Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A Schauder fixed point theorem in semilinear spaces and applications

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Abstract

In this paper we present existence and uniqueness results for a class of fuzzy fractional integral equations. To prove the existence result, we give a variant of the Schauder fixed point theorem in semilinear Banach spaces.

MSC: 34A07; 34A08

Keywords: fuzzy fractional differential equation; Schauder fixed point theorem; Ascoli-Arzelá-type theorem; existence result

1 Introduction

The topic of fuzzy differential equations has been extensively developed in recent years as a fundamental tool in the description of uncertain models that arise naturally in the real world. Fuzzy differential equations have become an important branch of differential equations with many applications in modeling real world phenomena in quantum optics, robotics, gravity, artificial intelligence, medicine, engineering and many other fields of science. The fundamental notions and results in the fuzzy differential equations can be found in the monographs [1] and [2].

The concept of fuzzy fractional differential equations has been recently introduced in some papers [3–10]. In [7], the authors established the existence and uniqueness of the solution for a class of fuzzy fractional differential equations, where a fuzzy derivative is used in the sense of Seikkala. In [5], the authors proposed the concept of Riemann-Liouville H-differentiability which is a direct extension of strongly generalized H-differentiability (see Bede and Gal [11]) to the fractional literature. They derived explicit solutions to fuzzy fractional differential equations under Riemann-Liouville H-differentiability. In [6], the authors established an existence result for fuzzy fractional integral equations using a compactness-type condition. In this paper, we present an existence result for a class of fuzzy fractional integral equations without a Lipschitz condition. For this we use a variant of the Schauder fixed point theorem. Since the space of continuous fuzzy functions is a semilinear Banach space, we prove a variant of the Schauder fixed point theorem in semilinear Banach spaces.

The paper is organized as follows. Section 2 includes the properties and results which we will use in the rest of the paper. We present an example which shows that a fuzzy fractional differential equation is generally not equivalent to a fuzzy fractional integral equation. In Section 3, we establish the Schauder fixed point theorem in semilinear Banach spaces. In Section 4, we prove an existence result for a class of fuzzy fractional integral equations without a Lipschitz condition. Finally, using Weissinger's fixed point theorem, we give an existence and uniqueness result.

2 Preliminaries

In the sequel, $\mathbb{R}^n$ will denote the n-dimensional Euclidean space with the norm $\|\cdot\|$. Let $K_c(\mathbb{R}^n)$ denote the family of all nonempty, compact and convex subsets of $\mathbb{R}^n$. A semilinear structure in $K_c(\mathbb{R}^n)$ is defined by

(i) $A + B = \{a + b : a \in A, b \in B\}$,

(ii) $\lambda A = \{\lambda a : a \in A\}$,

for all $A, B \in K_c(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$.

The distance between A and B is defined by the Hausdorff-Pompeiu metric

$$d_H(A,B) = \max\Bigl\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\Bigr\}.$$

$K_c(\mathbb{R}^n)$ is a complete and separable metric space with respect to the Hausdorff-Pompeiu metric [12].

In the following, we give some basic notions and results on fuzzy set theory. We denote by $\mathbb{E}^n$ the space of all fuzzy sets in $\mathbb{R}^n$, that is, $\mathbb{E}^n$ is the space of all functions $y : \mathbb{R}^n \to [0,1]$ with the following properties:

(i) y is normal, *i.e.*, there exists $x_0 \in \mathbb{R}^n$ such that $y(x_0) = 1$;

(ii) $[y]^0 = \overline{\{x \in \mathbb{R}^n : y(x) > 0\}}$ is compact;

(iii) y is a convex fuzzy function, *i.e.*, for all $x_1, x_2 \in \mathbb{R}^n$, and for all $\lambda \in (0,1)$, we have

$$y\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) \geq \min\bigl\{y(x_1), y(x_2)\bigr\};$$

(iv) y is an upper semi-continuous function.

The fuzzy null set is defined by

$$\widehat{0}(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

If $y \in \mathbb{E}^n$, then the set

$$[y]^\alpha = \bigl\{x \in \mathbb{R}^n; y(x) \geq \alpha\bigr\}, \quad \alpha \in (0,1],$$

is called the α-level set of y. Then from (i)-(iv) it follows that the set $[y]^\alpha \in K_c(\mathbb{R}^n)$ for all $\alpha \in [0,1]$.

