - \beta_{n,i}) \|z_{n,i} - S_i z_{n,i}\|^2 \\
&\leq \beta_{n,i} \|z_{n,i} - p\|^2 + (1 - \beta_{n,i})\big(\|z_{n,i} - p\|^2 + k\|z_{n,i} - S_i z_{n,i}\|^2 \big) \\
&\quad - \beta_{n,i}(1 - \beta_{n,i}) \|z_{n,i} - S_i z_{n,i}\|^2 \\
&\leq \|z_{n,i} - p\|^2 - (1 - \beta_{n,i})(\beta_{n,i} - k)\|z_{n,i} - S_i z_{n,i}\|^2 .
\end{aligned}
\tag{3.4}
$$

Reasoning as above and utilizing (3.3), the estimate (3.4) implies that

$$
\|y_{n,i} - p\| \leq \|x_n - p\|.
\tag{3.5}
$$

This implies that $p \in C_{n+1}$ for all $n \geq 1$. It suffices to show that $p \in Q_{n+1}$ for all $n \geq 1$. We prove this by induction. Note that $\mathcal{F} \subset Q_1 = C$ is obvious. Assume that $\mathcal{F} \subset Q_k$ also $\mathcal{F} \subset C_k \cap Q_k$ for some $k \geq 2$. This implies x_k is a projection of x_1 onto $C_k \cap Q_k$, and consequently we have

$$
\langle x_k - z, x_1 - x_k \rangle \geq 0 \quad \text{for all } z \in C_k \cap Q_k.
$$

Since $\mathcal{F} \subset C_k \cap Q_k$, we have

$$
\langle x_k - p, x_1 - x_k \rangle \geq 0 \quad \text{for all } p \in \mathcal{F}.
$$

Hence, $p \in Q_{k+1}$, and consequently $\mathcal{F} \subset C_{n+1} \cap Q_{n+1}$ for all $n \geq 1$. Since $\mathcal{F}$ is now closed and convex, so it follows from Lemma 2.1 that $P_{\mathcal{F}}$ is well defined. Note that $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x_1$, therefore $\|x_{n+1} - x_1\| \leq \|p - x_1\|$ for all $p \in \mathcal{F} \subset C_{n+1} \cap Q_{n+1}$. In particular, we have $\|x_{n+1} - x_1\| \leq \|P_{\mathcal{F}} x_1 - x_1\|$. This implies that $\{x_n\}$ is bounded. On the other hand, $x_n = P_{C_n \cap Q_n} x_1$ and $x_{n+1} \in C_{n+1} \cap Q_{n+1} \subset Q_{n+1}$, we have

$$
\|x_n - x_1\| \leq \|x_{n+1} - x_1\|.
$$

That is, the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. This implies $\lim_{n \to \infty} \|x_n - x_1\|$ exists. Note that

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 + x_1 - x_n\|^2 \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_1 \rangle \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n + x_n - x_1 \rangle \\
&= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n \rangle \\
&\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 .
\end{aligned}
$$

Taking lim sup on both sides of the above estimate, we have $\limsup_{n\to\infty}\|x_{n+1}-x_n\|^2=0$. That is,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.6}$$

Since $x_{n+1}\in C_{n+1}$, we have $\|y_{n,i}-x_{n+1}\|\leq\|x_n-x_{n+1}\|$. This implies

$$\lim_{n\to\infty}\|y_{n,i}-x_{n+1}\|=0. \tag{3.7}$$

Moreover, it follows from (3.6), (3.7) and the following inequality:

$$\|y_{n,i}-x_n\|\leq\|y_{n,i}-x_{n+1}\|+\|x_{n+1}-x_n\|$$

that

$$\lim_{n\to\infty}\|y_{n,i}-x_n\|=0. \tag{3.8}$$

Note that

$$
\begin{aligned}
\|u_{n,i}-p\|^2 &= \left\|V_{r_{n,i}}(I-r_{n,i}A_i)x_n - V_{r_{n,i}}(I-r_{n,i}A_i)p\right\|^2 \\
&= \left\|(I-r_{n,i}A_i)x_n - (I-r_{n,i}A_i)p\right\|^2 \\
&= \left\|x_n-p-r_{n,i}(A_ix_n-A_ip)\right\|^2 \\
&= \|x_n-p\|^2 - 2r_{n,i}\langle A_ix_n-A_ip, x_n-p\rangle + r_{n,i}^2\|A_ix_n-A_ip\|^2 \\
&\leq \|x_n-p\|^2 + r_{n,i}(r_{n,i}-2\delta)\|A_ix_n-A_ip\|^2.
\end{aligned}
\tag{3.9}
$$

