Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH | Open Access

Some generalizations of Suzuki and Edelstein type theorems

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Abstract
We prove some generalizations of Suzuki's fixed point theorem and Edelstein's theorem.

MSC: 54H25

Keywords: Banach principle; contraction; Suzuki's theorem; Edelstein's theorem

Introduction and preliminaries

Let (X,d) be a complete metric space and T be a selfmap of X. Then T is called a *contraction* if there exists $r \in [0,1)$ such that

$$d(Tx,Ty) \leq rd(x,y)$$

for all $x,y \in X$.

The following famous theorem is referred to as the Banach contraction principle.

Theorem 1 (Banach [1]) *Let (X,d) be a complete metric space, and let T be a contraction on X. Then T has a unique fixed point.*

This theorem is a very forceful and simple, and it has become a classical tool in nonlinear analysis. It has many generalizations, see [2–19].

In 2008, Suzuki [20] introduced a new type of mapping and presented a generalization of the Banach contraction principle in which the completeness can also be characterized by the existence of a fixed point of these mappings.

Theorem 2 [20] *Let (X,d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2, \\ (1-r)/r^2 & \text{if } (\sqrt{5}-1)/2 \leq r \leq 1/\sqrt{2}, \\ 1/(1+r) & \text{if } 1/\sqrt{2} \leq r < 1. \end{cases} \tag{1}$$

Assume that there exists $r \in [0,1)$ such that $\theta(r)d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq rd(x,y)$ for all $x,y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_n T^n x = z$ for all $x \in X$.

Its further outcomes by Altun and Erduran [21], Karapinar [22, 23], Kikkawa and Suzuki [24, 25], Moţ and Petruşel [26], Dhompongsa and Yingtaweesittikul [27], Popescu [28, 29], Singh and Mishra [30–32] are important contributions to metric fixed point theory.

Popescu [28] introduced a new type of contractive operator and proved the following theorem.

Theorem 3 [28] *Let (X, d) be a complete metric space and $T : X \to X$ be a (s, r)-contractive single-valued operator*:

$$x, y \in X \quad \text{with } d(y, Tx) \le sd(y, x) \quad \text{implies} \quad d(Tx, Ty) \le rM_T(x, y),$$

where $r \in [0, 1)$, $s > r$ and

$$M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Then T has a fixed point. Moreover, if $s \ge 1$, then T has a unique fixed point.

As a direct consequence of Theorem 3, we obtain the following result.

Theorem 4 *Let (X, d) be a complete metric space, and let T be a mapping on X. Assume that there exist $r \in [0, 1)$ and $s > r$ such that*

$$d(y, Tx) \le sd(y, x) \quad \text{implies} \quad d(Tx, Ty) \le rd(x, y) \tag{2}$$

for all $x, y \in X$. Then there exists a fixed point z of T. Further, if $s \ge 1$, then there exists a unique fixed point of T.

The following theorem is a well-known result in fixed point theory.

Theorem 5 (Edelstein [33]) *Let (X, d) be a compact metric space, and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \ne y$. Then T has a unique fixed point.*

Inspired by Theorem 2, Suzuki [34] proved a generalization of Edelstein's fixed point theorem (see also [35–38]).

Theorem 6 [34] *Let (X, d) be a compact metric space, and let T be a mapping on X. Assume that $(1/2)d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Then T has a unique fixed point.*

In this paper, we prove generalizations of Theorem 2, Theorem 4, Theorem 5 and extend Theorem 6. The direction of our extension is new, very simple and inspired by Theorem 3.

Main results

We start this section by proving the following theorem.

Theorem 7 *Let (X, d) be a complete metric space, and let T be a mapping on X. Assume that there exist $r \in [0, 1)$, $a \in [0, 1]$, $b \in [0, 1)$, $(a + b)r^2 + r \le 1$ if $r \in [1/2, 1/\sqrt{2})$, $a + (a + b)r \le$*

1 *if* $r \in [1/\sqrt{2}, 1)$ *such that*

$$ad(x, Tx) + bd(y, Tx) \leq d(y, x) \quad \textit{implies} \quad d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. *Then there exists a unique fixed point* z *of* T. *Moreover*, $\lim_n T^n x = z$ *for all* $x \in X$.

