Ceng et al. *Fixed Point Theory and Applications* 2013, 2013:313
http://www.fixedpointtheoryandapplications.com/content/2013/1/313

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Hybrid viscosity CQ method for finding a common solution of a variational inequality, a general system of variational inequalities, and a fixed point problem

Lu-Chuan Ceng[1], Sy-Ming Guu[2,3*] and Jen-Chih Yao[4,5]

*Correspondence:
iesmguu@mail.cgu.edu.tw
[2]Graduate Institute of Business and Management, College of Management, Chang Gung University, Kwei-Shan, Taoyuan, Taiwan R.O.C
[3]Medical Research Division, Chang Gung Memorial Hospital, Kwei-Shan, Taoyuan, Taiwan R.O.C
Full list of author information is available at the end of the article

Abstract

In the literature, various iterative methods have been proposed for finding a common solution of the classical variational inequality problem and a fixed point problem. Research along these lines is performed either by relaxing the assumptions on the mappings in the settings (for instance, commonly seen assumptions for the mapping involved in the fixed point problem are nonexpansive or strictly pseudocontractive) or by adding a general system of variational inequalities into the settings. In this paper, we consider both possible ways in our settings. Specifically, we propose an iterative method for finding a common solution of the classical variational inequality problem, a general system of variational inequalities and a fixed point problem of a uniformly continuous asymptotically strictly pseudocontractive mapping in the intermediate sense. Our iterative method is hybridized by utilizing the well-known extragradient method, the CQ method, the Mann-type iterative method and the viscosity approximation method. The iterates yielded by our method converge strongly to a common solution of these three problems. In addition, we propose a hybridized extragradient-like method to yield iterates converging weakly to a common solution of these three problems.

MSC: 49J30; 47H09; 47J20

Keywords: extragradient method; CQ method; Mann-type iterative method; viscosity approximation method; variational inequalities; asymptotically strictly pseudocontractive mapping in the intermediate sense; inverse-strong monotonicity

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, let C be a nonempty closed convex subset of H, and let P_C be the metric projection of H onto C. Let $S: C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A: C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$, then A is called a nonexpansive mapping [1]; if $L \in [0,1)$, then A is called a contraction. Also, a mapping $A: C \to H$ is called monotone if $\langle Ax - Ay, x - y\rangle \geq 0$

for all $x, y \in C$. A is called η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C.$$

A is called α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

For a given nonlinear operator $A : C \to H$, we consider the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, A)$. VIP (1.1) was first discussed by Lions [2] and now has many applications in computational mathematics, mathematical physics, operations research, mathematical economics, optimization theory, and other fields; see, *e.g.*, [3–6]. It is well known that if A is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (1.1) has a unique solution.

In the literature, there is a growing interest in studying how to find a common solution of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$. Under various assumptions imposed on A and S, iterative algorithms were derived to yield iterates which converge *strongly or weakly* to a common solution of these two problems.

1.1 Finding a common element and weak convergence

Consider that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse-strongly monotone. Takahashi and Toyoda [7] introduced the Mann-type iterative scheme:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to some $z \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$.

Motivated by Korpelevich's extragradient method [8], Nadezhkina and Takahashi [9] proposed an extragradient iterative method and showed the iterates converge weakly to a common element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \quad \forall n \geq 0, \end{cases}$$

where $A : C \to H$ is a monotone, L-Lipschitz continuous mapping and $S : C \to C$ is a nonexpansive mapping and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_n\} \subset [c, d]$ for some

$c, d \in (0,1)$. See also Zeng and Yao [10], in which a hybridized iterative method was proposed to yield a new weak convergence result.

1.2 Finding a common element and strong convergence

Let $C \subset H$ be a nonempty closed convex subset, let $S : C \to C$ be a nonexpansive mapping, and let $A : C \to H$ be an α-inverse strongly monotone mapping. Iiduka and Takahashi [11] introduced the following hybrid method:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \geq 0, \end{cases}$$

where $0 \leq \alpha_n \leq c < 1$ and $0 < a \leq \lambda_n \leq b < 2\alpha$. They showed that if $\mathrm{Fix}(S) \cap \mathrm{VI}(C,A) \neq \emptyset$, then the sequence $\{x_n\}$, generated by this iterative process, converges strongly to $P_{\mathrm{Fix}(S)\cap \mathrm{VI}(C,A)} x$. Recently, the method proposed by Nadezhkina and Takahashi [12] also demonstrated the strong convergence result. However, note that they assumed that A is monotone and L-Lipschitz-continuous while S is nonexpansive. For another strong convergence result, see Ceng and Yao [13] whose method is based on the extragradient method and the viscosity approximation method.

As we have seen, most of the papers were based on the different assumptions imposed on A while the mapping S is nonexpansive. In the following, we shall relax the nonexpansive requirement on S (for instance, κ-strictly pseudocontractive, asymptotically κ-strictly pseudocontractive mapping in the intermediate sense, *etc.*). Furthermore, we also consider adding a general system of variational inequalities to our settings.

1.3 Relaxation on nonexpansive S

Definition 1.1 Let C be a nonempty subset of a normed space X, and let $S : C \to C$ be a self-mapping on C.

(i) S is asymptotically nonexpansive (*cf.* [14]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\left\|S^n x - S^n y\right\| \leq k_n \|x - y\|, \quad \forall n \geq 1, \forall x, y \in C;$$

(ii) S is asymptotically nonexpansive in the intermediate sense [15] provided S is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y \in C} \left(\left\|S^n x - S^n y\right\| - \|x - y\|\right) \leq 0;$$

(iii) S is uniformly Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that

$$\left\|S^n x - S^n y\right\| \leq \mathcal{L} \|x - y\|, \quad \forall n \geq 1, \forall x, y \in C.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] as an important generalization of the class of nonexpansive mappings. The existence of fixed points of asymptotically nonexpansive mappings was proved by Goebel and Kirk [14] as follows.

Theorem GK (see [14, Theorem 1]) *If C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping $S : C \to C$ has a fixed point in C.*

Let C be a nonempty closed convex bounded subset of a Hilbert space H. An iterative method for the approximation of fixed points of an asymptotically nonexpansive mapping with sequence $\{k_n\}$ was developed by Schu [16] via the following Mann-type iterative scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n x_n, \quad \forall n \geq 1, \tag{1.3}$$

where $\delta \leq \alpha_n \leq 1 - \delta$ ($\forall n \geq 1$) for some $\delta > 0$. He proved the weak convergence of $\{x_n\}$ to a fixed point of S if $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Moreover, iterative methods for approximation of fixed points of asymptotically nonexpansive mappings have been further studied by other authors (see, *e.g.*, [16–18] and references therein).

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck *et al.* [15] and iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings were studied by Bruck *et al.* [15], Agarwal *et al.* [19], Chidume *et al.* [20], Kim and Kim [21] and many others.

Recently, Kim and Xu [22] introduced the concept of asymptotically κ-strictly pseudocontractive mappings in a Hilbert space as follows.

Definition 1.2 Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strictly pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exists a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|S^n x - S^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + \kappa \|x - S^n x - (y - S^n y)\|^2, \quad \forall n \geq 1, \forall x, y \in C. \tag{1.4}$$

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically κ-strictly pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup\{\frac{\kappa + \sqrt{1+(1-\kappa)\gamma_n}}{1+\kappa} : n \geq 1\}$.

Very recently, Sahu *et al.* [23] considered the concept of asymptotically κ-strictly pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 1.3 Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exists a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|S^n x - S^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - \kappa \|x - S^n x - (y - S^n y)\|^2\right) \leq 0. \tag{1.5}$$

Put $c_n := \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa\|x - S^n x - (y - S^n y)\|^2)\}$. Then $c_n \geq 0$ ($\forall n \geq 1$), $c_n \to 0$ ($n \to \infty$) and (1.5) reduces to the relation

$$\|S^n x - S^n y\|^2 \leq (1+\gamma_n)\|x-y\|^2 + \kappa\|x - S^n x - (y - S^n y)\|^2 + c_n, \quad \forall n \geq 1, \forall x, y \in C. \tag{1.6}$$

Whenever $c_n = 0$ for all $n \geq 1$ in (1.6), then S is an asymptotically κ-strictly pseudocontractive mapping with sequence $\{\gamma_n\}$.

For S to be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\mathrm{Fix}(S)$ is nonempty and bounded, Sahu *et al.* [23] proposed an iterative Mann-type CQ method in which the iterates converge strongly to a fixed point of S.

Theorem SXY (see [23, Theorem 4.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that* $\mathrm{Fix}(S)$ *is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in* $[0,1]$ *such that $0 < \delta \leq \alpha_n \leq 1-\kappa$ for all $n \geq 1$. Let $\{x_n\}$ be a sequence in C generated by the following* (*CQ*) *algorithm*:

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n S^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in \mathrm{Fix}(S)\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(S)}x$.

1.4 Common solution of three problems

Let $B_1, B_2 : C \to H$ be two mappings. Recently, Ceng *et al.* [24] introduced and considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.8}$$

which is called a general system of variational inequalities (GSVI), where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. The set of solutions of GSVI (1.8) is denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, if $B_1 = B_2$, then GSVI (1.8) reduces to the new system of variational inequalities (NSVI), introduced and studied by Verma [25]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to VIP (1.1). Moreover, they transformed GSVI (1.8) into a fixed point problem in the following way.