Since $\|y_{n,i}-p\|^2\leq\|z_{n,i}-p\|^2\leq\|u_{n,i}-p\|^2$, therefore utilizing (3.9), we get

$$\|y_{n,i}-p\|^2\leq\|x_n-p\|^2-r_{n,i}(2\delta-r_{n,i})\|A_ix_n-A_ip\|^2.$$

Re-arranging the terms in the above estimate and utilizing (C2), we have

$$r(2\delta-s)\|A_ix_n-A_ip\|^2\leq\|x_n-p\|^2-\|y_{n,i}-p\|^2\leq\left(\|x_n-p\|+\|y_{n,i}-p\|\right)\|x_n-y_{n,i}\|.$$

Hence, (3.8) implies that

$$\lim_{n\to\infty}\|A_ix_n-A_ip\|=0\quad\text{for all }i\geq1. \tag{3.10}$$

Since $k<a\leq\beta_{n,i}\leq b<1$, then the following variant of the estimate (3.4) implies that

$$
\begin{aligned}
(1-b)(a-k)\|z_{n,i}-S_iz_{n,i}\|^2 &\leq \|x_n-p\|^2-\|y_{n,i}-p\|^2 \\
&\leq \left(\|x_n-p\|+\|y_{n,i}-p\|\right)\|x_n-y_{n,i}\|.
\end{aligned}
$$

Again, utilizing (3.8), we have

$$\lim_{n\to\infty}\|z_{n,i}-S_iz_{n,i}\|=0\quad\text{for all }i\geq1. \tag{3.11}$$

Observe that $\|y_{n,i} - z_{n,i}\| = (1 - \beta_{n,i})\|S_i z_{n,i} - z_{n,i}\|$. It follows from the fact that $k < a \le \beta_{n,i} \le b < 1$ and (3.11) that

$$\lim_{n \to \infty} \|y_{n,i} - z_{n,i}\| = 0 \quad \text{for all } i \ge 1. \tag{3.12}$$

Moreover, $\|x_n - z_{n,i}\| \le \|x_n - y_{n,i}\| + \|y_{n,i} - z_{n,i}\|$. Letting $n \to \infty$ on both sides and utilizing (3.8) and (3.12), we have

$$\lim_{n \to \infty} \|x_n - z_{n,i}\| = 0 \quad \text{for all } i \ge 1. \tag{3.13}$$

Reasoning as above, that is, $k < a \le \alpha_{n,i} \le b < 1$, we consider the following variant of the estimate (3.2):

$$(1 - b)(a - k)\|u_{n,i} - T_i u_{n,i}\|^2 \le \|x_n - p\|^2 - \|z_{n,i} - p\|^2$$
$$\le \left(\|x_n - p\| + \|z_{n,i} - p\|\right)\|x_n - z_{n,i}\|.$$

Hence, we conclude from the above estimate and (3.13) that

$$\lim_{n \to \infty} \|u_{n,i} - T_i u_{n,i}\| = 0 \quad \text{for all } i \ge 1. \tag{3.14}$$

On the other hand, $\|z_{n,i} - u_{n,i}\| = (1 - \alpha_{n,i})\|T_i u_{n,i} - u_{n,i}\|$. Making use of the fact that $k < a \le \alpha_{n,i} \le b < 1$ and (3.14), we get

$$\lim_{n \to \infty} \|z_{n,i} - u_{n,i}\| = 0 \quad \text{for all } i \ge 1. \tag{3.15}$$