The following operations, based on a generalization of Zadeh's extension principle, define a semilinear structure on $\mathbb{E}^n$:

$$(y + z)(x) = \sup_{u+v=x} \min\bigl\{y(u), z(v)\bigr\},$$

$$(\lambda y)(x) = \begin{cases} y(x/\lambda), & \lambda \neq 0, \\ \chi_0(x), & \lambda = 0, \end{cases}$$

where $y, z \in \mathbb{E}^n$ and $\lambda \in \mathbb{R}$. The α-level set of fuzzy sets satisfy the following properties (see [2]):

(i) $[y+z]^\alpha = [y]^\alpha + [z]^\alpha$;
(ii) $[\lambda y]^\alpha = \lambda [y]^\alpha$

for all $y, z \in \mathbb{E}^n$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$.

We define a metric d on $\mathbb{E}^n$ by

$$d(y,z) = \sup_{0 \le \alpha \le 1} d_H\big([y]^\alpha, [z]^\alpha\big),$$

where d_H is the Hausdorff-Pompeiu metric. Then $(\mathbb{E}^n, d)$ is a complete metric space (see [13]).

Proposition 2.1 [2] *If $y, z, w, w' \in \mathbb{E}^n$, then*
(i) $d(y+w, z+w) = d(y,z)$,
(ii) $d(\lambda y, \lambda z) = \lambda d(y,z)$ *for all* $\lambda \ge 0$,
(iii) $d(y+w, z+w') \le d(y,z) + d(w,w')$.

Define $\mathbb{E}^n_c$ as the space of fuzzy sets $y \in E^n$ with the property that the function $\alpha \mapsto [y]^\alpha$ is continuous with respect to the Hausdorff-Pompeiu metric on $[0,1]$.

Let $T \subset \mathbb{R}$ be an interval. We denote by $C(T, E^n)$ the space of all continuous fuzzy functions on T.

It is known that $(\mathbb{E}^n_c, d)$ is a complete metric space (see [14]). Therefore, $(C([0,a], \mathbb{E}^n_c), D)$ is a complete metric space where

$$D(y,z) = \sup_{t \in [0,a]} d\big(y(t), z(t)\big).$$

A subset $A \subseteq \mathbb{E}^n_c$ is said to be compact-supported if there exists a compact set $K \subseteq \mathbb{R}^n$ such that $[y]^0 \subseteq K$ for all $y \in A$.

A subset $A \subseteq \mathbb{E}^n_c$ is said to be level-equicontinuous at $\alpha_0 \in [0,1]$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\alpha - \alpha_0| < \delta \quad \text{implies} \quad d_H\big([y]^\alpha, [y]^{\alpha_0}\big) < \epsilon \quad \text{for all } y \in A.$$

A is level-equicontinuous on $[0,1]$ if A is level-equicontinuous at α for all $\alpha \in [0,1]$.

Theorem 2.2 [14] *Let A be a compact-supported subset of $\mathbb{E}^n_c$. Then the following are equivalent*:
(a) *A is a relatively compact subset of* $(\mathbb{E}^n_c, d)$;
(b) *A is level-equicontinuous on* $[0,1]$.

Remark 2.3 [14] Let K be a compact subset of $\mathbb{R}^n$ and

$$\widetilde{K} := \{\chi_{\{t\}} : t \in K\}.$$

Then $\widetilde{K}$ is relatively compact in $\mathbb{E}^n_c$.

A continuous function $f : [0,a] \times \mathbb{E}^n_c \to \mathbb{E}^n_c$ is said to be compact if $I \subseteq [0,a]$ and $A \subseteq \mathbb{E}^n_c$ is bounded imply that $f(I \times A)$ is relatively compact in $\mathbb{E}^n_c$.

Let $L^1([0,a],\mathbb{R}^n)$ denote the space of Lebesgue integrable functions from $[0,a]$ to $\mathbb{R}^n$. Let $u \in L^1([0,a],\mathbb{R}^n)$. The fractional integral of order $q > 0$ of y is given by

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\,ds$$

provided the expression on the right-hand side is defined.

We denote by S_F^1 the set of all Lebesgue integrable selections of $F : [0,a] \to K_c(\mathbb{R}^n)$, that is,

$$S_F^1 = \{f \in L^1([0,a],\mathbb{R}^n) : f(t) \in F(t) \text{ a.e.}\}.$$

The Aumann integral of F is defined by

$$\int_0^a F(t)\,dt = \left\{ \int_0^a f(t)\,dt : f \in S_F^1 \right\}.$$

A function $F : [0,a] \to K_c(\mathbb{R}^n)$ is called measurable (see [15]) if $F^{-1}(B) \in \mathcal{B}$ for all closed set $B \subset \mathbb{R}^n$, where $\mathcal{B}$ denotes the Borel algebra of $[0,a]$. A function $F : [0,a] \to K_c(\mathbb{R}^n)$ is called integrably bounded if there exists a function $h \in L^1(\mathbb{R}_+)$ such that $\sup\{\|x\|; x \in F(t)\} \le h(t)$ for a.e. $t \in [0,a]$. If such F has measurable selectors, then they are also integrable and S_F^1 is nonempty.