Lemma CWY (see [24]) *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of GSVI* (1.8) *if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\big[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)\big], \quad \forall x \in C,$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

In particular, if the mapping $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$.

Utilizing Lemma CWY, they introduced and studied a relaxed extragradient method for solving GSVI (1.8). Throughout this paper, the set of fixed points of the mapping G is denoted by Ξ. Based on the relaxed extragradient method and the viscosity approximation method, Yao *et al.* [26] proposed and analyzed an iterative algorithm for finding a common solution of GSVI (1.8), and the fixed point problem of a κ-strictly pseudocontractive mapping $S : C \to C$ (namely, there exists a constant $\kappa \in [0, 1)$ such that $\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2$ for all $x, y \in C$).

The main theme of this paper is to study the problem of finding a common element of the solution set of VIP (1.1), the solution set of GSVI (1.8) and the fixed point set of a self-mapping $S : C \to C$. Ceng *et al.* [27] analyzed this problem by assuming the mapping S to be strictly pseudocontractive as follows.

Theorem CGY (see [27, Theorem 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone and $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a κ-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \neq \emptyset$. *Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \ge 0, \end{cases} \tag{1.9}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\lambda_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n$ *for all* $n \ge 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;
(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$;
(v) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\liminf_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ generated by (1.9) *converges strongly to $\bar{x} = P_{\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)} Q\bar{x}$ and $(\bar{x}, \bar{y})$ is a solution of GSVI* (1.8), *where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.*

In this paper, we study the problem of finding a common element of the solution set of VIP (1.1), the solution set of GSVI (1.8) and the fixed point set of a self-mapping $S : C \to C$, where the mapping S is assumed to be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$ is nonempty and bounded. Not surprisingly, our main points of proof come from the ideas in [23, Theorem 4.1] and [27, Theorem 3.1]. Our major contribution ensures a strong convergence result to the extent of involving uniformly continuous asymptotically κ-strictly pseudocontractive mappings in the intermediate sense. Moreover, in Section 4 we extend Ceng, Hadjisavvas and Wong's hybrid extragradient-like approximation method given in [28, Theorem 5] to establish a new weak convergence theorem for finding a common solution of VIP (1.1), GSVI (1.8) and the fixed point problem of S.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1 *For given $x \in H$ and $z \in C$:*
(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \le 0$, $\forall y \in C$;
(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;
(iii) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2$, $\forall y \in H$.
Consequently, P_C is nonexpansive and monotone.

We need some facts and tools which are listed as lemmas below.

Lemma 2.1 (see [29, demiclosedness principle]) *Let C be a nonempty closed and convex subset of a Hilbert space H, and let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed on C. That is, whenever $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I - S)x_n \to y$, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.2 ([19, Proposition 2.4]) *Let $\{x_n\}$ be a bounded sequence on a reflexive Banach space X. If $\omega_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.*

Lemma 2.3 *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem, the characterization of the projection* (*see Proposition* 2.1(i)) *implies*

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \forall \lambda > 0.$$

Lemma 2.4 *Let H be a real Hilbert space. Then the following hold*:
(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;
(b) $\|(1 - t)x + ty\|^2 = (1 - t)\|x\|^2 + t\|y\|^2 - t(1 - t)\|x - y\|^2$ *for all* $t \in [0,1]$ *and for all* $x, y \in H$;
(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

Lemma 2.5 ([23, Lemma 2.5]) *Let H be a real Hilbert space. Given a nonempty closed convex subset of H and points $x, y, z \in H$, and given also a real number $a \in \mathbf{R}$, the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (*and closed*).

Lemma 2.6 ([23, Lemma 2.6]) *Let C be a nonempty subset of a Hilbert space H, and let $S : C \to C$ be an asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|S^n x - S^n y\| \leq \frac{1}{1-\kappa}\left(\kappa\|x - y\| + \sqrt{(1 + (1-\kappa)\gamma_n)\|x - y\|^2 + (1-\kappa)c_n}\right)$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 2.7 ([23, Lemma 2.7]) *Let C be a nonempty subset of a Hilbert space H, and let $S : C \to C$ be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - S^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.8 (Demiclosedness principle [23, Proposition 3.1]) *Let C be a nonempty closed convex subset of a Hilbert space H, and let $S : C \to C$ be a continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I - S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - S^m x_n\| = 0$, then $(I - S)x = 0$.*

Lemma 2.9 ([23, Proposition 3.2]) *Let C be a nonempty closed convex subset of a Hilbert space H, and let $S : C \to C$ be a continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that* $\mathrm{Fix}(S) \neq \emptyset$. *Then* $\mathrm{Fix}(S)$ *is closed and convex.*

Remark 2.1 Lemmas 2.8 and 2.9 give some basic properties of an asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Moreover, Lemma 2.8 extends the demiclosedness principles studied for certain classes of nonlinear mappings in Kim and Xu [22], Gornicki [30], Marino and Xu [31] and Xu [32].

To prove a weak convergence theorem by the hybrid extragradient-like method for finding a common solution of VIP (1.1), GSVI (1.8) and the fixed point problem of an asymptotically κ-strictly pseudocontractive mapping in the intermediate sense, we need the following lemma by Osilike *et al.* [33].

Lemma 2.10 ([33, p.80]) *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Corollary 2.1 ([34, p.303]) *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

We need a technique lemma in the sequel, whose proof is an immediate consequence of Opial's property [35] of a Hilbert space and is hence omitted.

Lemma 2.11 *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H satisfying the properties*:

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each* $x \in K$;

(ii) $\omega_w(x_n) \subset K$.

Then $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to a point in K.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $\mathrm{Gph}(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y,g) \in \mathrm{Gph}(T)$ implies $f \in Tx$. Let $A : C \to H$ be a monotone and Lipschitzian mapping, and let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

It is known that in this case T is maximal monotone, and $0 \in Tv$ if and only if $v \in \Omega$; see [36].

3 Strong convergence theorem

In this section, we prove a strong convergence theorem for a hybrid viscosity CQ iterative algorithm for finding a common solution of VIP (1.1), GSVI (1.8) and the fixed point problem of a uniformly continuous asymptotically κ-strictly pseudocontractive mapping $S : C \to C$ in the intermediate sense. This iterative algorithm is based on the extragradient method, the CQ method, the Mann-type iterative method and the viscosity approximation method.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone, and let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, 1)$, and let $S : C \to C$ be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$ is nonempty and bounded. Let $\{\gamma_n\}$ and*

$\{c_n\}$ be defined as in (1.6). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = \alpha_n f(x_n) + (1-\alpha_n) P_C[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\ z_n = (1-\mu_n-\nu_n)x_n + \mu_n t_n + \nu_n S^n t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\theta_n = c_n + (\alpha_n + \gamma_n)\Delta_n$,

$$\Delta_n = \sup\left\{ \|x_n - z\|^2 + \frac{1+\gamma_n}{1-\rho} \big\|(I-f)z\big\|^2 : z \in \operatorname{Fix}(S) \cap \Xi \cap \operatorname{VI}(C, A)\right\} < \infty,$$

$\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and $\{\alpha_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ are three sequences in $[0,1]$ such that $\mu_n + \nu_n \le 1$ for all $n \ge 1$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) *$\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$*;

(iii) *$\kappa \le \mu_n$ for all $n \ge 1$*;

(iv) *$0 < \sigma \le \nu_n$ for all $n \ge 1$.*

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{\operatorname{Fix}(S)\cap\Xi\cap\operatorname{VI}(C,A)}x$.

Proof It is obvious that C_n is closed and Q_n is closed and convex for every $n = 1, 2, \ldots$. As the defining inequality in C_n is equivalent to the inequality

$$\big\langle 2(x_n - z_n), z\big\rangle \le \|x_n\|^2 - \|z_n\|^2 + \theta_n,$$

by Lemma 2.5 we also have that C_n is convex for every $n = 1, 2, \ldots$. As $Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}$, we have $\langle x_n - z, x - x_n\rangle \ge 0$ for all $z \in Q_n$ and, by Proposition 2.1(i), we get $x_n = P_{Q_n} x$.

Next, we divide the rest of the proof into several steps.

Step 1. $\operatorname{Fix}(S) \cap \Xi \cap \operatorname{VI}(C, A) \subset C_n \cap Q_n$ for all $n \ge 1$.

