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Convergence theorems for a generalized Φ-pseudo-contractive type mapping in real normal linear spaces

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Abstract
In this paper, we first give a new notion of generalized Φ-pseudo-contractive type mapping, and then we consider some convergence theorems for a fixed point of the mapping. Our results improve and extend the corresponding results due to (Chidume and Chidume in J. Math. Anal. Appl. 302:545-554, 2005) and other papers.

Keywords: convergence theorems; generalized Φ-pseudo-contractive type mappings; generalized Φ-accretive type mappings; real normal linear spaces

1 Introduction and statement of results

Let E be a real normed linear space and E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|x\| = \|f\|\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

Definition 1.1 [1, 2] Let $\phi : [0,\infty) \to [0,\infty)$ be a function for which $\phi(0) = 0$, $\forall r_0 > 0$, $\liminf_{r\to r_0} \phi(r) > 0$. A mapping $T : D(T) \subset E \to E$ is called ϕ-strongly accretive if for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \phi(\|x - y\|)\|x - y\|.$$

We also say that $T : D(T) \subset E \to E$ is ϕ-strongly pseudo-contractive if $I - T$ is ϕ-strongly accretive.

Definition 1.2 Let $\Phi : [0,\infty) \to [0,\infty)$ be a function for which $\Phi(0) = 0$, $\forall r_0 > 0$, $\liminf_{r\to r_0} \Phi(r) > 0$. A mapping $T : D(T) \subset E \to E$ is called generalized Φ-accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \Phi(\|x - y\|), \quad \forall x, y \in D(T).$$

We also say that $T : D(T) \subset E \to E$ is generalized Φ-pseudo-contractive if $I - T$ is generalized ϕ-accretive.

Remark 1.3 Definition 1.1 and Definition 1.2 do not assume that $\phi(r)$ ($\Phi(r)$) is strictly increasing. Clearly, ϕ-strongly accretive maps (ϕ-strongly pseudo-contractive maps) are generalized by generalized ϕ-accretive maps (generalized Φ-pseudo-contractive maps) with $\Phi(r) = r\phi(r)$.

Definition 1.4 $T : D(T) \subset E \to E$ is called a generalized Φ-accretive type mapping if there exists $x^* \in D(T)$ such that for all $x \in D(T)$, there exists $j(x - x^*) \in J(x - x^*)$ such that

$$\langle Tx - Tx^*, j(x - x^*)\rangle \geq \Phi(\|x - x^*\|),$$

where Φ is as in Definition 1.2. T is called a generalized Φ-pseudo-contractive type mapping if $I - T$ is a generalized Φ-accretive type mapping.

Recently, Chidume and Chidume proved the following theorems by using the conclusion that a uniformly continuous mapping on K is bounded.

Theorem CC1 [3] *Let E be a real normed linear space, K be a nonempty subset of E and $T : K \to E$ be a uniformly continuous generalized Φ-hemi-contractive mapping, i.e., there exist $x^* \in K$ and a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$ such that for all $x \in K$, there exists $j(x - x^*) \in J(x - x^*)$ such that*

$$\langle Tx - Tx^*, j(x - x^*)\rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|).$$

(a) *If $y^* \in K$ is a fixed point of T, then $y^* = x^*$ and so T has at most one fixed point in K.*
(b) *Suppose that there exists $x_0 \in K$ such that the sequence $\{x_n\}$ defined by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad \forall n \geq 0,$$

is contained in K, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $[0,1]$ satisfying the following conditions:
(i) $a_n + b_n + c_n = 1$;
(ii) $\sum_{n=0}^{\infty}(b_n + c_n) = \infty$;
(iii) $\sum_{n=0}^{\infty}(b_n + c_n)^2 < \infty$;
(iv) $\sum_{n=0}^{\infty} c_n < \infty$; *and $\{u_n\}$ is a bounded sequence in K.*
Then $\{x_n\}$ converges strongly to x^*. In particular, if y^* is a fixed point of T in K, then $\{x_n\}$ converges strongly to y^*.