The fractional integral of the function $F : [0,a] \to K_c(\mathbb{R}^n)$ of order $q > 0$ is defined by (see [16])

$$I^q F(t) = \{I^q f(t)\,dt : f \in S_F^1\}.$$

A fuzzy function $y : [0,a] \to E^n$ is measurable if, for all $\alpha \in [0,1]$, the set-valued function $y_\alpha : [0,a] \to K_c(\mathbb{R}^n)$, defined by

$$y_\alpha(t) := [y(t)]^\alpha = \{x \in \mathbb{R}^n : y(t)(x) \ge \alpha\},$$

is measurable.

A fuzzy function $y : [0,a] \to E^n$ is integrably bounded if there exists a function $h \in L^1(\mathbb{R}_+)$ such that $\|x\| \le h(t)$ for all $x \in [y(t)]^0$. A measurable and integrably bounded fuzzy function $y : [0,a] \to E^n$ is said to be integrable on $[0,a]$ if there exists $v \in E^n$ such that $v_\alpha = \int_0^a y_\alpha(t)\,dt$ for all $\alpha \in [0,1]$.

Lemma 2.4 [6] *Let $q \in (0,1]$, and let $y : [0,a] \to E^n$ be an integrable fuzzy function. Then for each $t \in [0,a]$ there exists a unique fuzzy set $v(t) \in E^n$ such that*

$$I^q y_\alpha(t) = \{x \in \mathbb{R}^n : v(t)(x) \ge \alpha\} \quad \textit{for all } \alpha \in [0,1].$$

Let $y : [0,a] \to E^n$ be an integrable fuzzy function. The fuzzy fractional integral of order $q > 0$ of the function y,

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\,ds,$$

is defined by (see [6])

$$I^q y(t)(x) = \sup\{\alpha \in [0,1] : x \in I^q y_\alpha\}.$$

Its level sets are given by

$$[I^q y(t)]^\alpha = \{x \in \mathbb{R}^n : I^q y(t)(x) \geq \alpha\}, \quad \alpha \in [0,1];$$

that is, we have

$$[I^q y(t)]^\alpha = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [y(s)]^\alpha \, ds.$$

Let $y \in [0,a] \to \mathbb{E}^n$. If the fuzzy function $t \mapsto \int_0^t (t-s)^{-q} y(s)\, ds$ is Hukuhara differentiable on $[0,a]$, then we define the fractional derivative of order $q \in (0,1)$ of y by

$$D^q y(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} y(s)\, ds,$$

provided that the equation defines a fuzzy number $D^q y(t) \in \mathbb{E}^n$. It is easy to see that $D^q y(t) = \frac{d}{dt} I^{1-q} y(t)$, $t \in [0,a]$.

Lemma 2.5 [6] *Let* $0 < q < 1$ *and* $y \in [0,a] \to \mathbb{E}_c^n$ *be integrable. Then*

$$D^q I^q y(t) = y(t), \quad t \in [0,a].$$

Remark 2.6 Let $0 < q < 1$. The equality

$$I^q D^q y_\alpha(t) = y_\alpha(t) - \frac{t^{q-1}}{\Gamma(q)} I^{1-q} y_\alpha(0), \quad t \in [0,a] \tag{2.1}$$

is not true in the fuzzy case. Indeed, let $y : [0,1] \to \mathbb{E}$ be a fuzzy function defined by

$$y(t)(x) = \begin{cases} 1 - \frac{x}{1-t}, & 0 < x \leq 1-t, t \in [0,1), \\ 1, & t = 1. \end{cases}$$

Then it is easy to see that

$$y_\alpha(t) = [y(t)]^\alpha = [0, (1-\alpha)(1-t)], \quad t \in (0,1], \alpha \in [0,1],$$

define the α-level intervals of $y(t)$.