Proof Since $ad(x, Tx) + bd(Tx, Tx) = ad(x, Tx) \leq d(Tx, x)$ holds for every $x \in X$, by hypothesis, we get

$$d(Tx, T^2 x) \leq rd(x, Tx) \tag{3}$$

for all $x \in X$. We now fix $u \in X$ and define a sequence $\{u_n\} \in X$ by $u_n = T^n u$. Then (3) yields $d(u_n, u_{n+1}) \leq r^n d(u, Tu)$, so $\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty$. Hence $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$. We next show that

$$d(Tx, z) \leq rd(x, z) \tag{4}$$

for all $x \in X$, $x \neq z$. Since $\lim_n d(u_n, Tu_n) = 0$, $\lim_n d(x, Tu_n) = \lim_n d(x, u_n) = d(x, z)$, there exists a positive integer ν such that $ad(u_n, Tu_n) + bd(x, Tu_n) \leq d(x, u_n)$ for all $n \geq \nu$. By hypothesis, we get $d(Tu_n, Tx) \leq rd(u_n, x)$. Letting n tend to ∞, we obtain $d(z, Tx) \leq rd(z, x)$. That is, we have shown (4).

Now we assume that $T^j z \neq z$ for every integer $j \geq 1$. Then (4) yields

$$d(T^{j+1} z, z) \leq r^j d(Tz, z) \tag{5}$$

for every integer $j \geq 1$. We consider the following three cases:

(a) $0 \leq r < 1/2$,
(b) $1/2 \leq r < 1/\sqrt{2}$,
(c) $1/\sqrt{2} \leq r < 1$.

In the case (a) we note that $2r < 1$. Then, by (3) and (5), we have

$$d(z, Tz) \leq d(z, T^2 z) + d(Tz, T^2 z) \leq rd(z, Tz) + rd(z, Tz) = 2rd(z, Tz) < d(z, Tz).$$

This is a contradiction.

In the case (b), we note that $2r^2 < 1$. If we assume $ad(T^2 z, T^3 z) + bd(z, T^3 z) > d(z, T^2 z)$, then we have, in view of (3) and (5),

$$\begin{aligned} d(z, Tz) &\leq d(z, T^2 z) + d(Tz, T^2 z) \\ &< ad(T^2 z, T^3 z) + bd(z, T^3 z) + d(Tz, T^2 z) \\ &\leq ar^2 d(z, Tz) + br^2 d(z, Tz) + rd(z, Tz) \\ &= [(a + b)r^2 + r] d(z, Tz) \\ &\leq d(z, Tz). \end{aligned}$$

This is a contradiction. Hence $ad(T^2z, T^3z) + bd(z, T^3z) \leq d(z, T^2z)$. By hypothesis and (5), we have

$$\begin{aligned} d(z, Tz) &\leq d(z, T^3z) + d(Tz, T^3z) \\ &\leq r^2 d(z, Tz) + rd(z, T^2z) \\ &\leq r^2 d(z, Tz) + r^2 d(z, Tz) \\ &= 2r^2 d(z, Tz) \\ &< d(z, Tz). \end{aligned}$$

This is also a contradiction.

In the case (c), we assume there exists an integer $\nu \geq 1$ such that

$$ad(u_n, u_{n+1}) + bd(z, u_{n+1}) > d(z, u_n)$$

for all $n \geq \nu$. Then

$$\begin{aligned} d(z, u_n) &< ad(u_n, u_{n+1}) + b\big[ad(u_{n+1}, u_{n+2}) + bd(z, u_{n+2})\big] \\ &\leq (a + abr)d(u_n, u_{n+1}) + b^2 d(z, u_{n+2}) \\ &< (a + abr)d(u_n, u_{n+1}) + b^2\big[ad(u_{n+2}, u_{n+3}) + bd(z, u_{n+3})\big] \\ &\leq \big(a + abr + ab^2r^2\big)d(u_n, u_{n+1}) + b^3 d(z, u_{n+3}). \end{aligned}$$