Theorem CC2 [3] *Let E be a real normed linear space, $A : E \to E$ be a uniformly continuous generalized Φ-quasi-contractive mapping, i.e., there exists $x^* \in D(A)$ such that for all $x \in E$, there exist $j(x - x^*) \in J(x - x^*)$ and a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$ such that*

$$\langle Ax - Ax^*, j(x - x^*)\rangle \geq \Phi(\|x - x^*\|).$$

For arbitrary $x_0 \in D(A)$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = a_n x_n + b_n S x_n + c_n u_n, \quad \forall n \geq 0,$$

where $S : E \to E$ is defined by $Sx := x - Ax$ for all $x \in E$; and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = 1$;
(ii) $\sum_{n=0}^{\infty}(b_n + c_n) = \infty$;
(iii) $\sum_{n=0}^{\infty}(b_n + c_n)^2 < \infty$;
(iv) $\sum_{n=0}^{\infty} c_n < \infty$; *and $\{u_n\}$ is a bounded sequence in K.*

Then $\{x_n\}$ converges strongly to x^*.

Remark 1.5 In Theorem CC1 and Theorem CC2, the condition that K is convex is needed. Since $K \subset E$ is a nonempty subset without assuming that K is convex, then a uniformly continuous mapping T on K is not necessarily bounded. See the following example.

Let $\{e_n\}$ be an orthonormal set of l^2, $K = \{x \in l^2 \mid x = te_n + (1-t)e_{n+1}, t \in [0,1]\}$. Let $T : K \to l^2$ be a mapping defined by

$$Tx = (n+t)e_n + (n+1-t)e_{n+1}, \quad \text{where } x = te_n + (1-t)e_{n+1} \in K.$$

Then T is uniformly continuous on a bounded and nonconvex set K. But T is not bounded.

Proof Clearly K is bounded and nonconvex. Let $x_m, y_m \in K$ such that $\|x_m - y_m\| \to 0$ $(m \to \infty)$. Then this implies that there exist $n_0 \in N$ and $t_m, t'_m \in [0,1]$ such that

$$x_m = t_m e_{n_0} + (1-t_m)e_{n_0+1},$$
$$y_m = t'_m e_{n_0} + \left(1-t'_m\right)e_{n_0+1},$$
$$\left\|t_m - t'_m\right\| \to 0.$$

So,

$$\begin{aligned}\|Tx_m - Ty_m\| &= \left\|(n_0+t_m)e_{n_0} + (n_0+1-t_m)e_{n_0+1} - \left(n_0+t'_m\right)e_{n_0} - \left(n_0+1-t'_m\right)e_{n_0+1}\right\| \\ &= \left|t_m - t'_m\right|\|e_{n_0} + e_{n_0+1}\| \\ &= \sqrt{2}\left|t_m - t'_m\right| \to 0 \quad (m \to \infty).\end{aligned}$$

Hence T is uniformly continuous.

Let $x \in K$, then

$$\begin{aligned}\|Tx\| &= \left\|(n+t)e_n + (n+1-t)e_{n+1}\right\| \\ &= \left((n+t)^2 + (n+1-t)^2\right)^{\frac{1}{2}} \to \infty \quad (n \to \infty).\end{aligned}$$

This says that T is unbounded and completes the proof. □

In 1999, Morales and Chidume proved the following theorem.

Theorem MC [1] *Let E be a uniformly smooth Banach space, and let $A : E \to E$ be a bounded demicontinuous ϕ-strongly accretive mapping for some $x_0 \in E$, $\liminf_{r\to\infty}\phi(r) > \|Ax_0\|$. Let $\{c_n\}$ be a real sequence in $[0,1]$ satisfying the following conditions*: (i) $\sum_{n=0}^{\infty} c_n =$

∞; (ii) $\sum_{n=0}^{\infty} c_n b(c_n) < \infty$. *Let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = x_n - c_n A x_n, \quad \forall n \geq 0.$$

Then there exists a constant $r_0 > 0$ *such that when* $c_n < r_0$ $(\forall n \geq 0)$, *the sequence* $\{x_n\}$ *converges strongly to the unique zero of* A.