Indeed, take $x^* \in \operatorname{Fix}(S) \cap \Xi \cap \operatorname{VI}(C, A)$ arbitrarily. Then $Sx^* = x^*$, $x^* = P_C(x^* - \lambda_n A x^*)$ and

$$x^* = P_C\big[P_C\big(x^* - \mu_2 B_2 x^*\big) - \mu_1 B_1 P_C\big(x^* - \mu_2 B_2 x^*\big)\big]. \tag{3.2}$$

Since $A : C \to H$ is α-inverse strongly monotone and $0 < \lambda_n \le 2\alpha$, we have, for all $n \ge 1$,

$$\begin{aligned} \big\|y_n - x^*\big\|^2 &= \big\|P_C(x_n - \lambda_n A x_n) - P_C\big(x^* - \lambda_n A x^*\big)\big\|^2 \\ &\le \big\|(x_n - \lambda_n A x_n) - \big(x^* - \lambda_n A x^*\big)\big\|^2 \\ &= \big\|\big(x_n - x^*\big) - \lambda_n\big(A x_n - A x^*\big)\big\|^2 \\ &\le \big\|x_n - x^*\big\|^2 - \lambda_n(2\alpha - \lambda_n)\big\|A x_n - A x^*\big\|^2 \\ &\le \big\|x_n - x^*\big\|^2. \end{aligned} \tag{3.3}$$

For simplicity, we write $y^* = P_C(x^* - \mu_2 B_2 x^*)$, $u_n = P_C(y_n - \mu_2 B_2 y_n)$ and

$$v_n := P_C(u_n - \mu_1 B_1 u_n) = P_C\bigl[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\bigr]$$

for all $n \geq 1$. Since $B_i : C \to H$ is β_i-inverse strongly monotone and $0 < \mu_i < 2\beta_i$ for $i = 1, 2$, we know that for all $n \geq 1$,

$$\begin{aligned}
&\|v_n - x^*\| \\
&\quad = \bigl\|P_C\bigl[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\bigr] - x^*\bigr\|^2 \\
&\quad = \bigl\|P_C\bigl[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\bigr] \\
&\qquad - P_C\bigl[P_C\bigl(x^* - \mu_2 B_2 x^*\bigr) - \mu_1 B_1 P_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr]\bigr\|^2 \\
&\quad \leq \bigl\|\bigl[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\bigr] \\
&\qquad - \bigl[P_C\bigl(x^* - \mu_2 B_2 x^*\bigr) - \mu_1 B_1 P_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr]\bigr\|^2 \\
&\quad = \bigl\|\bigl[P_C(y_n - \mu_2 B_2 y_n) - P_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr] \\
&\qquad - \mu_1\bigl[B_1 P_C(y_n - \mu_2 B_2 y_n) - B_1 P_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr]\bigr\|^2 \\
&\quad \leq \bigl\|P_C(y_n - \mu_2 B_2 y_n) - P_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr\|^2 \\
&\qquad - \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 P_C(y_n - \mu_2 B_2 y_n) - B_1 P_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr\|^2 \\
&\quad \leq \bigl\|(y_n - \mu_2 B_2 y_n) - \bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr\|^2 - \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2 \\
&\quad = \bigl\|\bigl(y_n - x^*\bigr) - \mu_2\bigl(B_2 y_n - B_2 x^*\bigr)\bigr\|^2 - \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2 \\
&\quad \leq \bigl\|y_n - x^*\bigr\|^2 - \mu_2(2\beta_2 - \mu_2)\bigl\|B_2 y_n - B_2 x^*\bigr\|^2 - \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2 \\
&\quad \leq \bigl\|x_n - x^*\bigr\|^2 - \lambda_n(2\alpha - \lambda_n)\bigl\|A x_n - A x^*\bigr\|^2 \\
&\qquad - \mu_2(2\beta_2 - \mu_2)\bigl\|B_2 y_n - B_2 x^*\bigr\|^2 - \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2 \\
&\quad \leq \bigl\|x_n - x^*\bigr\|^2.
\end{aligned} \tag{3.4}$$

Hence we get

$$\begin{aligned}
&\bigl\|t_n - x^*\bigr\|^2 \\
&\quad = \bigl\|\alpha_n\bigl(f(x_n) - x^*\bigr) + (1 - \alpha_n)\bigl(P_C\bigl[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\bigr] - x^*\bigr)\bigr\|^2 \\
&\quad \leq \bigl[\alpha_n\bigl\|f(x_n) - x^*\bigr\| + (1 - \alpha_n)\bigl\|P_C\bigl[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\bigr] - x^*\bigr\|\bigr]^2 \\
&\quad \leq \bigl[\alpha_n\bigl(\bigl\|f(x_n) - f\bigl(x^*\bigr)\bigr\| + \bigl\|f\bigl(x^*\bigr) - x^*\bigr\|\bigr) + (1 - \alpha_n)\bigl\|x_n - x^*\bigr\|\bigr]^2 \\
&\quad \leq \bigl[\alpha_n\bigl(\rho\bigl\|x_n - x^*\bigr\| + \bigl\|f\bigl(x^*\bigr) - x^*\bigr\|\bigr) + (1 - \alpha_n)\bigl\|x_n - x^*\bigr\|\bigr]^2 \\
&\quad = \left[\bigl(1 - (1 - \rho)\alpha_n\bigr)\bigl\|x_n - x^*\bigr\| + (1 - \rho)\alpha_n \frac{\|f(x^*) - x^*\|}{1 - \rho}\right]^2 \\
&\quad \leq \bigl(1 - (1 - \rho)\alpha_n\bigr)\bigl\|x_n - x^*\bigr\|^2 + \alpha_n \frac{\|f(x^*) - x^*\|^2}{1 - \rho} \\
&\quad \leq \bigl\|x_n - x^*\bigr\|^2 + \alpha_n \frac{\|f(x^*) - x^*\|^2}{1 - \rho}.
\end{aligned} \tag{3.5}$$

Therefore, from (3.5), $z_n = (1 - \mu_n - \nu_n)x_n + \mu_n t_n + \nu_n S^n t_n$, and $x^* = Sx^*$, we have

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
&= \|(1 - \mu_n - \nu_n)x_n + \mu_n t_n + \nu_n S^n t_n - x^*\|^2 \\
&= \|(1 - \mu_n - \nu_n)(x_n - x^*) + \mu_n(t_n - x^*) + \nu_n(S^n t_n - x^*)\|^2 \\
&\leq (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\left\| \frac{\mu_n}{\mu_n + \nu_n}(t_n - x^*) + \frac{\nu_n}{\mu_n + \nu_n}(S^n t_n - x^*) \right\|^2 \\
&= (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\left\{ \frac{\mu_n}{\mu_n + \nu_n}\|t_n - x^*\|^2 \right. \\
&\quad \left. + \frac{\nu_n}{\mu_n + \nu_n}\|S^n t_n - x^*\|^2 - \frac{\mu_n \nu_n}{(\mu_n + \nu_n)^2}\|t_n - S^n t_n\|^2 \right\} \\
&\leq (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\left\{ \frac{\mu_n}{\mu_n + \nu_n}\|t_n - x^*\|^2 \right. \\
&\quad + \frac{\nu_n}{\mu_n + \nu_n}\left[(1 + \gamma_n)\|t_n - x^*\|^2 + \kappa\|t_n - S^n t_n\|^2 + c_n\right] \\
&\quad \left. - \frac{\mu_n \nu_n}{(\mu_n + \nu_n)^2}\|t_n - S^n t_n\|^2 \right\} \\
&\leq (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\left\{ (1 + \gamma_n)\|t_n - x^*\|^2 \right. \\
&\quad \left. + \frac{\nu_n}{\mu_n + \nu_n}\left(\kappa - \frac{\mu_n}{\mu_n + \nu_n}\right)\|t_n - S^n t_n\|^2 + \frac{\nu_n c_n}{\mu_n + \nu_n} \right\} \\
&\leq (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\left\{ (1 + \gamma_n)\|t_n - x^*\|^2 + \frac{\nu_n c_n}{\mu_n + \nu_n} \right\} \\
&\leq (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\left\{ (1 + \gamma_n)\left[\|x_n - x^*\|^2 \right.\right. \\
&\quad \left.\left. + \alpha_n \frac{\|f(x^*) - x^*\|^2}{1 - \rho} \right] + \frac{\nu_n c_n}{\mu_n + \nu_n} \right\} \\
&\leq \|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 + \alpha_n(1 + \gamma_n)\frac{\|f(x^*) - x^*\|^2}{1 - \rho} + c_n \\
&\leq \|x_n - x^*\|^2 + (\gamma_n + \alpha_n)\left(\|x_n - x^*\|^2 + \frac{1 + \gamma_n}{1 - \rho}\|(I - f)x^*\|^2 \right) + c_n \\
&\leq \|x_n - x^*\|^2 + c_n + (\alpha_n + \gamma_n)\Delta_n
\end{aligned} \tag{3.6}$$

for every $n = 1, 2, \ldots$, and hence $x^* \in C_n$. So, $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \subset C_n$ for every $n = 1, 2, \ldots$. Now, let us show by mathematical induction that $\{x_n\}$ is well defined and $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \subset C_n \cap Q_n$ for every $n = 1, 2, \ldots$. For $n = 1$, we have $Q_1 = C$. Hence we obtain $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \subset C_1 \cap Q_1$. Suppose that x_k is given and $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \subset C_k \cap Q_k$ for some integer $k \geq 1$. Since $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$ is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of C. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for every $z \in C_k \cap Q_k$. Since $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for every $z \in \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$, and hence $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \subset Q_{k+1}$. Therefore, we obtain $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A) \subset C_{k+1} \cap Q_{k+1}$.

Step 2. $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - x_{n+1}\| = \lim_{n\to\infty}\|x_n - z_n\| = 0$.

Indeed, let $q = P_{\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)}x$. From $x_{n+1} = P_{C_n\cap Q_n}x$ and $q \in \mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|q - x\| \tag{3.7}$$

for every $n = 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.3)-(3.6) we also obtain that $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}x$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\|$$

for every $n = 1, 2, \ldots$. Therefore, there exists $\lim_{n\to\infty}\|x_n - x\|$. Since $x_n = P_{Q_n}x$ and $x_{n+1} \in Q_n$, using Proposition 2.1(ii), we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n,$$

which implies that

$$\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n}.$$

Hence we get

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_{n+1} - x_n\| + \sqrt{\theta_n}$$

for every $n = 1, 2, \ldots$. From $\|x_{n+1} - x_n\| \to 0$ and $\theta_n \to 0$, we have $\|x_n - z_n\| \to 0$.