Continuing this process, we get

$$\begin{aligned} d(z, u_n) &< \big(a + abr + ab^2r^2 + \cdots + ab^{p-1}r^{p-1}\big)d(u_n, u_{n+1}) + b^p d(z, u_{n+p}) \\ &\leq a\frac{1 - (br)^p}{1 - br} d(u_n, u_{n+1}) + b^p d(z, u_{n+p}) \end{aligned}$$

for all $n \geq \nu$, $p \geq 1$. Letting p tend to ∞, we obtain

$$d(z, u_n) \leq \frac{a}{1 - br} d(u_n, u_{n+1})$$

for all $n \geq \nu$. Thus,

$$d(z, u_{n+1}) \leq \frac{a}{1 - br} d(u_{n+1}, u_{n+2}) \leq \frac{ar}{1 - br} d(u_n, u_{n+1})$$

for all $n \geq \nu$, so

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(z, u_n) + d(z, u_{n+1}) \\ &< \frac{a}{1 - br} d(u_n, u_{n+1}) + \frac{ar}{1 - br} d(u_n, u_{n+1}) \\ &= \frac{a + ar}{1 - br} d(u_n, u_{n+1}) \\ &\leq d(u_n, u_{n+1}) \end{aligned}$$

for all $n \geq \nu$. This is a contradiction. Hence there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$ad(u_{n(k)}, u_{n(k)+1}) + bd(z, u_{n(k)+1}) \leq d(z, u_{n(k)})$$

for all $k \geq 1$. By hypothesis, we get $d(Tz, Tu_{n(k)}) \leq rd(z, u_{n(k)})$ for all $k \geq 1$. Letting k tend to ∞, we get $d(z, Tz) = 0$, that is, $z = Tz$. This is a contradiction.

Thus there exists an integer $j \geq 1$ such that $T^j z = z$. By (3) we get $d(z, Tz) = d(T^j z, T^{j+1} z) \leq r^j d(z, Tz)$, so $d(z, Tz) = 0$, that is, $Tz = z$.

Now we suppose that y is another fixed point of T, that is, $Ty = y$. Then

$$ad(y, Ty) + bd(z, Ty) = bd(z, y) \leq d(z, y),$$

so, by hypothesis, $d(y, z) = d(Ty, Tz) \leq rd(y, z)$. Hence $d(y, z) = 0$. This is a contradiction. □

Remark 1 For $r \in [0, 1/2)$, taking $a = 1$, $b = 0$, we obtain Suzuki's condition from Theorem 2. Moreover, from our condition and the triangle inequality, we get

$$ad(x, Tx) + b\big[d(x, Tx) - d(y, x)\big] \leq d(y, x),$$

that is,

$$\frac{a + b}{1 + b} d(x, Tx) \leq d(y, x).$$

If $r \in [1/\sqrt{2}, 1)$, we have

$$\frac{a + b}{1 + b} = \frac{1}{1 + r} = \theta(r),$$

hence our condition implies Suzuki's condition. We also note that if we take $a = (1 - r)/r^2$, $b = 0$ for $r \in [1/2, 1/\sqrt{2})$, we get Suzuki's condition. Therefore, our theorem generalizes, extends and complements Suzuki's theorem.

Example 1 Define a complete metric space X by $X = \{-1, 0, 1, 2\}$ and a mapping T on X by $Tx = 0$ if $x \in \{-1, 0, 1\}$ and $T2 = -1$. Then T satisfies our condition from Theorem 7 for every $r \in [0, 1/3) \cup [1/2, 1)$, but T does not satisfy Suzuki's condition from Theorem 2.