Moreover, we conclude from the estimates (3.12) and (3.15) that

$$\lim_{n \to \infty} \|y_{n,i} - u_{n,i}\| = 0 \quad \text{for all } i \ge 1. \tag{3.16}$$

As a direct consequence of the estimates (3.8) and (3.16), we have

$$\lim_{n \to \infty} \|x_n - u_{n,i}\| = 0 \quad \text{for all } i \ge 1. \tag{3.17}$$

Next, we show that $\omega(x_n) \subset \mathcal{F}$, where $\omega(x_n)$ is the set of all weak ω-limit of $\{x_n\}$. Since $\{x_n\}$ is bounded, therefore $\omega(x_n) \ne \emptyset$. Let $q \in \omega(x_n)$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$. It follows from the estimate (3.17) that $u_{n_j,i} \rightharpoonup q$. We first show that $q \in \mathrm{GEP}(f_1, A_1)$, where $f_1 = f_{n_j}$ for some $j \ge 1$. From $u_{n_j,i} = V_{r_{n_j,i}}(I - r_{n_j,i} A_i)x_n$, for all $n \ge 1$, we have

$$f_1(u_{n_j,i}, y) + \langle A_1 x_{n_j}, y - u_{n_j,i} \rangle + \frac{1}{r_{n_j,i}} \langle y - u_{n_j,i}, u_{n_j,i} - x_{n_j} \rangle \ge 0 \quad \text{for all } y \in C.$$

From (A2), we have

$$\langle A_1 x_{n_j}, y - u_{n_j,i} \rangle + \frac{1}{r_{n_j,i}} \langle y - u_{n_j,i}, u_{n_j,i} - x_{n_j} \rangle \ge f_1(y, u_{n_j,i}) \quad \text{for all } y \in C. \tag{3.18}$$

Let $y_t = ty + (1-t)q$ for $0 < t < 1$ and $y \in C$. Since $q \in C$, this implies that $y_t \in C$. It follows from the estimate (3.18) that

$$
\begin{aligned}
\langle y_t - u_{n_j,i}, A_1 y_t \rangle \geq{} & \langle y_t - u_{n_j,i}, A_1 y_t \rangle - \langle A_1 x_{n_j}, y_t - u_{n_j,i} \rangle \\
& - \left\langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j,i}} \right\rangle + f_1(y_t, u_{n_j,i}) \\
={} & \langle y_t - u_{n_j,i}, A_1 y_t - A_1 u_{n_j,i} \rangle + \langle y_t - u_{n_j,i}, A_1 u_{n_j,i} - A_1 x_{n_j} \rangle \\
& - \left\langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j,i}} \right\rangle + f_1(y_t, u_{n_j,i}).
\end{aligned}
\tag{3.19}
$$

Since $\lim_{n \to \infty} \|x_{n_j} - u_{n_j,i}\| = 0$, therefore $\lim_{n \to \infty} \|A_1 x_{n_j} - A_1 u_{n_j,i}\| = 0$. Moreover, it follows from the monotonicity of A that $\langle y_t - u_{n_j,i}, A_1 y_t - A_1 u_{n_j,i} \rangle \geq 0$. Hence, (A4) implies that

$$
\langle y_t - q, A_1 y_t \rangle \geq f_1(y_t, q).
\tag{3.20}
$$

Using (3.20), (A1) and (A4), the following estimate:

$$
\begin{aligned}
0 = f_1(y_t, y_t) &\leq t f_1(y_t, y) + (1-t) f_1(y_t, q) \\
&\leq t f_1(y_t, y) + (1-t)\langle y_t - q, A_1 y_t \rangle \\
&= t f_1(y_t, y) + (1-t)t\langle y - q, A_1 y_t \rangle,
\end{aligned}
$$