Step 3. $\lim_{n\to\infty}\|Ax_n - Ax^*\| = \lim_{n\to\infty}\|B_2y_n - B_2x^*\| = \lim_{n\to\infty}\|B_1u_n - B_1y^*\| = 0$.

Indeed, from (3.1), (3.4) and (3.6) it follows that

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
&\quad\le (1-\mu_n-\nu_n)\|x_n - x^*\|^2 + (\mu_n+\nu_n)\left\{(1+\gamma_n)\|t_n - x^*\|^2 + \frac{\nu_n c_n}{\mu_n+\nu_n}\right\} \\
&\quad\le (1-\mu_n-\nu_n)\|x_n - x^*\|^2 + (\mu_n+\nu_n)\left\{(1+\gamma_n)\big[\alpha_n\|f(x_n) - x^*\|^2\right. \\
&\qquad\left. + (1-\alpha_n)\|P_C[P_C(y_n-\mu_2B_2y_n) - \mu_1B_1P_C(y_n-\mu_2B_2y_n)] - x^*\|^2\big] + \frac{\nu_n c_n}{\mu_n+\nu_n}\right\} \\
&\quad\le (1-\mu_n-\nu_n)\|x_n - x^*\|^2 + (\mu_n+\nu_n)\left\{(1+\gamma_n)\big[\alpha_n\|f(x_n) - x^*\|^2\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \bigl\|P_C\bigl[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)\bigr] - x^*\bigr\|^2\bigr] + \frac{\nu_n c_n}{\mu_n + \nu_n}\biggr\} \\
&\le (1 - \mu_n - \nu_n)\bigl\|x_n - x^*\bigr\|^2 + (\mu_n + \nu_n)\biggl\{(1 + \gamma_n)\bigl[\alpha_n\bigl\|f(x_n) - x^*\bigr\|^2 \\
&\quad + \bigl\|x_n - x^*\bigr\|^2 - \lambda_n(2\alpha - \lambda_n)\bigl\|Ax_n - Ax^*\bigr\|^2 \\
&\quad - \mu_2(2\beta_2 - \mu_2)\bigl\|B_2 y_n - B_2 x^*\bigr\|^2 - \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2\bigr] + \frac{\nu_n c_n}{\mu_n + \nu_n}\biggr\} \\
&\le \bigl\|x_n - x^*\bigr\|^2 + \gamma_n\bigl\|x_n - x^*\bigr\|^2 + c_n + \alpha_n(1 + \gamma_n)\bigl\|f(x_n) - x^*\bigr\|^2 \\
&\quad - (\mu_n + \nu_n)(1 + \gamma_n)\bigl[\lambda_n(2\alpha - \lambda_n)\bigl\|Ax_n - Ax^*\bigr\|^2 \\
&\quad + \mu_2(2\beta_2 - \mu_2)\bigl\|B_2 y_n - B_2 x^*\bigr\|^2 + \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2\bigr],
\end{aligned} \tag{3.8}
$$

which hence implies that

$$
\begin{aligned}
&(\kappa + \sigma)(1 + \gamma_n)\bigl[a(2\alpha - b)\bigl\|Ax_n - Ax^*\bigr\|^2 \\
&\qquad + \mu_2(2\beta_2 - \mu_2)\bigl\|B_2 y_n - B_2 x^*\bigr\|^2 + \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2\bigr] \\
&\quad \le (\mu_n + \nu_n)(1 + \gamma_n)\bigl[\lambda_n(2\alpha - \lambda_n)\bigl\|Ax_n - Ax^*\bigr\|^2 \\
&\qquad + \mu_2(2\beta_2 - \mu_2)\bigl\|B_2 y_n - B_2 x^*\bigr\|^2 + \mu_1(2\beta_1 - \mu_1)\bigl\|B_1 u_n - B_1 y^*\bigr\|^2\bigr] \\
&\quad \le \bigl\|x_n - x^*\bigr\|^2 - \bigl\|z_n - x^*\bigr\|^2 + \gamma_n\bigl\|x_n - x^*\bigr\|^2 + c_n + \alpha_n(1 + \gamma_n)\bigl\|f(x_n) - x^*\bigr\|^2 \\
&\quad \le \bigl(\bigl\|x_n - x^*\bigr\| + \bigl\|z_n - x^*\bigr\|\bigr)\|x_n - z_n\| + \gamma_n\bigl\|x_n - x^*\bigr\|^2 + c_n + \alpha_n(1 + \gamma_n)\bigl\|f(x_n) - x^*\bigr\|^2.
\end{aligned}
$$

Since $\alpha_n \to 0$, $\gamma_n \to 0$, $c_n \to 0$ and $\|x_n - z_n\| \to 0$, from the boundedness of $\{x_n\}$ and $\{z_n\}$ we obtain that

$$
\lim_{n\to\infty}\bigl\|Ax_n - Ax^*\bigr\| = \lim_{n\to\infty}\bigl\|B_2 y_n - B_2 x^*\bigr\| = \lim_{n\to\infty}\bigl\|B_1 u_n - B_1 y^*\bigr\| = 0. \tag{3.9}
$$

Step 4. $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Indeed, utilizing Proposition 2.1(iii), we deduce from (3.1) that

$$
\begin{aligned}
\bigl\|y_n - x^*\bigr\|^2 &= \bigl\|P_C(x_n - \lambda_n Ax_n) - P_C\bigl(x^* - \lambda_n Ax^*\bigr)\bigr\|^2 \\
&\le \bigl\langle (x_n - \lambda_n Ax_n) - \bigl(x^* - \lambda_n Ax^*\bigr), y_n - x^*\bigr\rangle \\
&= \frac{1}{2}\bigl[\bigl\|x_n - x^* - \lambda_n\bigl(Ax_n - Ax^*\bigr)\bigr\|^2 + \bigl\|y_n - x^*\bigr\|^2 \\
&\quad - \bigl\|\bigl(x_n - x^*\bigr) - \lambda_n\bigl(Ax_n - Ax^*\bigr) - \bigl(y_n - x^*\bigr)\bigr\|^2\bigr] \\
&\le \frac{1}{2}\bigl[\bigl\|x_n - x^*\bigr\|^2 + \bigl\|y_n - x^*\bigr\|^2 - \bigl\|(x_n - y_n) - \lambda_n\bigl(Ax_n - Ax^*\bigr)\bigr\|^2\bigr] \\
&= \frac{1}{2}\bigl[\bigl\|x_n - x^*\bigr\|^2 + \bigl\|y_n - x^*\bigr\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\bigl\langle x_n - y_n, Ax_n - Ax^*\bigr\rangle \\
&\quad - \lambda_n^2\bigl\|Ax_n - Ax^*\bigr\|^2\bigr] \\
&\le \frac{1}{2}\bigl[\bigl\|x_n - x^*\bigr\|^2 + \bigl\|y_n - x^*\bigr\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\bigl\|Ax_n - Ax^*\bigr\|\bigr].
\end{aligned}
$$

Thus,

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\lambda_n \|x_n - y_n\| \|Ax_n - Ax^*\|. \tag{3.10}$$

Similarly to the above argument, utilizing Proposition 2.1(iii), we conclude from $u_n = P_C(y_n - \mu_2 B_2 y_n)$ that

$$\begin{aligned}
&\|u_n - y^*\|^2 \\
&\quad = \|P_C(y_n - \mu_2 B_2 y_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad \le \langle (y_n - \mu_2 B_2 y_n) - (x^* - \mu_2 B_2 x^*), u_n - y^* \rangle \\
&\quad = \frac{1}{2}\big[\|y_n - x^* - \mu_2(B_2 y_n - B_2 x^*)\|^2 + \|u_n - y^*\|^2 \\
&\qquad - \|(y_n - x^*) - \mu_2(B_2 y_n - B_2 x^*) - (u_n - y^*)\|^2\big] \\
&\quad \le \frac{1}{2}\big[\|y_n - x^*\|^2 + \|u_n - y^*\|^2 - \|(y_n - u_n) - \mu_2(B_2 y_n - B_2 x^*) - (x^* - y^*)\|^2\big] \\
&\quad = \frac{1}{2}\big[\|y_n - x^*\|^2 + \|u_n - y^*\|^2 - \|y_n - u_n - (x^* - y^*)\|^2 \\
&\qquad + 2\mu_2 \langle y_n - u_n - (x^* - y^*), B_2 y_n - B_2 x^* \rangle - \mu_2^2 \|B_2 y_n - B_2 x^*\|^2\big],
\end{aligned}$$

that is,

$$\begin{aligned}
\|u_n - y^*\|^2 \le{}& \|y_n - x^*\|^2 - \|y_n - u_n - (x^* - y^*)\|^2 \\
&+ 2\mu_2 \|y_n - u_n - (x^* - y^*)\| \|B_2 y_n - B_2 x^*\|.
\end{aligned} \tag{3.11}$$

Substituting (3.10) in (3.11), we have

$$\begin{aligned}
&\|u_n - y^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\lambda_n \|x_n - y_n\| \|Ax_n - Ax^*\| \\
&\qquad - \|y_n - u_n - (x^* - y^*)\|^2 + 2\mu_2 \|y_n - u_n - (x^* - y^*)\| \|B_2 y_n - B_2 x^*\|.
\end{aligned} \tag{3.12}$$