Proof Since $\theta(r) d(1, T1) \leq 1 = d(1, 2)$ for every $r \in [0, 1)$, and $d(T1, T2) = 1 = d(1, 2)$, T does not satisfy Suzuki's condition. If $r \in [1/2, (\sqrt{5} - 1)/2)$, we have $r^2 + r < 1$, so taking $a + b = (1 - r)/r^2$, we get $a + b > 1$. Hence $ad(1, T1) + bd(1, T2) = a + 2b > 1 = d(1, 2)$ and $ad(2, T2) + bd(2, T1) = 3a + 2b > 1 = d(1, 2)$. Now it is obvious that T satisfies our condition. If $r \in [(\sqrt{5} - 1)/2, 1)$, we take $b = 1/2$. We have two cases: $r \in [(\sqrt{5} - 1)/2, 1/\sqrt{2})$ and $r \in [1/\sqrt{2}, 1)$. In the first case we put $a = (2 - 2r - r^2)/(2r^2)$ and in the second $a = (2 - r)/(2 + 2r)$. We have $a + 2b = 1 + a > 1$ in both cases, so T satisfies our condition. If $r \in [0, 1/3)$ for $a = 1$, $b = 1/2$, it is obvious that T satisfies our condition. □

The following theorem is a generalization of Theorem 4.

Theorem 8 *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume that there exist $r \in [0,1)$, $s > r$ such that*

$$\frac{s-r}{1+r}d(x,Tx) + d(y,Tx) \leq sd(y,x) \quad \textit{implies} \quad d(Tx,Ty) \leq rd(x,y)$$

for all $x,y \in X$. Then T has a unique fixed point. Moreover, if $s \geq 1$, then T has a unique fixed point.

Proof Let $u_1 \in X$ and the sequence u_n be defined by $u_{n+1} = Tu_n$. Since

$$0 = d(u_{n+1}, Tu_n) \leq sd(u_{n+1}, u_n) - \frac{s-r}{1+r}d(u_n, Tu_n),$$

we get from hypothesis $d(u_{n+1}, u_{n+2}) \leq rd(u_{n+1}, u_n)$ for all $n \geq 1$. Therefore, $d(u_{n+1}, u_{n+2}) \leq r^n d(u_1, u_2)$ for all $n \geq 1$. Thus

$$\sum_{n=1}^{\infty} d(u_{n+1}, u_n) \leq \sum_{n=1}^{\infty} r^{n-1} d(u_1, u_2) < \infty.$$

Hence $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$.

Now, we will show that there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$d(z, Tu_{n(k)}) \leq sd(z, u_{n(k)}) - \frac{s-r}{1+r}d(u_{n(k)}, Tu_{n(k)})$$

for all $k \geq 1$. Arguing by contradiction, we suppose that there exists a positive integer ν such that

$$d(z, Tu_n) > sd(z, u_n) - \frac{s-r}{1+r}d(u_n, Tu_n)$$

for all $n \geq \nu$. Then we have

$$\begin{aligned}
d(z, u_{n+2}) &> sd(z, u_{n+1}) - \frac{s-r}{1+r}d(u_{n+1}, u_{n+2}) \\
&> s^2 d(z, u_n) - s \cdot \frac{s-r}{1+r}d(u_n, u_{n+1}) - \frac{s-r}{1+r}d(u_{n+1}, u_{n+2}) \\
&\geq s^2 d(z, u_n) - \frac{s-r}{1+r}\bigl[sd(u_n, u_{n+1}) + rd(u_n, u_{n+1})\bigr] \\
&= s^2 d(z, u_n) - \frac{s-r}{1+r}(s+r)d(u_n, u_{n+1}).
\end{aligned}$$

By induction, we get for all $n \geq \nu$, $p \geq 1$ that

$$d(z, u_{n+p}) > s^p d(z, u_n) - \frac{s-r}{1+r}\bigl(s^{p-1} + s^{p-2}r + \cdots + r^{p-1}\bigr)d(u_n, u_{n+1}).$$

Then we have

$$\begin{aligned}
d(z, u_{n+p}) &> s^p d(z, u_n) - \frac{s-r}{1+r} \cdot s^{p-1} \cdot \frac{1-(r/s)^p}{1-r/s}d(u_n, u_{n+1}) \\
&= s^p \left[d(z, u_n) - \frac{s-r}{1+r} \cdot \frac{1-(r/s)^p}{s-r}d(u_n, u_{n+1}) \right].
\end{aligned}$$