Now take $q = 1/2$. Then

$$I^{1/2} D^{1/2} y_\alpha(t) = [0, (1-\alpha)(1-t)] = y_\alpha(t).$$

Since

$$\frac{t^{-1/2}}{\Gamma(\frac{1}{2})} I^{1/2} y_\alpha(0) = \left[0, \frac{2}{\pi}(1-\alpha)\right],$$

then

$$y_\alpha(t) - \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} I^{1/2} y_\alpha(0) = \left[0, (1-\alpha)\left(1 - t - \frac{2}{\pi}\right)\right],$$

which is a fuzzy number for $t \in [0, 1 - \frac{2}{\pi}]$. However, it is not a fuzzy number for $t > 1 - \frac{2}{\pi}$. Thus $y_\alpha(t)$ does not satisfy equation (2.1).

3 Schauder fixed point theorem for semilinear spaces

In this section, we prove the Schauder fixed point theorem for semilinear Banach spaces. First, we recall the Schauder fixed point theorem.

Theorem 3.1 ([17], Schauder fixed point theorem) *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

We recall that a semilinear metric space is a semilinear space S with a metric $d : S \times S \to \mathbb{R}_+$ which is translation invariant and positively homogeneous, that is,

(i) $d(a + c, b + c) = d(a, b)$,

(ii) $d(\lambda a, \lambda b) = \lambda d(a, b)$ for all $\lambda \geq 0$,

for all $a, b, c \in S$ and $\lambda \geq 0$.

In this case, we can define a norm on S by $\|x\| = d(x, \widetilde{0})$, where $\widetilde{0}$ is the zero element in S. If S is a semilinear metric space, then addition and scalar multiplication on S are continuous. If S is a complete metric space, then we say that S is a semilinear Banach space.

Let S be a semilinear space having the cancelation property. Define an equivalence relation $\sim$ on $S \times S$ by

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad a + d = b + c$$

for all $(a, b), (c, d) \in S \times S$, and let $\langle a, b\rangle$ denote the equivalence class containing (a, b). Let G denote the collection of all equivalence classes of $S \times S$. On G define addition and scalar multiplication as follows:

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$$

and

$$\lambda\langle a, b\rangle = \begin{cases} \langle \lambda a, \lambda b\rangle, & \lambda \geq 0, \\ \langle -\lambda a, -\lambda b\rangle, & \text{otherwise} \end{cases}$$

for all $(a, b), (c, d) \in S \times S$, and $\lambda \in \mathbb{R}$. Further define a map $j : S \to G$ by

$$j(a) := \langle a, \widetilde{0}\rangle$$

for all $a \in S$. Let S be a semilinear metric space. On G, define a norm $\|\cdot\| : G \to \mathbb{R}_+$ by

$$\big\|\langle a, b\rangle\big\| := d(a, b)$$

for all $\langle a, b\rangle \in G$.

Theorem 3.2 [18] *Suppose that S is a semilinear space having the cancelation property. Then G is a vector space satisfying $G = j(S) - j(S)$ and j is an injection such that*
(i) $j(a + b) = j(a) + j(b)$;
(ii) $j(\lambda a) = \lambda j(a)$
for all $a, b \in S$ and $\lambda \geq 0$.

Theorem 3.3 [18] *Suppose that S is a semilinear metric space. Then the set all equivalence classes G, constructed above, is a metric vector space and j is an isometry.*

Now, we are able to prove a variant of the Schauder fixed point theorem in semilinear Banach spaces.

Theorem 3.4 (Schauder fixed point theorem for semilinear spaces) *Let B be a nonempty, closed, bounded and convex subset of a semilinear Banach space S having the cancelation property, and suppose that $P : B \to B$ is a compact operator. Then P has at least one fixed point in B.*

Proof By Theorem 3.3, there exists an embedding $j : S \to G$. Let B be a nonempty, closed, bounded and convex subset of S. Since j is isometry, it follows that $j(B)$ is also a closed and bounded subset of G. For convexity, let $u, v \in j(B)$ and $\lambda \geq 0$. Then there exist $\overline{u}, \overline{v} \in B$ such that $u = j(\overline{u})$ and $v = j(\overline{v})$. By Theorem 3.2, we obtain

$$\lambda u + (1 - \lambda)v = \lambda j(\overline{u}) + (1 - \lambda)j(\overline{v}) = j\bigl(\lambda\overline{u} + (1 - \lambda)\overline{v}\bigr).$$