Inspired and motivated by these facts, we will give convergence theorems for a fixed point of the generalized Φ-pseudo-contractive type mapping. Our result generalizes the corresponding results in [1–9].

2 Main results

Let $F(T) = \{x \in K : Tx = x\}$, $N(A) = \{x \in D(A) : Ax = 0\}$.

We shall make use of the following well-known inequality.

Lemma 2.1 *Let* E *be a real normed linear space. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E, \forall j(x + y) \in J(x + y).$$

Theorem 2.2 *Let* E *be a real normed linear space,* K *be a nonempty subset of* E *and* $T : K \to E$ *be a uniformly continuous generalized* Φ*-pseudo-contractive type mapping, i.e., there exist* $x^* \in K$ *and a function* $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$ *such that for all* $x \in K$, *there exists* $j(x - x^*) \in J(x - x^*)$ *such that*

$$\langle Tx - Tx^*, j(x - x^*)\rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|). \tag{2.1}$$

(a) *If* $y^* \in K$ *is a fixed point of* T, *then* $y^* = x^*$ *and so* T *has at most one fixed point in* K.

(b) *Let the above* $x^\star \in F(T)$, $x_0 \in K$, $Tx_0 \neq x_0$, $x_0 \neq x^*$. *Suppose that the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad \forall n \geq 0, \tag{2.2}$$

is contained in K, *where* $\{u_n\}$ *is a bounded sequence in* K *and* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are real sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $a_n + b_n + c_n = 1$;
(ii) $\sum_{n=0}^{\infty}(b_n + c_n) = \infty$;
(iii) $b_n + c_n \to 0$ *as* $n \to \infty$;
(iv) $c_n \leq b_n^2$.

If $\liminf_{r\to\infty} \frac{\Phi(r)}{1+r} > \|x_0 - Tx_0\|$ *and* $\{x_n - Tx_n\}$ *is bounded, then there exists a constant* $d_0 > 0$ *such that when* $0 < b_n + c_n \leq d_0$, *the sequence* $\{x_n\}$ *converges strongly to* x^*.

Proof The proof of (a) is the same as the proof of Theorem CC1 [3].

(b) Define $a = \sup\{r \in R^+ : \frac{\Phi(r)}{1+r} \leq \|x_0 - Tx_0\|\}$. Then, by $\Phi(0) = 0$ and $\|x_0 - Tx_0\| > 0$, we have $a > 0$. We show that $a \neq \infty$. If $a = \infty$, then there exists $\{r_n\} \subset [0, \infty)$, $r_n \to \infty$ as $n \to \infty$, $\frac{\Phi(r_n)}{1+r_n} \leq \|x_0 - Tx_0\|$, and hence $\|x_0 - Tx_0\| < \liminf_{r\to\infty} \frac{\Phi(r)}{1+r} \leq \|x_0 - Tx_0\|$, a contradiction. Therefore, $a < \infty$.

Let $N^* = \sup_n \|u_n - x^*\|$ and $M = \sup_n \|x_n - Tx_n\| + N^*$. Since T is uniformly continuous on K, for $\epsilon = \frac{\|x_0 - Tx_0\|}{6a}$, there exists $\delta > 0$ such that $x, y \in K$ implies $\|Tx - Ty\| < \epsilon$.

Let

$$d_0 = \frac{1}{2(a+M)} \min\left\{\delta, a, \frac{\|x_0 - Tx_0\|}{24a}\right\}. \tag{2.3}$$