Hence

$$s^p\left[d(z,u_n) - \frac{1-(r/s)^p}{1+r}d(u_n,u_{n+1})\right] < d(z,u_{n+p}). \tag{6}$$

On the other hand,

$$\begin{aligned} d(u_{n+p},u_n) &\le d(u_n,u_{n+1}) + d(u_{n+1},u_{n+2}) + \cdots + d(u_{n+p-1},u_{n+p}) \\ &\le \left(1+r+\cdots+r^{p-1}\right)d(u_n,u_{n+1}) \\ &= \frac{1-r^p}{1-r}d(u_n,u_{n+1}). \end{aligned}$$

Letting $p \to \infty$, we get for all $n \ge 1$ that $d(z,u_n) \le d(u_n,u_{n+1})/(1-r)$. Thus

$$d(z,u_{n+p}) \le d(u_{n+p},u_{n+p+1})/(1-r) \le r^p d(u_n,u_{n+1})/(1-r). \tag{7}$$

By (6) and (7) we have for all $n \ge \nu$, $p \ge 1$ that

$$\frac{r^p}{1-r}d(u_n,u_{n+1}) > s^p\left[d(z,u_n) - \frac{1-(r/s)^p}{1+r}d(u_n,u_{n+1})\right],$$

so

$$\frac{(r/s)^p}{1-r}d(u_n,u_{n+1}) > d(z,u_n) - \frac{1-(r/s)^p}{1+r}d(u_n,u_{n+1}).$$

Taking the limit as $p \to \infty$, we obtain that $d(z,u_n) \le d(u_n,u_{n+1})/(1+r)$ for all $n \ge \nu$. Then we have

$$d(z,u_{n+1}) \le d(u_{n+1},u_{n+2})/(1+r) \le rd(u_n,u_{n+1})/(1+r)$$

and

$$rd(u_n,u_{n+1})/(1+r) > sd(z,u_n) - (s-r)d(u_n,u_{n+1})/(1+r).$$

This implies $d(z,u_n) < d(u_n,u_{n+1})/(1+r)$ for all $n \ge \nu$. Thus,

$$d(u_n,u_{n+1}) \le d(z,u_n) + d(z,u_{n+1}) < d(u_n,u_{n+1})/(1+r) + rd(u_n,u_{n+1})/(1+r) = d(u_n,u_{n+1}).$$

This is a contradiction. Therefore there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$d(z,Tu_{n(k)}) \le sd(z,u_{n(k)}) - \frac{s-r}{1+r}d(u_{n(k)},Tu_{n(k)})$$

for all $k \ge 1$. By hypothesis, we get $d(Tz,Tu_{n(k)}) \le rd(z,u_{n(k)})$. Letting $k \to \infty$, we obtain $d(Tz,z) = 0$, that is, $z = Tz$.

If $s \ge 1$, we assume that y is another fixed point of T. Then $d(z,Ty) = d(z,y) \le sd(z,y) - (s-r)d(y,Ty)/(1+r) = sd(z,y)$, so, by hypothesis, $d(z,y) = d(Tz,Ty) \le rd(z,y)$. Since $r < 1$, this is a contradiction. □

Edelstein's theorem

The following theorem extends Theorem 6 and generalizes Theorem 5.

Theorem 9 *Let (X, d) be a compact metric space, and let T be a mapping on X. Assume that*

$$ad(x, Tx) + bd(y, Tx) < d(y, x) \quad \textit{implies} \quad d(Tx, Ty) < d(x, y) \tag{8}$$

for $x, y \in X$, where $a > 0$, $b > 0$, $2a + b < 1$. Then T has a unique fixed point.