Since B is convex, we have $\lambda\overline{u} + (1 - \lambda)\overline{v} \in B$, which implies $\lambda u + (1 - \lambda)v \in j(B)$. Hence $j(B)$ is convex. Let $\widetilde{P} : j(B) \to j(B)$ be defined by $\widetilde{P} = j \circ P \circ j^{-1}$, that is, $P = j^{-1} \circ \widetilde{P} \circ j$. First we show that $\widetilde{P}$ is a compact operator. Note that $\widetilde{P}$ is a continuous operator because P, j and j^{-1} are continuous. Further, we have

$$\widetilde{P}\bigl(j(B)\bigr) = \bigl(j \circ P \circ j^{-1}\bigr)\bigl(j(B)\bigr) = j\bigl(P(B)\bigr).$$

Since $P(B)$ is relatively compact, it follows that $j(P(B))$ is relatively compact. Hence, by the Schauder fixed point theorem, $\widetilde{P}$ has a fixed point $u_0 \in j(B)$, that is, $\widetilde{P}(u_0) = u_0$. Let $v_0 = j^{-1}(u_0) \in B$. Then

$$P(v_0) = \bigl(j^{-1} \circ \widetilde{P} \circ j\bigr)\bigl(j^{-1}(u_0)\bigr) = j^{-1}\bigl(\widetilde{P}(u_0)\bigr) = j^{-1}(u_0) = v_0.$$

Thus $v_0 \in B$ is a fixed point of P. □

Remark 3.5 The space of fuzzy sets $\mathbb{E}^n$ is a semilinear Banach space S having the cancelation property. Therefore, the Schauder fixed point theorem holds true for fuzzy metric spaces.

4 Existence and uniqueness

Consider the following fuzzy fractional integral equation:

$$y(t) = y_0(t) + I^q f\bigl(t, y(t)\bigr), \tag{4.1}$$

where $0 < q < 1$, $y_0(t) \in C([0, a], \mathbb{E}_c^n)$ and $f : [0, a] \times \mathbb{E}_c^n \to \mathbb{E}_c^n$ is continuous on $[0, a] \times \mathbb{E}_c^n$.

A function $y \in C([0,a],\mathbb{E}_c^n)$ is called a solution for (4.1) if

$$y(t) = y_0(t) + I^q f\big(t, y(t)\big)$$

holds for all $t \in [0,a]$.

Remark 4.1 Let $0 < q < 1$. Consider the following fuzzy fractional differential equation:

$$D^q y(t) = f\big(t, y(t)\big), \qquad \lim_{t\to 0^+} t^{1-q} y(t) = y_0, \quad t \in [0,a]. \tag{4.2}$$

If $y : [0,a] \to \mathbb{E}_c^n$ is a solution of the integral equation

$$y(t) = t^{q-1} y_0 + I^q f\big(t, y(t)\big), \tag{4.3}$$

then by Lemma 2.5 $y(t)$ is a solution of (4.2), but the converse is not true.

In [19], the authors showed that the space $\mathbb{E}_c^n$ can be embedded in $C([0,1] \times S^{n-1})$, the Banach space of continuous real-valued functions defined on $[0,1] \times S^{n-1}$, where $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$ is the unit ball. In [14], an Ascoli-Arzelá-type theorem was proved. We use this theorem to establish an existence theorem for fuzzy fractional integral equations. Let $\overline{0}$ be the zero function in $C([0,a],\mathbb{E}_c^n)$.

Theorem 4.2 *Let* $0 < q < 1$, $R > 0$ *and* $a^* > 0$. *Define*

$$G = \big\{(t,y) \in \big[0,a^*\big] \times \mathbb{E}_c^n : d(y,\widehat{0}) \le R\big\}.$$

Suppose that $f : G \to \mathbb{E}_c^n$ *is a compact function and* $M = \sup_{(t,y)\in G} d(f(t,y),\widehat{0})$. *Let* $y_0(t) \in C([0,a],\mathbb{E}_c^n)$ *such that* $y_0([0,a^*])$ *is compact-supported and* $N = D(y_0,\overline{0})$. *Then integral equation* (4.1) *has at least one solution* $y(t) \in C([0,a],\mathbb{E}_c^n)$, *where* $a \in (0,a^*]$ *is chosen such that*

$$N + \frac{Ma^q}{\Gamma(q+1)} \le R.$$

Proof Define the set

$$\Omega = \big\{y \in C\big([0,a],\mathbb{E}_c^n\big) : D(y,\overline{0}) \le R\big\}.$$

It is evident that Ω is a closed, bounded and convex subset of the Banach space $C([0,a],\mathbb{E}_c^n)$. On the set Ω, we define the operator $T : \Omega \to C([0,a],\mathbb{E}_c^n)$ by

$$(Ty)(t) = y_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, y(s)\big)\, ds.$$