Claim 1 *$\{x_n\}$ is bounded, i.e.,*

$$\|x_n - x^*\| \le 2a, \quad \forall n \ge 0. \tag{2.4}$$

We show this by induction. By (2.1),

$$\frac{\Phi(\|x_0 - x^*\|)}{1 + \|x_0 - x^*\|} \le \|x_0 - Tx_0\|.$$

Therefore, $\|x_0 - x^*\| \le a < 2a$. Suppose $\|x_n - x^*\| \le 2a$, we show that $\|x_{n+1} - x^*\| \le 2a$. Suppose not, then $\|x_{n+1} - x^*\| > 2a > a$ and from the definition of a, we have

$$\frac{\Phi(\|x_{n+1} - x^*\|)}{1 + \|x_{n+1} - x^*\|} > \|x_0 - Tx_0\|,$$

and hence

$$\Phi(\|x_{n+1} - x^*\|) > \|x_0 - Tx_0\|. \tag{2.5}$$

Set $\alpha_n = b_n + c_n$. Then Eq. (2.2) becomes

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n + c_n U_n, \tag{2.6}$$

where $U_n = u_n - Tx_n$. Observe that

$$\|U_n\| \le \|u_n - x^*\| + \|x_n - x^*\| + \|x_n - Tx_n\| \le 2a + M. \tag{2.7}$$

Furthermore,

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \|x_n - x^*\| + \alpha_n\|x_n - Tx_n\| + c_n\|U_n\| \\ &\le 2a + d_0(2a + 2M) \le 3a. \end{aligned} \tag{2.8}$$

Also,

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \alpha_n\{\|x_n - Tx_n\| + \|U_n\|\} \\ &\le \alpha_n(2a + 2M) < d_0(2a + 2M) \le \delta, \end{aligned} \tag{2.9}$$

so that $\|Tx_{n+1} - Tx_n\| < \epsilon$. Using Lemma 2.1, (2.1), (2.3), (2.5), (2.7)-(2.9) and recursion formula (2.6), we now obtain the following estimates:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|x_n - x^* - \alpha_n(x_n - Tx_n) + c_n U_n\|^2 \\ &\le \|x_n - x^*\|^2 - 2\alpha_n\langle x_n - Tx_n, j(x_{n+1} - x^*)\rangle + 2c_n\|U_n\| \cdot \|x_{n+1} - x^*\| \end{aligned}$$

$$\begin{aligned}
&\le \|x_n - x^*\|^2 - 2\alpha_n\langle x_{n+1} - Tx_{n+1} - x_{n+1} + Tx_{n+1} + x_n - Tx_n, j(x_{n+1} - x^*)\rangle \\
&\quad + 6c_n(2a + M)a \\
&\le \|x_n - x^*\|^2 - 2\alpha_n\Phi(\|x_{n+1} - x^*\|) + 2\alpha_n\|x_{n+1} - x_n\| \cdot \|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\|Tx_{n+1} - Tx_n\| \cdot \|x_{n+1} - x^*\| + 6\alpha_n^2(2a + M)a \\
&\le \|x_n - x^*\|^2 - 2\alpha_n\|x_0 - Tx_0\| + 2\alpha_n^2(2a + 2M) \cdot 3a + 2\alpha_n \cdot \frac{3a\|x_0 - Tx_0\|}{6a} \\
&\quad + 6\alpha_n^2(2a + M)a \\
&\le \|x_n - x^*\|^2 - \frac{\alpha_n}{2}\|x_0 - Tx_0\| < \|x_n - x^*\|^2,
\end{aligned}$$

and hence $\|x_{n+1} - x^*\| < 2a$, a contraction. Hence $\{x_n\}$ is bounded.

Claim 2 $\liminf_{n\to\infty} \|x_n - x^*\| = 0$.

Suppose this is not true. Let $\liminf_{n\to\infty} \|x_n - x^*\| = \sigma > 0$. Then there exists an integer N_0 such that

$$\|x_n - x^*\| \ge \frac{\sigma}{2}, \quad \forall n \ge N_0. \tag{2.10}$$

Since, for any $r_0 > 0$, $\liminf_{r\to r_0} \Phi(r) > 0$, then $\liminf_{n\to\infty} \Phi(\|x_n - x^*\|) \triangleq \beta > 0$. Hence there exists an integer $N_1 > N_0$ such that

$$\Phi(\|x_n - x^*\|) \ge \frac{\beta}{2}, \quad \forall n \ge N_1. \tag{2.11}$$

Since $\{x_n - Tx_n\}$, $\{u_n\}$ and $\{x_n\}$ are bounded,

$$\|x_{n+1} - x_n\| \le \alpha_n\|x_n - Tx_n\| + c_n\|u_n - Tx_n\| \to 0 \quad \text{as } n \to \infty.$$