Proof We put

$$\beta = \inf\{d(x, Tx) : x \in X\}$$

and choose a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, Tx_n) = \beta$. Since X is compact, without loss of generality, we may assume that $\{x_n\}$ and $\{Tx_n\}$ converge to some elements $v, w \in X$, respectively. We have

$$\lim_{n\to\infty} d(x_n, w) = \lim_{n\to\infty} d(Tx_n, v) = d(v, w) = \beta.$$

We shall show $\beta = 0$. Arguing by contradiction, we assume $\beta > 0$. Since

$$\lim_{n\to\infty} \bigl[ad(x_n, Tx_n) + bd(w, Tx_n)\bigr] = a\beta < \beta = \lim_{n\to\infty} d(w, x_n),$$

we can choose a positive integer ν such that

$$ad(x_n, Tx_n) + bd(w, Tx_n) < d(w, x_n)$$

for all $n \geq \nu$. By hypothesis, $d(Tw, Tx_n) < d(w, x_n)$ holds for $n \geq \nu$. This implies

$$d(w, Tw) = \lim_{n\to\infty} d(Tw, Tx_n) \leq \lim_{n\to\infty} d(w, x_n) = \beta.$$

From the definition of β, we obtain $d(w, Tw) = \beta$. Since $ad(w, Tw) + bd(Tw, Tw) < d(Tw, w)$, we have

$$d\bigl(Tw, T^2 w\bigr) < d(w, Tw) = \beta,$$

which contradicts the definition of β. Therefore we obtain $\beta = 0$. We have $\lim_{n\to\infty} d(x_n, w) = \lim_{n\to\infty} d(Tx_n, v) = \lim_{n\to\infty} d(Tx_n, x_n) = d(v, w) = 0$, so $v = w$. Thus, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = w$.

We next show that T has a fixed point. Arguing by contradiction, we assume that T does not have a fixed point. Since $ad(x_n, Tx_n) + bd(Tx_n, Tx_n) < d(Tx_n, x_n)$ for all $n \geq 1$, we get $d(T^2 x_n, Tx_n) < d(Tx_n, x_n)$, so $\lim_{n\to\infty} T^2 x_n = w$. By induction, we obtain that $d(T^p x_n, T^{p+1} x_n) < d(T^{p-1} x_n, T^p x_n) < \cdots < d(x_n, Tx_n)$ and $\lim_{n\to\infty} T^p x_n = w$ for all integers $p \geq 1$. If there exist an integer $p \geq 1$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$ad\bigl(T^{p-1} x_{n(k)}, T^p x_{n(k)}\bigr) + bd\bigl(w, T^p x_{n(k)}\bigr) < d\bigl(w, T^{p-1} x_{n(k)}\bigr)$$

for all $k \geq 1$, by hypothesis we get $d(Tw, T^p x_{n(k)}) < d(w, T^{p-1}x_{n(k)})$. Taking the limit as $k \to \infty$, we obtain $d(w, Tw) = 0$, that is, $Tw = w$, which is a contradiction. Hence, we can assume that for every $m \geq 1$, there exists an integer $n(m) \geq 1$ such that

$$ad(T^{m-1}x_n, T^m x_n) + bd(w, T^m x_n) \geq d(w, T^{m-1}x_n) \tag{9}$$

for all $n \geq n(m)$. Since

$$\lim_{p\to\infty} \frac{pb^p}{1-b^p} = 0,$$

and

$$\frac{2a}{1-b} < 1,$$

we can choose p satisfying

$$\frac{pb^p}{1-b^p} + \frac{(p-1)b^{p-1}}{1-b^{p-1}} + \frac{2a}{1-b} < 1. \tag{10}$$

We put $\nu = \max\{n(1), n(2), \dots, n(p)\}$. Then by (9) we have

$$\begin{aligned} d(w,x_n) &\leq ad(x_n, Tx_n) + bd(w, Tx_n) \\ &\leq ad(x_n, Tx_n) + b\big[ad(Tx_n, T^2x_n) + bd(w, T^2x_n)\big] \\ &= ad(x_n, Tx_n) + abd(Tx_n, T^2x_n) + b^2 d(w, T^2x_n) \\ &\leq \cdots \\ &\leq ad(x_n, Tx_n) + abd(Tx_n, T^2x_n) + \cdots \\ &\quad + ab^{p-1}d(T^{p-2}x_n, T^{p-1}x_n) + b^p d(w, T^p x_n) \\ &\leq (a + ab + \cdots + ab^{p-1})d(x_n, Tx_n) + b^p d(w, T^p x_n) \\ &\leq \big[a(1-b^p)/(1-b)\big]d(x_n, Tx_n) + b^p d(w, T^p x_n) \end{aligned}$$