In order to prove our desired existence result, we show that T has a fixed point. First we show that the operator T is continuous on Ω. For this, let $y_n \to y$ in Ω. Then we have

$$
\begin{aligned}
& d\big(Ty_n(t), Ty(t)\big) \\
&\quad = \frac{1}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1} f\big(s, y_n(s)\big)\, ds, \int_0^t (t-s)^{q-1} f\big(s, y(s)\big)\, ds\right) \\
&\quad \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sup_{s\in[0,t]} d\big(f\big(s, y_n(s)\big), f\big(s, y(s)\big)\big)\, ds \\
&\quad \le \frac{1}{\Gamma(q)} \sup_{t\in[0,a]} d\big(f\big(t, y_n(t)\big), f\big(t, y(t)\big)\big) \int_0^t (t-s)^{q-1}\, ds \\
&\quad \le \frac{t^q}{\Gamma(q+1)} \sup_{t\in[0,a]} d\big(f\big(t, y_n(t)\big), f\big(t, y(t)\big)\big) \\
&\quad \le \frac{a^q}{\Gamma(q+1)} \sup_{t\in[0,a]} d\big(f\big(t, y_n(t)\big), f\big(t, y(t)\big)\big) \to 0, \quad n \to \infty.
\end{aligned}
$$

This implies that T is a continuous operator on Ω. For $y \in \Omega$ and $t \in [0,a]$, we have

$$
\begin{aligned}
d\big(Ty(t), \widehat{0}\big) &= d\left(y_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, y(s)\big)\, ds, \widehat{0}\right) \\
&\le d\big(y_0(t), \widehat{0}\big) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d\big(f\big(s, y(s)\big), \widehat{0}\big)\, ds \\
&\le D(y_0, \overline{0}) + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds \\
&\le N + \frac{Mt^q}{\Gamma(q+1)}.
\end{aligned}
$$

It follows that

$$
D(Ty, \overline{0}) \le N + \frac{Ma^q}{\Gamma(q+1)} \le R.
$$

Thus, T maps the set Ω to itself. Now we will prove that $T(\Omega)$ is relatively compact in $C([0,a], \mathbb{E}_c^n)$. Using the Arzela-Ascoli theorem, we just need to prove:

(i) $T(\Omega)$ is an equicontinuous subset of $C([0,a], \mathbb{E}_c^n)$;
(ii) $T(\Omega)(t)$ is relatively compact in $\mathbb{E}_c^n$ for each $t \in [0,a]$.

Let $t_1, t_2 \in [0,a]$, $t_1 < t_2$ and $y \in \Omega$, we obtain

$$
\begin{aligned}
& d\big(Ty(t_2), Ty(t_1)\big) \\
&\quad \le d\big(y_0(t_2), y_0(t_1)\big) + \frac{1}{\Gamma(q)} d\left(\int_0^{t_2} (t_2-s)^{q-1} f\big(s, y(s)\big)\, ds, \int_0^{t_1} (t_1-s)^{q-1} f\big(s, y(s)\big)\, ds\right) \\
&\quad \le d\big(y_0(t_2), y_0(t_1)\big) + \frac{1}{\Gamma(q)} d\left(\int_0^{t_1} (t_2-s)^{q-1} f\big(s, y(s)\big)\, ds, \int_0^{t_1} (t_1-s)^{q-1} f\big(s, y(s)\big)\, ds\right) \\
&\qquad + \frac{1}{\Gamma(q)} d\left(\int_{t_1}^{t_2} (t_2-s)^{q-1} f\big(s, y(s)\big)\, ds, \widehat{0}\right) \\
&\quad \le d\big(y_0(t_2), y_0(t_1)\big) + \frac{M}{\Gamma(q)} \left(\int_0^{t_1} \big((t_1-s)^{q-1} - (t_2-s)^{q-1}\big)\, ds + \int_{t_1}^{t_2} (t_2-s)^{q-1}\, ds\right)
\end{aligned}
$$

$$\begin{aligned}
&\le d\big(y_0(t_2), y_0(t_1)\big) + \frac{M}{\Gamma(q+1)}\big(2(t_2-t_1)^q + t_1^q - t_2^q\big) \\
&\le d\big(y_0(t_2), y_0(t_1)\big) + \frac{2M}{\Gamma(q+1)}(t_2-t_1)^q,
\end{aligned}$$

so $d(Ty(t_2), Ty(t_1)) \to 0$ when $t_1 \to t_2$ for all $y \in \Omega$. This implies that $T(\Omega)$ is equicontinuous on $[0,a]$. Now we show that $T(\Omega)(t)$ is relatively compact in $\mathbb{E}_c^n$ and by Theorem 2.2 this is equivalent to proving that $T(\Omega)(t)$ is a level-equicontinuous and compact-supported subset of $\mathbb{E}_c^n$.