Therefore, there exists an integer $N_2 > N_1$ such that

$$\|x_{n+1} - x_n\| < \frac{\beta}{16a}, \quad \forall n > N_2. \tag{2.12}$$

Since T is uniformly continuous, then there exists an integer $N_3 > N_2$ such that

$$\|Tx_{n+1} - Tx_n\| < \frac{\beta}{16a}, \quad \forall n > N_3. \tag{2.13}$$

Also, since $\alpha_n \to 0$ as $n \to \infty$, there exists an integer $N_4 > N_3$ such that

$$\alpha_n < \frac{\beta}{16a(2a + M)}, \quad \forall n > N_4. \tag{2.14}$$

By Lemma and (2.11)-(2.14), we obtain the following estimates:

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - 2\alpha_n\langle x_n - Tx_n, j(x_{n+1} - x^*)\rangle + 2c_n\langle U_n, j(x_{n+1} - x^*)\rangle \\
&\le \|x_n - x^*\|^2 - 2\alpha_n\Phi(\|x_{n+1} - x^*\|) + 2\alpha_n\|x_{n+1} - x_n\| \cdot \|x_{n+1} - x^*\|
\end{aligned}$$

$$\begin{aligned}
&\quad + 2\alpha_n \|Tx_{n+1} - Tx_n\| \cdot \|x_{n+1} - x^*\| + 2\alpha_n^2(2a + M)\|x_{n+1} - x^*\| \\
&\le \|x_n - x^*\|^2 - 2\alpha_n \cdot \frac{\beta}{2} + 2\alpha_n \cdot \frac{\beta}{16a} \cdot 2a + 2\alpha_n \cdot \frac{\beta}{16a} \cdot 2a \\
&\quad + 2\alpha_n \cdot \frac{\beta}{16a(2a+M)} \cdot (2a + M) \cdot 2a \\
&= \|x_n - x^*\|^2 - \frac{1}{4}\alpha_n\beta
\end{aligned} \tag{2.15}$$

for all $n \ge N_4$, and this implies $\sum_{n=0}^{\infty} \alpha_n < \infty$, a contraction to condition (ii) of Theorem 2.2. Hence Claim 2 holds.

Thus, there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to x^*$ as $n \to \infty$, *i.e.*, for any $\epsilon > 0$, there exists some integer n_{j_0} such that $\|x_{n_{j_0}} - x^*\| < \epsilon$.

Claim 3 $\|x_{n_{j_0}+m} - x^*\| < \epsilon$, $m = 1, 2, \ldots$.

Let $r_0 = \inf\{\Phi(r) : r \ge \epsilon\}$, then $r_0 > 0$.

Since $\|x_{n+1} - x_n\| \to 0$, $\|Tx_{n+1} - Tx_n\| \to 0$ and $\alpha_n \to 0$ as $n \to \infty$, then there exists an integer $N > 0$ such that for all $n \ge N$, the following inequalities hold:

$$\begin{aligned}
&\|x_{n+1} - x_n\| \le \frac{r_0}{16a}, \\
&\|Tx_{n+1} - Tx_n\| \le \frac{r_0}{16a}, \\
&\alpha_n < \frac{r_0}{4a(2a+M)}.
\end{aligned}$$