for all $n \geq \nu$. Since

$$\begin{aligned} d(w, T^p x_n) &\leq d(w,x_n) + d(x_n, Tx_n) + \cdots + d(T^{p-1}x_n, T^p x_n) \\ &< d(w,x_n) + pd(x_n, Tx_n), \end{aligned}$$

we get

$$d(w,x_n) < \big[a(1-b^p)/(1-b)\big]d(x_n, Tx_n) + b^p\big[d(w,x_n) + pd(x_n, Tx_n)\big],$$

so

$$d(w,x_n) < \left(\frac{a}{1-b} + \frac{pb^p}{1-b^p}\right)d(x_n, Tx_n) \tag{11}$$

for all $n \geq \nu$. Similarly, we can obtain

$$d(w, Tx_n) < \left[\frac{a}{1-b} + \frac{(p-1)b^{p-1}}{1-b^{p-1}}\right] d\big(Tx_n, T^2x_n\big)$$
$$< \left[\frac{a}{1-b} + \frac{(p-1)b^{p-1}}{1-b^{p-1}}\right] d(x_n, Tx_n)$$

for all $n \geq \nu$. Using (11), we get

$$d(x_n, Tx_n) \leq d(w, x_n) + d(w, Tx_n) < \left[\frac{2a}{1-b} + \frac{pb^p}{1-b^p} + \frac{(p-1)b^{p-1}}{1-b^{p-1}}\right] d(x_n, Tx_n)$$

for all $n \geq \nu$. Thus, by (10), we obtain $d(x_n, Tx_n) < d(x_n, Tx_n)$, which is a contradiction. Therefore there exists $z \in X$ such that $Tz = z$. Fix $y \in X$ with $y \neq x$. Then since $ad(x, Tx) + bd(y, Tx) = bd(y, x) < d(y, x)$, we have $d(Ty, x) = d(Ty, Tx) < d(y, x)$ and hence y is not a fixed point of T. Therefore, the fixed point of T is unique. □

Remark 2 The proof of Theorem 9 is available for $a = 1/2$, $b = 0$. In this case we obtained Theorem 6. We do not know if Theorem 9 is still correct for $a = 0$, $b = 1$, or, more generally, for $2a + b = 1$. This is an open question.

Example 2 Define a complete metric space X by $X = \{A, B, C, D, E\}$ such that $d(A, B) = d(A, C) = d(B, D) = d(C, D) = 2$, $d(A, D) = d(B, C) = 3$, $d(A, E) = d(C, E) = 5/2$, $d(B, E) = d(D, E) = 1$ and a mapping T on X by $TA = B$, $TB = E$, $TC = D$, $TD = E$, $TE = E$. Then T satisfies our condition from Theorem 9 for $a = 1/8$, $b = 2/3$, but T does not satisfy Suzuki's condition from Theorem 6.

Proof We have $d(A, C) = 2 = d(TA, TC)$ and $(1/2)d(A, TA) = 1 < d(A, C) = 2$, so T does not satisfy Suzuki's condition from Theorem 6. Moreover, we have $ad(A, TA) + bd(C, TA) = ad(C, TC) + bd(A, TC) = 2a + 3b = 9/4 > d(A, C)$. It is now obvious that T satisfies our condition from Theorem 9. □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The author is highly indebted to the referees for their careful reading of the manuscript and valuable suggestions.

Received: 29 July 2013 Accepted: 28 October 2013 Published: 25 Nov 2013

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10.1186/1687-1812-2013-319
Cite this article as: Popescu: Some generalizations of Suzuki and Edelstein type theorems. *Fixed Point Theory and Applications* 2013, 2013:319