Similarly to the above argument, utilizing Proposition 2.1(iii), we conclude from $v_n = P_C(u_n - \mu_1 B_1 u_n)$ that

$$\begin{aligned}
&\|v_n - x^*\|^2 \\
&\quad = \|P_C(u_n - \mu_1 B_1 u_n) - P_C(y^* - \mu_1 B_1 y^*)\|^2 \\
&\quad \le \langle (u_n - \mu_1 B_1 u_n) - (y^* - \mu_1 B_1 y^*), v_n - x^* \rangle \\
&\quad = \frac{1}{2}\big[\|u_n - y^* - \mu_1(B_1 u_n - B_1 y^*)\|^2 + \|v_n - x^*\|^2 \\
&\qquad - \|(u_n - y^*) - \mu_1(B_1 u_n - B_1 y^*) - (v_n - x^*)\|^2\big] \\
&\quad \le \frac{1}{2}\big[\|u_n - y^*\|^2 + \|v_n - x^*\|^2 - \|(u_n - v_n) - \mu_1(B_1 u_n - B_1 y^*) - (y^* - x^*)\|^2\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\bigl[\|u_n - y^*\|^2 + \|v_n - x^*\|^2 - \|u_n - v_n - (y^* - x^*)\|^2 \\
&\quad + 2\mu_1\langle u_n - v_n - (y^* - x^*), B_1u_n - B_1y^*\rangle - \mu_1^2\|B_1u_n - B_1y^*\|^2\bigr],
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|v_n - x^*\|^2 &\le \|u_n - y^*\|^2 - \|u_n - v_n - (y^* - x^*)\|^2 \\
&\quad + 2\mu_1\|u_n - v_n - (y^* - x^*)\|\|B_1u_n - B_1y^*\|.
\end{aligned} \tag{3.13}
$$

Substituting (3.12) in (3.13), we have

$$
\begin{aligned}
\|v_n - x^*\|^2 \le{}& \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|Ax_n - Ax^*\| \\
&- \|y_n - u_n - (x^* - y^*)\|^2 + 2\mu_2\|y_n - u_n - (x^* - y^*)\|\|B_2y_n - B_2x^*\| \\
&- \|u_n - v_n + (x^* - y^*)\|^2 + 2\mu_1\|u_n - v_n + (x^* - y^*)\|\|B_1u_n - B_1y^*\|.
\end{aligned}
$$

This together with (3.4) and (3.8) implies that

$$
\begin{aligned}
&\|z_n - x^*\|^2 \\
&\quad\le (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\biggl\{(1 + \gamma_n)\bigl[\alpha_n\|f(x_n) - x^*\|^2 \\
&\qquad + \bigl\|P_C\bigl[P_C(y_n - \mu_2B_2y_n) - \mu_1B_1P_C(y_n - \mu_2B_2y_n)\bigr] - x^*\bigr\|^2\bigr] + \frac{\nu_nc_n}{\mu_n + \nu_n}\biggr\} \\
&\quad\le (1 - \mu_n - \nu_n)\|x_n - x^*\|^2 + (\mu_n + \nu_n)\biggl\{(1 + \gamma_n)\bigl[\alpha_n\|f(x_n) - x^*\|^2 \\
&\qquad + \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|Ax_n - Ax^*\| \\
&\qquad - \|y_n - u_n - (x^* - y^*)\|^2 + 2\mu_2\|y_n - u_n - (x^* - y^*)\|\|B_2y_n - B_2x^*\| \\
&\qquad - \|u_n - v_n + (x^* - y^*)\|^2 + 2\mu_1\|u_n - v_n + (x^* - y^*)\|\|B_1u_n - B_1y^*\|\bigr] + \frac{\nu_nc_n}{\mu_n + \nu_n}\biggr\} \\
&\quad\le \|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 + c_n + \alpha_n(1 + \gamma_n)\|f(x_n) - x^*\|^2 \\
&\qquad + 2(1 + \gamma_n)\bigl[\lambda_n\|x_n - y_n\|\|Ax_n - Ax^*\| + \mu_2\|y_n - u_n - (x^* - y^*)\|\|B_2y_n - B_2x^*\| \\
&\qquad + \mu_1\|u_n - v_n + (x^* - y^*)\|\|B_1u_n - B_1y^*\|\bigr] - (\mu_n + \nu_n)(1 + \gamma_n)\bigl[\|x_n - y_n\|^2 \\
&\qquad + \|y_n - u_n - (x^* - y^*)\|^2 + \|u_n - v_n + (x^* - y^*)\|^2\bigr].
\end{aligned} \tag{3.14}
$$

So, we have

$$
\begin{aligned}
&(\kappa + \sigma)(1 + \gamma_n)\bigl[\|x_n - y_n\|^2 + \|y_n - u_n - (x^* - y^*)\|^2 + \|u_n - v_n + (x^* - y^*)\|^2\bigr] \\
&\quad\le (\mu_n + \nu_n)(1 + \gamma_n)\bigl[\|x_n - y_n\|^2 + \|y_n - u_n - (x^* - y^*)\|^2 + \|u_n - v_n + (x^* - y^*)\|^2\bigr] \\
&\quad\le \|x_n - x^*\|^2 - \|z_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 + c_n + \alpha_n(1 + \gamma_n)\|f(x_n) - x^*\|^2 \\
&\qquad + 2(1 + \gamma_n)\bigl[\lambda_n\|x_n - y_n\|\|Ax_n - Ax^*\| + \mu_2\|y_n - u_n - (x^* - y^*)\|\|B_2y_n - B_2x^*\| \\
&\qquad + \mu_1\|u_n - v_n + (x^* - y^*)\|\|B_1u_n - B_1y^*\|\bigr]
\end{aligned}
$$

$$\begin{aligned}&\le \big(\|x_n - x^*\| + \|z_n - x^*\|\big)\|x_n - z_n\| + \gamma_n\|x_n - x^*\|^2 + c_n + \alpha_n(1+\gamma_n)\|f(x_n) - x^*\|^2 \\&\quad + 2(1+\gamma_n)\big[\lambda_n\|x_n - y_n\|\|Ax_n - Ax^*\| + \mu_2\|y_n - u_n - (x^* - y^*)\|\|B_2y_n - B_2x^*\| \\&\quad + \mu_1\|u_n - v_n + (x^* - y^*)\|\|B_1u_n - B_1y^*\|\big].\end{aligned}$$

Since $\alpha_n \to 0$, $\gamma_n \to 0$, $c_n \to 0$, $\|x_n - z_n\| \to 0$, $\|Ax_n - Ax^*\| \to 0$, $\|B_2y_n - B_2x^*\| \to 0$ and $\|B_1u_n - B_1y^*\| \to 0$, from the boundedness of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ we obtain that

$$\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|y_n - u_n - (x^* - y^*)\| = \lim_{n\to\infty}\|u_n - v_n + (x^* - y^*)\| = 0,$$

and hence

$$\lim_{n\to\infty}\|y_n - v_n\| = \lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.15}$$

Step 5. $\lim_{n\to\infty}\|x_n - t_n\| = \lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

Indeed, it follows from (3.1) that

$$\|t_n - y_n\| \le \alpha_n\|f(x_n) - y_n\| + (1-\alpha_n)\|v_n - y_n\| \le \alpha_n\|f(x_n) - y_n\| + \|v_n - y_n\|.$$

Since $\alpha_n \to 0$ and $\|v_n - y_n\| \to 0$, from the boundedness of $\{x_n\}$ and $\{y_n\}$ we know that $\|t_n - y_n\| \to 0$ as $n \to \infty$. Also, from $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$ we also have $\|x_n - t_n\| \to 0$. Since $z_n = (1-\mu_n-\nu_n)x_n + \mu_n t_n + \nu_n S^n t_n$, we have $\nu_n(S^nt_n - t_n) = (1-\mu_n-\nu_n)(t_n - x_n) + (z_n - t_n)$. Then

$$\begin{aligned}\sigma\|S^nt_n - t_n\| &\le \nu_n\|S^nt_n - t_n\| \\&\le (1-\mu_n-\nu_n)\|t_n - x_n\| + \|z_n - t_n\| \\&\le (1-\mu_n-\nu_n)\|t_n - x_n\| + \|z_n - x_n\| + \|x_n - t_n\| \\&\le 2\|t_n - x_n\| + \|z_n - x_n\|,\end{aligned}$$

and hence $\|t_n - S^nt_n\| \to 0$. Furthermore, observe that

$$\|x_n - S^nx_n\| \le \|x_n - t_n\| + \|t_n - S^nt_n\| + \|S^nt_n - S^nx_n\|. \tag{3.16}$$

Utilizing Lemma 2.6, we have

$$\|S^nt_n - S^nx_n\| \le \frac{1}{1-\kappa}\Big(\kappa\|t_n - x_n\| + \sqrt{\big(1+(1-\kappa)\gamma_n\big)\|t_n - x_n\|^2 + (1-\kappa)c_n}\Big)$$

for every $n = 1, 2, \ldots$. Hence it follows from $\|x_n - t_n\| \to 0$ that $\|S^nt_n - S^nx_n\| \to 0$. Thus from (3.16) and $\|t_n - S^nt_n\| \to 0$ we get $\|x_n - S^nx_n\| \to 0$. Since $\|x_{n+1} - x_n\| \to 0$, $\|x_n - S^nx_n\| \to 0$ as $n \to \infty$ and S is uniformly continuous, we obtain from Lemma 2.7 that $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.