Fixing $t \in [0,a]$, we see that $T(\Omega)(t) \in \mathbb{E}_c^n$ and if $v \in T(\Omega)(t)$, then

$$v = y_0(t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\big(s, y(s)\big)\, ds \quad \text{for some } y \in \Omega.$$

Since $f([0,a] \times \Omega)$ is relatively compact in $\mathbb{E}_c^n$, Theorem 2.2 implies that $f([0,a] \times \Omega)$ is level-equicontinuous. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\alpha - \beta| < \delta \quad \Rightarrow \quad d_H\big(\big[f\big(s,y(s)\big)\big]^\alpha, \big[f\big(s,y(s)\big)\big]^\beta\big) < \frac{\Gamma(q+1)\varepsilon}{2a^q} \quad \text{for all } (s,y) \in [0,a] \times \Omega.$$

Also, $|\alpha - \beta| < \delta$ implies

$$d_H\big(\big[y_0(t)\big]^\alpha, \big[y_0(t)\big]^\beta\big) \le \frac{\varepsilon}{2} \quad \text{for all } t \in [0,a].$$

Hence, we obtain

$$\begin{aligned}
&d_H\big([v]^\alpha, [v]^\beta\big) \\
&\quad = d_H\big(\big[T(y)(t)\big]^\alpha, \big[T(y)(t)\big]^\beta\big) \\
&\quad \le d_H\big(\big[y_0(t)\big]^\alpha, \big[y_0(t)\big]^\beta\big) \\
&\qquad + \frac{1}{\Gamma(q)} d_H\left(\left[\int_0^t (t-s)^{q-1} f\big(s,y(s)\big)\, ds\right]^\alpha, \left[\int_0^t (t-s)^{q-1} f\big(s,y(s)\big)\, ds\right]^\beta\right) \\
&\quad \le d_H\big(\big[y_0(t)\big]^\alpha, \big[y_0(t)\big]^\beta\big) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} d_H\big(\big[f\big(s,y(s)\big)\big]^\alpha, \big[f\big(s,y(s)\big)\big]^\beta\big)\, ds \\
&\quad \le \varepsilon \quad \text{for all } |\alpha - \beta| < \delta.
\end{aligned}$$

Therefore $T(\Omega)(t)$ is level-equicontinuous in $\mathbb{E}_c^n$. Finally, due to the relative compactness of $f([0,a] \times \Omega)$ and $y_0([0,a])$, we have that there exist compact sets $K_1, K_2 \subset \mathbb{R}^n$ such that $[f(s,y(s))]^0 \subseteq K_1$ for all $(s,y) \in [0,a] \times \Omega$ and $[y_0(t)]^0 \subseteq K_2$ for all $t \in [0,a]$. Thus, we have

$$\begin{aligned}
&\left[y_0(t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\big(s,y(s)\big)\, ds\right]^0 \\
&\quad = \big[y_0(t)\big]^0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[f\big(s,y(s)\big)\big]^0\, ds \\
&\quad \subseteq \frac{K_1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\, ds + K_2 \\
&\quad = \frac{t^q K_1}{\Gamma(q+1)} + K_2.
\end{aligned}$$

Since t^q is bounded on $[0,a]$, hence there exists a compact set $K_0 \subseteq \mathbb{R}^n$ such that

$$\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\big(s,y(s)\big)\,ds\right]^0 \subseteq K_0,$$

which proves that $T(\Omega)(t)$ is compact-supported. Thus, T is a compact operator. Hence, by Theorem 3.4, it follows that T has a fixed point in Ω, which is a solution of integral equation (4.1). □

The following Weissinger fixed point theorem will be used to prove an existence and uniqueness result.

Theorem 4.3 [20] *Let (U,d) be a nonempty complete metric space, and let $\gamma_n \geq 0$ for all $n \in \{0,1,2,\dots\}$ be such that $\sum_{n=0}^{\infty}\gamma_n$ converges. Moreover, let the mapping $T : U \to U$ satisfy the inequality*

$$d\big(T^n u, T^n v\big) \leq \gamma_n d(u,v)$$

for all $n \in \mathbb{N}$ and for all $u,v \in U$. Then the operator T has a unique fixed point $u^* \in U$. Furthermore, for any $u_0 \in U$, the sequence $\{T^n u_0\}_{n=1}^{\infty}$ converges to the above fixed point u^*.