implies that

$$
f_1(y_t, y) + (1-t)\langle y - q, A_1 y_t \rangle \geq 0.
\tag{3.21}
$$

Letting $t \to 0$, we have $f_1(q, y) + \langle y - q, A_1 q \rangle \geq 0$ for all $y \in C$. Thus, $q \in \mathrm{GEP}(f_1, A_1)$. In a similar fashion, we have some $k \geq 1$ such that $f_2 = f_{n_k}$ and $q \in \mathrm{GEP}(f_2, A_2)$. Therefore, $q \in \bigcap_{i=1}^N \mathrm{GEP}(f_i, A_i)$. Since $u_{n_j,i} \rightharpoonup q$, so it follows from (3.14) and Lemma 2.2 that $q \in \bigcap_{i=1}^N F(T_i)$. Reasoning as above, we can show that $q \in \bigcap_{i=1}^N F(S_i)$. Hence, $q \in \mathcal{F}$. Let $x = P_{\mathcal{F}} x_1$, which implies that $x = P_{\mathcal{F}} x_1 \in C_{n+1}$. Since $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x_1 \in C_{n+1}$, we have

$$
\|x_{n+1} - x_1\| \leq \|x - x_1\|.
$$

On the other hand, we have

$$
\begin{aligned}
\|x - x_1\| &\leq \|q - x_1\| \\
&\leq \liminf_{j \to \infty} \|x_{n_j} - x_1\| \\
&\leq \limsup_{j \to \infty} \|x_{n_j} - x_1\| \\
&\leq \|x - x_1\|.
\end{aligned}
$$

That is,

$$
\|q - x_1\| = \lim_{j \to \infty} \|x_{n_j} - x_1\| = \|x - x_1\|.
$$

Therefore, we conclude that $\lim_{j\to\infty} x_{n_j} = q = P_{\mathcal{F}}x_1$. From the arbitrariness of $\{x_{n_j}\}$, we get that $\lim_{n\to\infty} x_n = P_{\mathcal{F}}x_1$. This completes the proof. $\qquad\square$

In particular, if T_i and S_i - in algorithm (3.1) - are two finite families of nonexpansive maps, then the following result holds.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i(\mathrm{mod}\,N), S_i(\mathrm{mod}\,N) : C \to C$ be two finite families of nonexpansive maps. Let $f_i(\mathrm{mod}\,N) : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying* (A1)-(A4) *and let $A_i(\mathrm{mod}\,N) : C \to H$ be a finite family of δ-inverse-strongly monotone maps. Let $\{r_{n,i}\} \subset (0,\infty)$ and $\{\alpha_{n,i}\}, \{\beta_{n,i}\}$ be two control sequences such that*

(C1) $0 \le k < a \le \alpha_{n,i}, \beta_{n,i} \le b < 1$;

(C2) $0 < r < r_{n,i} < s < 2\delta$ *for all $i \ge 1$.*

Assume that $\mathcal{F} := [\bigcap_{i=1}^{N} F(T_i)] \cap [\bigcap_{i=1}^{N} F(S_i)] \cap [\bigcap_{i=1}^{N} \mathrm{GEP}(f_i, A_i)] \ne \emptyset$, then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x = P_{\mathcal{F}}x_1$, where $P_{\mathcal{F}}$ is the metric projection of H onto $\mathcal{F}$.

In order to address variational inequality problems coupled with the fixed point problems, we prove the following result with a slight modification of algorithm (3.1).

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i(\mathrm{mod}\,N), S_i(\mathrm{mod}\,N) : C \to C$ be two finite families of k-strict pseudo-contractions. Let $f_i(\mathrm{mod}\,N) : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying* (A1)-(A4) *and let $A_i(\mathrm{mod}\,N) : C \to H$ be a finite family of δ-inverse-strongly monotone maps. Let $\{r_{n,i}\} \subset (0,\infty)$ and $\{\alpha_{n,i}\}, \{\beta_{n,i}\}$ be two control sequences such that*

(C1) $0 \le k < a \le \alpha_{n,i}, \beta_{n,i} \le b < 1$;