Step 6. $\omega_w(\{x_n\}) \subset \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$.

Indeed, by the boundedness of $\{x_n\}$, we know that $\omega_w(\{x_n\}) \neq \emptyset$. Take $\hat{x} \in \omega_w(\{x_n\})$ arbitrarily. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $\hat{x}$. We can assert that $\hat{x} \in \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$. First, note that S is uniformly continuous and

$\|x_n - Sx_n\| \to 0$. Hence it is easy to see that $\|x_n - S^m x_n\| \to 0$ for all $m \geq 1$. By Lemma 2.8, we obtain $\hat{x} \in \mathrm{Fix}(S)$. Now let us show that $\hat{x} \in \Xi$. We note that

$$\begin{aligned}
&\|t_n - G(t_n)\| \\
&\quad \leq \alpha_n \|f(x_n) - G(t_n)\| \\
&\qquad + (1-\alpha_n)\|P_C[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] - G(t_n)\| \\
&\quad = \alpha_n \|f(x_n) - G(t_n)\| + (1-\alpha_n)\|G(y_n) - G(t_n)\| \\
&\quad \leq \alpha_n \|f(x_n) - G(t_n)\| + (1-\alpha_n)\|y_n - t_n\| \to 0.
\end{aligned} \tag{3.17}$$

Since $x_{n_i} \rightharpoonup \hat{x}$ and $\|x_n - t_n\| \to 0$, it follows that $t_{n_i} \rightharpoonup \hat{x}$. Thus, according to Lemma 2.1 we get $\hat{x} \in \Xi$. Furthermore, we show $\hat{x} \in \mathrm{VI}(C,A)$. Since $x_n - y_n \to 0$ and $x_n - t_n \to 0$, we have $y_{n_i} \rightharpoonup \hat{x}$ and $t_{n_i} \rightharpoonup \hat{x}$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C,A)$; see [36] for more details. Let $\mathrm{Gph}(T)$ be the graph of T, and let $(v,w) \in \mathrm{Gph}(T)$. Then we have $w \in Tv = Av + N_C v$, and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$. On the other hand, from $y_n = P_C(x_n - \lambda_n A x_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n A x_n - y_n, y_n - v\rangle \geq 0,$$

and hence

$$\left\langle v - y_n, \frac{y_n - x_n}{\lambda_n} + Ax_n \right\rangle \geq 0.$$

Therefore, from $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$ and $y_{n_i} \in C$, we have

$$\begin{aligned}
\langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} \right\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned}$$

Thus, we obtain $\langle v - \hat{x}, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $\hat{x} \in T^{-1}0$ and hence $\hat{x} \in \mathrm{VI}(C,A)$. Consequently, $\hat{x} \in \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A)$. This implies that $\omega_w(\{x_n\}) \subset \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A)$.

Step 7. $x_n \to q = P_{\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)} x$.

Indeed, from $q = P_{\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)}x$, $\hat{x} \in \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A)$ and (3.7), we have

$$\|q - x\| \le \|\hat{x} - x\| \le \liminf_{i\to\infty} \|x_{n_i} - x\| \le \limsup_{i\to\infty} \|x_{n_i} - x\| \le \|q - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|\hat{x} - x\|.$$

From $x_{n_i} - x \rightharpoonup \hat{x} - x$ we have $x_{n_i} - x \to \hat{x} - x$ due to the Kadec-Klee property of a real Hilbert space [29]. So, it is clear that $x_{n_i} \to \hat{x}$. Since $x_n = P_{Q_n}x$ and $q \in \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A) \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|q - x_{n_i}\|^2 = \langle q - x_{n_i}, x_{n_i} - x\rangle + \langle q - x_{n_i}, x - q\rangle \ge \langle q - x_{n_i}, x - q\rangle.$$

As $i \to \infty$, we obtain $-\|q - \hat{x}\|^2 \ge \langle q - \hat{x}, x - q\rangle \ge 0$ by $q = P_{\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)}x$ and $\hat{x} \in \mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A)$. Hence we have $\hat{x} = q$. This implies that $x_n \to q$. It is easy to see that $y_n \to q$ and $z_n \to q$. This completes the proof. □

4 Weak convergence theorem

In this section, we prove a new weak convergence theorem by the hybrid extragradient-like method for finding a common element of the solution set of VIP (1.1), the solution set of GSVI (1.8) and the fixed point set of a uniformly continuous asymptotically κ-strictly pseudocontractive mapping $S : C \to C$ in the intermediate sense.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone, let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1,2$, let $f : C \to C$ be a ρ-contraction with $\rho \in [0,1)$, and let $S : C \to C$ be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that* $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A)$ *is nonempty and bounded. Let $\{\gamma_n\}$ and $\{c_n\}$ be defined as in* (1.6). *Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = \alpha_n f(x_n) + (1-\alpha_n)P_C[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\ x_{n+1} = (1 - \mu_n - \nu_n)x_n + \mu_n t_n + \nu_n S^n t_n, \end{cases} \tag{4.1}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1,2$, $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$ and $\{\alpha_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ are three sequences in $[0,1]$ such that $\mu_n + \nu_n \le 1$ for all $n \ge 1$. Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(ii) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0,2\alpha)$;
(iii) $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;
(iv) *and for all* $n \ge 1$, $\kappa \le \mu_n$, $\sigma \le \nu_n$ *and* $\mu_n + \nu_n \le c$ *for some* $c, \sigma \in (0,1)$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element of $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A)$.

Proof First of all, take $x^* \in \operatorname{Fix}(S) \cap \Xi \cap \operatorname{VI}(C,A)$ arbitrarily. Then, repeating the same arguments as in (3.3) and (3.5), we deduce from (4.1) that

$$\|y_n - x^*\| \le \|x_n - x^*\|, \tag{4.2}$$

and

$$\|t_n - x^*\|^2 \le \|x_n - x^*\|^2 + \alpha_n \frac{\|f(x^*) - x^*\|^2}{1-\rho}. \tag{4.3}$$

Repeating the same arguments as in (3.6), we can obtain that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 + \alpha_n(1+\gamma_n)\frac{\|f(x^*) - x^*\|^2}{1-\rho} + c_n \\ &= (1+\gamma_n)\|x_n - x^*\|^2 + \alpha_n(1+\gamma_n)\frac{\|f(x^*) - x^*\|^2}{1-\rho} + c_n. \end{aligned} \tag{4.4}$$

Since $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ it follows that

$$\sum_{n=1}^{\infty} \left\{ \alpha_n(1+\gamma_n)\frac{\|f(x^*) - x^*\|^2}{1-\rho} + c_n \right\} < \infty.$$

So, by Lemma 2.10 we know that

$$\lim_{n\to\infty} \|x_n - x^*\| \quad \text{exists for all } x^* \in \operatorname{Fix}(S) \cap \Xi \cap \operatorname{VI}(C,A).$$

This implies that $\{x_n\}$ is bounded and hence $\{t_n\}$, $\{y_n\}$ are bounded due to (4.2) and (4.3).

Repeating the same arguments as in (3.8), we can conclude that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le{}& \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 + c_n + \alpha_n(1+\gamma_n)\|f(x_n) - x^*\|^2 \\ &- (\mu_n + \nu_n)(1+\gamma_n)\big[\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 \\ &+ \mu_2(2\beta_2 - \mu_2)\|B_2 y_n - B_2 x^*\|^2 + \mu_1(2\beta_1 - \mu_1)\|B_1 u_n - B_1 y^*\|^2\big], \end{aligned} \tag{4.5}$$

which hence implies that

$$\begin{aligned} &(\kappa + \sigma)(1+\gamma_n)\big[a(2\alpha - b)\|Ax_n - Ax^*\|^2 \\ &\qquad + \mu_2(2\beta_2 - \mu_2)\|B_2 y_n - B_2 x^*\|^2 + \mu_1(2\beta_1 - \mu_1)\|B_1 u_n - B_1 y^*\|^2\big] \\ &\quad \le (\mu_n + \nu_n)(1+\gamma_n)\big[\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 \\ &\qquad + \mu_2(2\beta_2 - \mu_2)\|B_2 y_n - B_2 x^*\|^2 + \mu_1(2\beta_1 - \mu_1)\|B_1 u_n - B_1 y^*\|^2\big] \\ &\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 + c_n + \alpha_n(1+\gamma_n)\|f(x_n) - x^*\|^2. \end{aligned}$$

Since $\alpha_n \to 0$, $\gamma_n \to 0$, $c_n \to 0$ and $\lim_{n\to\infty} \|x_n - x^*\|$ exists, from the boundedness of $\{x_n\}$ we conclude that

$$\lim_{n\to\infty} \|Ax_n - Ax^*\| = \lim_{n\to\infty} \|B_2 y_n - B_2 x^*\| = \lim_{n\to\infty} \|B_1 u_n - B_1 y^*\| = 0. \tag{4.6}$$