If $\|x_{n_{j_0}+1} - x^*\| \ge \epsilon$, then $\Phi(\|x_{n_{j_0}+1} - x^*\|) \ge r_0$. Using recursion formula (2.15), we obtain the following estimate:

$$\begin{aligned}
\|x_{n_{j_0}+1} - x^*\|^2 &\le \|x_{n_{j_0}} - x^*\|^2 - 2\alpha_n r_0 + 2\alpha_n \cdot \frac{r_0}{16a} \cdot 2a + 2\alpha_n \cdot \frac{r_0}{16a} \cdot 2a \\
&\quad + 2\alpha_n \cdot \frac{r_0}{4a(2a+M)} \cdot (2a + M) \cdot 2a \\
&= \|x_{n_{j_0}} - x^*\|^2 - \alpha_n r_0 + \frac{1}{2}\alpha_n r_0 \\
&= \|x_{n_{j_0}} - x^*\|^2 - \frac{1}{2}\alpha_n r_0 < \|x_{n_{j_0}} - x^*\|^2 < \epsilon,
\end{aligned}$$

a contradiction. Hence Claim 3 holds for $m = 1$. Assume now that it holds for $m = k$. From the above argument, one easily proves that it holds for $m = k + 1$. Hence, Claim 3 holds. This shows that $\{x_n\}$ converges strongly to x^* as $n \to \infty$, completing the proof of Theorem 2.2. □

Theorem 2.3 *Let E be a real normed linear space, and let $A : D(A) \subset E \to E$ be a uniformly continuous generalized Φ-accretive type mapping, i.e., there exists $x^* \in N(A)$ such that for all $x \in E$, there exist $j(x - x^*) \in J(x - x^*)$ and a function $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$ such that*

$$\langle Ax - Ax^*, j(x - x^*)\rangle \ge \Phi(\|x - x^*\|).$$

For arbitrary $x_0 \in D(A)$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = a_n x_n + b_n S x_n + c_n u_n, \quad \forall n \geq 0,$$

where $S : E \to E$ is defined by $Sx := x - Ax$ for all $x \in D(A)$; and $\{u_n\}$ is a bounded sequence in E, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = 1$;
(ii) $\sum_{n=0}^{\infty}(b_n + c_n) = \infty$;
(iii) $b_n + c_n \to 0$ *as* $n \to \infty$;
(iv) $c_n \leq b_n^2$.

If $\liminf_{r\to\infty} \frac{\Phi(r)}{1+r} > \|Ax_0\|$ *and* $\{Ax_n\}$ *is bounded, then there exists a constant* $d_0 > 0$ *such that when* $0 < b_n + c_n \leq d_0$, *the sequence* $\{x_n\}$ *converges strongly to* x^*.

Proof We simply observe that S is a uniformly continuous and generalized Φ-pseudo-contractive type mapping of $D(A)$ into E. The result can follow from Theorem 2.2. □

Remark 2.4 (1) Our theorems extend and improve Theorem CC1 and Theorem CC2 in the following ways:

(i) Our theorems do not assume that $\Phi(t)$ is a strictly increasing function.
(ii) The conditions $\sum_{n=0}^{\infty}(b_n + c_n)^2 < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ are replaced by $b_n + c_n \to 0$ as $n \to \infty$, $c_n \leq b_n^2$, respectively. Our theorems enlarge the range of b_n and c_n values.
(iii) We do not need the condition that K is convex. We added the condition that $\{x_n - Tx_n\}$ is bounded. It is readily seen that $\{x_n\}$ converges strongly to x^* if and only if $\{x_n - Tx_n\}(\{Ax_n\})$ is bounded under the assumptions of Theorem 2.2 (Theorem 2.3).

(2) Since the class of generalized Φ-accretive maps (generalized Φ-pseudo-contractive maps) includes the class of ϕ-strongly accretive maps (ϕ-strongly pseudo-contractive maps), our results unify and extend many known results. In particular, since $\liminf_{r\to\infty} \phi(r) > \|Ax_0\|$ in Theorem MC implies $\liminf_{r\to\infty} \frac{\Phi(r)}{1+r} = \liminf_{r\to\infty} \frac{\phi(r)r}{1+r} = \liminf_{r\to\infty} \phi(r) > \|Ax_0\|$, our Theorem 2.3 extends Theorem MC from uniformly smooth Banach spaces to arbitrary normed linear spaces.

(3) Our results also improve and extend the corresponding results in [2, 4–9].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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