(C2) $0 < r < r_{n,i} < s < 2\delta$ *for all $i \ge 1$.*

Assume that $\mathcal{F} := [\bigcap_{i=1}^{N} F(T_i)] \cap [\bigcap_{i=1}^{N} F(S_i)] \cap [\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)] \ne \emptyset$, then the sequence $\{x_n\}$ generated by

$$
\begin{aligned}
&x_1 \in C_1 = Q_1 = C, \\
&h_{n,i} = P_C(I - r_{n,i}A_i)x_n, \\
&z_{n,i} = \alpha_{n,i}h_{n,i} + (1 - \alpha_{n,i})T_i h_{n,i}, \\
&y_{n,i} = \beta_{n,i}z_{n,i} + (1 - \beta_{n,i})S_i z_{n,i}, \\
&C_{n+1} = \left\{ z \in C_n : \sup_{i\ge 1} \|y_{n,i} - z\| \le \|x_n - z\| \right\}, \\
&Q_{n+1} = \left\{ z \in Q_n : \langle z - x_n, x_1 - x_n \rangle \ge 0 \right\}, \\
&x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x_1, \quad n \ge 1,
\end{aligned}
\tag{3.22}
$$

converges strongly to $x = P_{\mathcal{F}}x_1$, where $P_{\mathcal{F}}$ is the metric projection of H onto $\mathcal{F}$.

Proof Set $f_i(x, y) \equiv 0$ for each $i \ge 1$, then

$$
\langle A_i x_n, y - u_{n,i} \rangle + \frac{1}{r_{n,i}} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \ge 0
$$

is equivalent to

$$\langle x_n - r_{n,i} A_i x_n - u_{n,i}, u_{n,i} - y \rangle \geq 0.$$

This implies that $h_{n,i} = u_{n,i} := P_C(x_n - r_{n,i} A_i x_n)$. The desired result then follows from Theorem 3.1 immediately. □

As an application of Theorem 3.1 - by substituting $A_i \equiv 0$ for all $i \geq 1$ in algorithm (3.1) - we have the following result for a finite family of equilibrium problems.

Theorem 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i (\mathrm{mod}\, N), S_i (\mathrm{mod}\, N) : C \to C$ be two finite families of k-strict pseudo-contractions. Let $f_i (\mathrm{mod}\, N) : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying* (A1)-(A4). *Let $\{\alpha_{n,i}\}, \{\beta_{n,i}\}$ be two control sequences such that*

(C1) $0 \leq k < a \leq \alpha_{n,i}, \beta_{n,i} \leq b < 1$.

Assume that $\mathcal{F} := [\bigcap_{i=1}^{N} F(T_i)] \cap [\bigcap_{i=1}^{N} F(S_i)] \cap [\bigcap_{i=1}^{N} \mathrm{EP}(f_i)] \neq \emptyset$, then the sequence $\{x_n\}$ generated by

$$
\begin{aligned}
&x_1 \in C_1 = Q_1 = C, \\
&f_i(u_{n,i}, y) + \frac{1}{r_{n,i}} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \geq 0, \quad \forall y \in C, \\
&y_{n,i} = \beta_{n,i} z_{n,i} + (1 - \beta_{n,i}) S_i z_{n,i}, \\
&z_{n,i} = \alpha_{n,i} u_{n,i} + (1 - \alpha_{n,i}) T_i u_{n,i}, \\
&C_{n+1} = \Big\{ z \in C_n : \sup_{i \geq 1} \| y_{n,i} - z \| \leq \| x_n - z \| \Big\}, \\
&Q_{n+1} = \big\{ z \in Q_n : \langle z - x_n, x_1 - x_n \rangle \geq 0 \big\}, \\
&x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x_1, \quad n \geq 1,
\end{aligned}
\tag{3.23}
$$

converges strongly to $x = P_{\mathcal{F}} x_1$, where $P_{\mathcal{F}}$ is the metric projection of H onto $\mathcal{F}$.

Remark 3.5 Additionally - in Theorem 3.4 - if we set $\beta_{n,i} \equiv 0$, then Theorem 3.4 sets analogue [17, Theorem 3.3] in the following aspects:

  (i) from a single k-strict pseudo-contraction to a finite family of maps;

 (ii) from an equilibrium problem to a finite family of generalized equilibrium problems.

Author details
[1]Department of Mathematics, Technische Universitat Darmstadt, Schlossgartenstrasse 7, Darmstadt, 64289, Germany.
[2]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan. [3]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 3126, Saudi Arabia.

Acknowledgements

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