Repeating the same arguments as in (3.14), we can conclude that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad \le \|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2+c_n+\alpha_n(1+\gamma_n)\|f(x_n)-x^*\|^2 \\
&\qquad +2(1+\gamma_n)\bigl[\lambda_n\|x_n-y_n\|\|Ax_n-Ax^*\|+\mu_2\|y_n-u_n-(x^*-y^*)\|\|B_2y_n-B_2x^*\| \\
&\qquad +\mu_1\|u_n-v_n+(x^*-y^*)\|\|B_1u_n-B_1y^*\|\bigr]-(\mu_n+\nu_n)(1+\gamma_n)\bigl[\|x_n-y_n\|^2 \\
&\qquad +\|y_n-u_n-(x^*-y^*)\|^2+\|u_n-v_n+(x^*-y^*)\|^2\bigr],
\end{aligned}$$

which hence implies that

$$\begin{aligned}
&(\kappa+\sigma)(1+\gamma_n)\bigl[\|x_n-y_n\|^2+\|y_n-u_n-(x^*-y^*)\|^2+\|u_n-v_n+(x^*-y^*)\|^2\bigr] \\
&\quad \le (\mu_n+\nu_n)(1+\gamma_n)\bigl[\|x_n-y_n\|^2+\|y_n-u_n-(x^*-y^*)\|^2+\|u_n-v_n+(x^*-y^*)\|^2\bigr] \\
&\quad \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\gamma_n\|x_n-x^*\|^2+c_n+\alpha_n(1+\gamma_n)\|f(x_n)-x^*\|^2 \\
&\qquad +2(1+\gamma_n)\bigl[\lambda_n\|x_n-y_n\|\|Ax_n-Ax^*\|+\mu_2\|y_n-u_n-(x^*-y^*)\|\|B_2y_n-B_2x^*\| \\
&\qquad +\mu_1\|u_n-v_n+(x^*-y^*)\|\|B_1u_n-B_1y^*\|\bigr].
\end{aligned}$$

Since $\alpha_n\to 0$, $\gamma_n\to 0$, $c_n\to 0$, $\|Ax_n-Ax^*\|\to 0$, $\|B_2y_n-B_2x^*\|\to 0$, $\|B_1u_n-B_1y^*\|\to 0$ and $\lim_{n\to\infty}\|x_n-x^*\|$ exists, from the boundedness of $\{x_n\}$ and $\{y_n\}$ we obtain that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|y_n-u_n-(x^*-y^*)\|=\lim_{n\to\infty}\|u_n-v_n+(x^*-y^*)\|=0,$$

and hence

$$\lim_{n\to\infty}\|y_n-v_n\|=\lim_{n\to\infty}\|x_n-v_n\|=0. \tag{4.7}$$

On the other hand, it follows from (4.1) that

$$\|t_n-y_n\|\le\alpha_n\|f(x_n)-y_n\|+(1-\alpha_n)\|v_n-y_n\|\le\alpha_n\|f(x_n)-y_n\|+\|v_n-y_n\|.$$

Since $\alpha_n\to 0$ and $\|v_n-y_n\|\to 0$, from the boundedness of $\{x_n\}$ and $\{y_n\}$ we know that $\|t_n-y_n\|\to 0$ as $n\to\infty$. Also, from $\|x_n-t_n\|\le\|x_n-y_n\|+\|y_n-t_n\|$ we also have $\|x_n-t_n\|\to 0$. Repeating the same arguments as in (3.6), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad =\|(1-\mu_n-\nu_n)(x_n-x^*)+\mu_n(t_n-x^*)+\nu_n(S^nt_n-x^*)\|^2 \\
&\quad =(1-\mu_n-\nu_n)\|x_n-x^*\|^2+(\mu_n+\nu_n)\left\|\frac{\mu_n}{\mu_n+\nu_n}(t_n-x^*)+\frac{\nu_n}{\mu_n+\nu_n}(S^nt_n-x^*)\right\|^2 \\
&\qquad -(1-\mu_n-\nu_n)(\mu_n+\nu_n)\left\|\frac{\mu_n}{\mu_n+\nu_n}(t_n-x_n)+\frac{\nu_n}{\mu_n+\nu_n}(S^nt_n-x_n)\right\|^2 \\
&\quad =(1-\mu_n-\nu_n)\|x_n-x^*\|^2+(\mu_n+\nu_n)\left\|\frac{\mu_n}{\mu_n+\nu_n}(t_n-x^*)+\frac{\nu_n}{\mu_n+\nu_n}(S^nt_n-x^*)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&- (1-\mu_n-\nu_n)(\mu_n+\nu_n)\left\| \frac{1}{\mu_n+\nu_n}(x_{n+1}-x_n)\right\|^2 \\
&\le \left\|x_n-x^*\right\|^2+\gamma_n\left\|x_n-x^*\right\|^2+\alpha_n(1+\gamma_n)\frac{\|f(x^*)-x^*\|^2}{1-\rho} \\
&\quad + c_n - \frac{(1-\mu_n-\nu_n)}{\mu_n+\nu_n}\|x_{n+1}-x_n\|^2,
\end{aligned}
$$

which hence implies that

$$
\begin{aligned}
&\frac{(1-c)}{c}\|x_{n+1}-x_n\|^2 \\
&\quad \le \frac{(1-\mu_n-\nu_n)}{\mu_n+\nu_n}\|x_{n+1}-x_n\|^2 \\
&\quad \le \left\|x_n-x^*\right\|^2-\left\|x_{n+1}-x^*\right\|^2+\gamma_n\left\|x_n-x^*\right\|^2+\alpha_n(1+\gamma_n)\frac{\|f(x^*)-x^*\|^2}{1-\rho}+c_n.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_n-x^*\|$ exists, $\alpha_n\to 0$, $\gamma_n\to 0$, $c_n\to 0$ and the sequence $\{x_n\}$ is bounded, we obtain that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.
$$

Also, since $x_{n+1}=(1-\mu_n-\nu_n)x_n+\mu_n t_n+\nu_n S^n t_n$, we have $\nu_n(S^n t_n-t_n)=(1-\mu_n-\nu_n)(t_n-x_n)+(x_{n+1}-t_n)$. Then

$$
\begin{aligned}
\sigma\left\|S^n t_n-t_n\right\| &\le \nu_n\left\|S^n t_n-t_n\right\| \\
&\le (1-\mu_n-\nu_n)\|t_n-x_n\|+\|x_{n+1}-t_n\| \\
&\le (1-\mu_n-\nu_n)\|t_n-x_n\|+\|x_{n+1}-x_n\|+\|x_n-t_n\| \\
&\le 2\|t_n-x_n\|+\|x_{n+1}-x_n\|,
\end{aligned}
$$

and hence $\|t_n-S^n t_n\|\to 0$. Furthermore, observe that

$$
\left\|x_n-S^n x_n\right\| \le \|x_n-t_n\|+\left\|t_n-S^n t_n\right\|+\left\|S^n t_n-S^n x_n\right\|. \tag{4.8}
$$

Utilizing Lemma 2.6, we have

$$
\left\|S^n t_n-S^n x_n\right\| \le \frac{1}{1-\kappa}\left(\kappa\|t_n-x_n\|+\sqrt{\bigl(1+(1-\kappa)\gamma_n\bigr)\|t_n-x_n\|^2+(1-\kappa)c_n}\right)
$$

for every $n=1,2,\ldots$. Hence it follows from $\|x_n-t_n\|\to 0$ that $\|S^n t_n-S^n x_n\|\to 0$. Thus from (4.8) and $\|t_n-S^n t_n\|\to 0$ we get $\|x_n-S^n x_n\|\to 0$. Since $\|x_{n+1}-x_n\|\to 0$, $\|x_n-S^n x_n\|\to 0$ as $n\to\infty$ and S is uniformly continuous, we obtain from Lemma 2.7 that $\|x_n-Sx_n\|\to 0$ as $n\to\infty$.

Further, repeating the same arguments as in the proof of Theorem 3.1, we can derive that $\omega_w(\{x_n\})\subset \mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)$. Utilizing Lemma 2.11, from the existence of $\lim_{n\to\infty}\|x_n-x^*\|$ for each $x^*\in\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)$, we infer that $\{x_n\}$ converges weakly to an element $\hat{x}\in\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)$. Since $\|x_n-y_n\|\to 0$ as $n\to\infty$, it is clear that $\{y_n\}$ converges weakly to $\hat{x}\in\mathrm{Fix}(S)\cap\Xi\cap\mathrm{VI}(C,A)$. □

In the following, we present a numerical example to illustrate how Theorem 4.1 works.