Theorem 4.4 *Let $0 < q < 1$. Suppose that $f : G \to \mathbb{E}_c^n$ is continuous and satisfies a Lipschitz condition, that is, there exists $L > 0$ such that*

$$d\big(f(t,u), f(t,v)\big) \leq L d(u,v), \tag{4.4}$$

for all $(t,u),(t,v) \in G$, where $G = \{(t,u) \in [0,a] \times \mathbb{E}_c^n : d(u,\widehat{0}) \leq R\}$. Then there exists a unique solution $y(t) \in C([0,a],\mathbb{E}_c^n)$ to integral equation (4.1).

Proof From Theorem 4.2, we have that the integral equation has a solution. In order to prove uniqueness of this solution, we prove that the operator T has a unique fixed point. For this, we shall first prove that, for all $n \in \{0,1,2,\dots\}$, $\tau \in [0,a]$ and $u,v \in \Omega$, the following inequality holds:

$$\sup_{t\in[0,\tau]} d\big(T^{n+1}y(t), T^{n+1}z(t)\big) \leq \sup_{t\in[0,\tau]} \frac{(Lt^q)^{n+1}}{\Gamma(1+q(n+1))} d\big(y(t),z(t)\big). \tag{4.5}$$

For $n = 0$, this statement is trivially true. Suppose that (4.5) is true for some $n \geq 1$. Then from inequality (4.4) we have

$$\begin{aligned}
&\sup_{t\in[0,\tau]} d\big(T^{n+1}y(t), T^{n+1}z(t)\big)\\
&\quad= \sup_{t\in[0,\tau]} d\big(TT^n y(t), TT^n z(t)\big)\\
&\quad= \sup_{t\in[0,\tau]} \frac{1}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1} f\big(s,T^n y(s)\big)\,ds, \int_0^t (t-s)^{q-1} f\big(s,T^n z(s)\big)\,ds\right)\\
&\quad\leq \sup_{t\in[0,\tau]} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d\big(f\big(s,T^n y(s)\big), f\big(s,T^n z(s)\big)\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{L}{\Gamma(q)} \sup_{t\in[0,\tau]} \int_0^t (t-s)^{q-1} d\bigl(T^n y(s), T^n z(s)\bigr)\, ds \\
&\leq \frac{L}{\Gamma(q)} \int_0^\tau (\tau-s)^{q-1} \sup_{t\in[0,s]} d\bigl(T^n y(t), T^n z(t)\bigr)\, ds \\
&\leq \frac{L^{n+1}}{\Gamma(q)\Gamma(1+qn)} \int_0^\tau (\tau-s)^{q-1} s^{qn} \sup_{t\in[0,s]} d\bigl(y(t), z(t)\bigr)\, ds \\
&\leq \frac{L^{n+1}}{\Gamma(q)\Gamma(1+qn)} \sup_{t\in[0,\tau]} d\bigl(y(t), z(t)\bigr) \int_0^\tau (\tau-s)^{q-1} s^{qn}\, ds \\
&= \frac{L^{n+1}}{\Gamma(q)\Gamma(1+qn)} \sup_{t\in[0,\tau]} d\bigl(y(t), z(t)\bigr) \frac{\Gamma(q)\Gamma(1+qn)}{\Gamma(1+q(n+1))} t^{q(n+1)}.
\end{aligned}
$$

Taking the supremum over $[0,a]$, we get

$$
D\bigl(T^{n+1}y, T^{n+1}z\bigr) \leq \frac{(La^q)^{n+1}}{\Gamma(1+q(n+1))} D(y,z).
$$

The series $\sum_{n=0}^{\infty} \gamma_n$ with $\gamma_n = \frac{(La^q)^n}{\Gamma(1+qn)}$ is a convergent series (see Theorem 4.1 in [21]). Thus by Theorem 4.3 we deduce the uniqueness of the solution of our integral equation. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
The second author acknowledges the financial support of Higher Education Commission (HEC) of Pakistan.

Received: 24 May 2013 Accepted: 10 October 2013 Published: 22 Nov 2013

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10.1186/1687-1812-2013-306
Cite this article as: Agarwal et al.: A Schauder fixed point theorem in semilinear spaces and applications. *Fixed Point Theory and Applications* 2013, 2013:306