Example 4.1 Let $H = \mathbf{R}^2$ with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ which are defined by

$$\langle x,y\rangle = ac + bd \quad \text{and} \quad \|x\| = \sqrt{a^2+b^2}$$

for all $x,y \in \mathbf{R}^2$ with $x = (a,b)$ and $y = (c,d)$. Let $C = \{(a,a) : a \in \mathbf{R}\}$. Clearly, C is a nonempty closed convex subset of a real Hilbert space $H = \mathbf{R}^2$. Let $A : C \to H$ be α-inverse strongly monotone, let $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1,2$, let $f : C \to C$ be a ρ-contraction with $\rho \in [0,1)$, and let $S : C \to C$ be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A)$ is nonempty bounded; for instance, putting $A = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$, $S = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, $B_1 = I - A = \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} \end{bmatrix}$, $B_2 = I - S = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $f = \frac{1}{2}S = \begin{bmatrix} \frac{2}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{6} \end{bmatrix}$. It is easy to see that $\|A\| = \|S\| = 1$, $\|f\| = \frac{1}{2}\|S\| = \frac{1}{2}$, and that A is α-inverse strongly monotone with $\alpha = \frac{1}{2}$, that B_1 and B_2 are $\frac{1}{2}$-inverse strongly monotone, f is a $\frac{1}{2}$-contraction, S is a nonexpansive mapping, *i.e.*, a uniformly continuous asymptotically 0-strictly pseudocontractive mapping in the intermediate sense with sequences $\{\gamma_n\}$ ($\gamma_n \equiv 0$) and $\{c_n\}$ ($c_n \equiv 0$). Moreover, it is clear that $\mathrm{Fix}(S) = C$, $\mathrm{VI}(C,A) = \{0\}$ and $\Xi = C$. Hence, $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C,A) = \{0\}$. In this case, from iterative scheme (4.1) in Theorem 4.1, we obtain that for any given $x_1 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n, \\ t_n = \alpha_n f(x_n) + (1-\alpha_n)P_C(I-\mu_1 B_1)P_C(I-\mu_2 B_2)y_n \\ \quad = \frac{1}{2}\alpha_n S x_n + (1-\alpha_n)P_C(I-\mu_1 B_1)P_C(I-\mu_2 B_2)(1-\lambda_n)x_n \\ \quad = \frac{1}{2}\alpha_n x_n + (1-\alpha_n)P_C(I-\mu_1 B_1)(1-\lambda_n)x_n \\ \quad = \frac{1}{2}\alpha_n x_n + (1-\alpha_n)(1-\lambda_n)x_n \\ \quad = [\frac{1}{2}\alpha_n + (1-\alpha_n)(1-\lambda_n)]x_n, \\ x_{n+1} = (1-\mu_n-\nu_n)x_n + \mu_n t_n + \nu_n S^n t_n \\ \qquad = (1-\mu_n-\nu_n)x_n + \mu_n[\frac{1}{2}\alpha_n + (1-\alpha_n)(1-\lambda_n)]x_n \\ \qquad\quad + \nu_n S^n[\frac{1}{2}\alpha_n + (1-\alpha_n)(1-\lambda_n)]x_n \\ \qquad = \{(1-\mu_n-\nu_n) + (\mu_n+\nu_n)[\frac{1}{2}\alpha_n + (1-\alpha_n)(1-\lambda_n)]\}x_n \\ \qquad = [1-(\mu_n+\nu_n)(\frac{1}{2}\alpha_n + \lambda_n(1-\alpha_n))]x_n. \end{cases}$$

Whenever $\sum_{n=1}^\infty \alpha_n < \infty$, $\{\lambda_n\} \subset [a,b] \subset (0,1)$, and $\{\mu_n+\nu_n\} \subset [\sigma,c] \subset (0,1)$, we have

$$\begin{aligned} \|x_{n+1}\| &= \left[1-(\mu_n+\nu_n)\left(\frac{1}{2}\alpha_n + \lambda_n(1-\alpha_n)\right)\right]\|x_n\| \\ &\le \left[1-(\mu_n+\nu_n)\left(\frac{1}{2}\lambda_n\alpha_n + \lambda_n(1-\alpha_n)\right)\right]\|x_n\| \\ &= \left[1-(\mu_n+\nu_n)\lambda_n\left(1-\frac{1}{2}\alpha_n\right)\right]\|x_n\| \\ &\le \left[1-\sigma a\left(1-\frac{1}{2}\alpha_n\right)\right]\|x_n\| \end{aligned}$$

$$
\begin{aligned}
&\le \|x_n\| \exp\left\{-\sigma a\left(1-\frac{1}{2}\alpha_n\right)\right\} \\
&\cdots \\
&\le \|x_n\| \exp\left\{-n\sigma a + \frac{1}{2}\sigma a \sum_{i=1}^{n} \alpha_i\right\} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

This shows that $\{x_n\}$ converges to the unique element 0 of $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$. Note that as $n \to \infty$,

$$
\|x_n - y_n\| = \lambda_n \|x_n\| \le \|x_n\| \to 0.
$$

Hence, $\{y_n\}$ also converges to the unique element 0 of $\mathrm{Fix}(S) \cap \Xi \cap \mathrm{VI}(C, A)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LC conceived of the study and drafted the manuscript initially. SM participated in its design, coordination and finalized the manuscript. JC outlined the scope and design of the study. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shanghai Normal University, and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, 200234, China. [2]Graduate Institute of Business and Management, College of Management, Chang Gung University, Kwei-Shan, Taoyuan, Taiwan R.O.C. [3]Medical Research Division, Chang Gung Memorial Hospital, Kwei-Shan, Taoyuan, Taiwan R.O.C. [4]Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung, 807, Taiwan. [5]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, Saudi Arabia.

Acknowledgements
Lu-Chuan Ceng was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133) and Ph.D. Program Foundation of Ministry of Education of China (20123127110002), Sy-Ming Guu was partially supported by NSC 100-2221-E-182-072-MY2, and Jen-Chih Yao was partially supported by the grant NSC 99-2115-M-037-002-MY3.

Received: 22 June 2013 Accepted: 29 October 2013 Published: 25 Nov 2013

References

1. Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. **20**, 197-228 (1967)
2. Lions, JL: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod, Paris (1969)
3. Glowinski, R: Numerical Methods for Nonlinear Variational Problems. Springer, New York (1984)
4. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
5. Oden, JT: Quantitative Methods on Nonlinear Mechanics. Prentice Hall, Englewood Cliffs (1986)
6. Zeidler, E: Nonlinear Functional Analysis and Its Applications. Springer, New York (1985)
7. Takahashi, W, Toyoda, M: Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **118**, 417-428 (2003)
8. Korpelevich, GM: The extragradient method for finding saddle points and other problems. Matecon **12**, 747-756 (1976)
9. Nadezhkina, N, Takahashi, W: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **128**, 191-201 (2006)
10. Zeng, LC, Yao, JC: Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems. Taiwan. J. Math. **10**(5), 1293-1303 (2006)
11. Iiduka, H, Takahashi, W: Strong convergence theorem by a hybrid method for nonlinear mappings of nonexpansive and monotone type and applications. Adv. Nonlinear Var. Inequal. **9**, 1-10 (2006)
12. Nadezhkina, N, Takahashi, W: Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings. SIAM J. Optim. **16**, 1230-1241 (2006)
13. Ceng, LC, Yao, JC: An extragradient-like approximation method for variational inequality problems and fixed point problems. Appl. Math. Comput. **190**, 205-215 (2007)
14. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**(1), 171-174 (1972)
15. Bruck, RE, Kuczumow, T, Reich, S: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**, 169-179 (1993)

16. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mapping. J. Math. Anal. Appl. **159**, 407-413 (1991)
17. Ceng, LC, Xu, HK, Yao, JC: The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces. Nonlinear Anal. **69**(4), 1402-1412 (2008)
18. Ceng, LC, Wong, NC, Yao, JC: Fixed point solutions of variational inequalities for a finite family of asymptotically nonexpansive mappings without common fixed point assumption. Comput. Math. Appl. **56**, 2312-2322 (2008)
19. Agarwal, RP, O'Regan, D, Sahu, DR: Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal. **8**(1), 61-79 (2007)
20. Chidume, CE, Shahzad, N, Zegeye, H: Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense. Numer. Funct. Anal. Optim. **25**, 239-257 (2004)
21. Kim, GE, Kim, TH: Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces. Comput. Math. Appl. **42**, 1565-1570 (2001)
22. Kim, TH, Xu, HK: Convergence of the modified Mann's iteration method for asymptotically strict pseudocontractions. Nonlinear Anal. **68**, 2828-2836 (2008)
23. Sahu, DR, Xu, HK, Yao, JC: Asymptotically strictly pseudocontractive mappings in the intermediate sense. Nonlinear Anal. **70**, 3502-3511 (2009)
24. Ceng, LC, Wang, CY, Yao, JC: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math. Methods Oper. Res. **67**, 375-390 (2008)
25. Verma, RU: On a new system of nonlinear variational inequalities and associated iterative algorithms. Math. Sci. Res. Hot-Line **3**(8), 65-68 (1999)
26. Yao, Y, Liou, YC, Kang, SM: Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. Comput. Math. Appl. **59**, 3472-3480 (2010)
27. Ceng, LC, Guu, SM, Yao, JC: Finding common solutions of a variational inequality, a general system of variational inequalities, and a fixed-point problem via a hybrid extragradient method. Fixed Point Theory Appl. **2011**, Article ID 626159 (2011). doi:10.1155/2011/626159
28. Ceng, LC, Hadjisavvas, N, Wong, NC: Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems. J. Glob. Optim. **46**, 635-646 (2010)
29. Goebel, K, Kirk, WA: Topics on Metric Fixed-Point Theory. Cambridge University Press, Cambridge (1990)
30. Gornicki, J: Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces. Comment. Math. Univ. Carol. **30**(2), 249-252 (1989)
31. Marino, G, Xu, HK: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **329**, 336-346 (2007)
32. Xu, HK: Existence and convergence for fixed points for mappings of asymptotically nonexpansive type. Nonlinear Anal. **16**, 1139-1146 (1991)
33. Osilike, MO, Aniagbosor, SC, Akuchu, BG: Fixed points of asymptotically demicontractive mappings in arbitrary Banach space. Panam. Math. J. **12**, 77-88 (2002)
34. Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)
35. Opial, Z: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591-597 (1967)
36. Rockafellar, RT: On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 75-88 (1970)

10.1186/1687-1812-2013-313
Cite this article as: Ceng et al.: Hybrid viscosity CQ method for finding a common solution of a variational inequality, a general system of variational inequalities, and a fixed point problem. *Fixed Point Theory and Applications* 2013, 2013:313