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Fixed point results for G^m-Meir-Keeler contractive and G-(α,ψ)-Meir-Keeler contractive mappings

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Abstract
In this paper, first we introduce the notion of a G^m-Meir-Keeler contractive mapping and establish some fixed point theorems for the G^m-Meir-Keeler contractive mapping in the setting of G-metric spaces. Further, we introduce the notion of a G_c^m-Meir-Keeler contractive mapping in the setting of G-cone metric spaces and obtain a fixed point result. Later, we introduce the notion of a G-(α,ψ)-Meir-Keeler contractive mapping and prove some fixed point theorems for this class of mappings in the setting of G-metric spaces.

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1 Introduction

In nonlinear functional analysis, the study of fixed points of given mappings satisfying certain contractive conditions in various abstract spaces has been at the center of vigorous research activity in the last decades. The Banach contraction mapping principle is one of the initial and crucial results in this direction: In a complete metric space each contraction has a unique fixed point. Following this celebrated result, many authors have devoted their attention to generalizing, extending and improving this theory. For this purpose, the authors consider to extend some well-known results to different abstract spaces such as symmetric spaces, quasi-metric spaces, fuzzy metric, partial metric spaces, probabilistic metric spaces and a G-metric space (see, *e.g.*, [1–9]). Several authors have reported interesting (common) fixed point results for various classes of functions in the setting of such abstract spaces (see, *e.g.*, [6, 7, 10–32]).

In this paper, we consider especially a G-metric space and cone metric spaces which are introduced by Mustafa-Sims [9] and Huang-Zhang [3], respectively. Roughly speaking, a G-metric assigns a real number to every triplet of an arbitrary set. On the other hand, a cone metric space is obtained by replacing the set of real numbers by an ordered Banach space. Very recently, a number of papers on these concepts have appeared [9, 33–48].

One of the remarkable notions in fixed point theory is Meir-Keeler contractions [49] which have been studied by many authors (see, *e.g.*, [50–56]). In this paper, first we introduce the notion of a G^m-Meir-Keeler contractive mapping and establish some fixed point theorems for the G^m-Meir-Keeler contractive mapping in the setting of G-metric spaces. In Section 4, we introduce the notion of a G_c^m-Meir-Keeler contractive mapping in the setting of cone G-metric spaces and establish a fixed point result. Later, we introduce the notion of a G-(α, ψ)-Meir-Keeler contractive mapping and prove some fixed point theorems for this class of mappings in the setting of G-metric spaces.

2 Preliminaries

We present now the necessary definitions and results in G-metric spaces which will be useful; for more details, we refer to [9, 57]. In the sequel, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of real numbers, the set of nonnegative real numbers and the set of positive integers, respectively.

Definition 1 Let X be a nonempty set. A function $G : X \times X \times X \longrightarrow \mathbb{R}_+$ is called a G-metric if the following conditions are satisfied:

(G1) if $x = y = z$, then $G(x, y, z) = 0$;
(G2) $0 < G(x, y, y)$ for any $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for any points $x, y, z \in X$, with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for any $x, y, z, a \in X$.

Then the pair (X, G) is called a G-metric space.

Definition 2 Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to+\infty} G(x, x_m, x_n) = 0$, and we say that the sequence $\{x_n\}$ is G-convergent to x and denote it by $x_n \longrightarrow x$.

We have the following useful results.

Proposition 3 (see [44]) *Let (X, G) be a G-metric space. Then the following are equivalent*:

(1) *$\{x_n\}$ is G-convergent to x*;
(2) $\lim_{n\to+\infty} G(x_n, x_n, x) = 0$;
(3) $\lim_{n\to+\infty} G(x_n, x, x) = 0$.

Definition 4 ([44]) Let (X, G) be a G-metric space, the sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq k$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 5 ([44]) *Let (X, G) be a G-metric space. Then the following are equivalent*:

(1) *the sequence $\{x_n\}$ is G-Cauchy*;
(2) *for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq k$.*

Definition 6 ([44]) A G-metric space (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 7 (see [44]) *Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that*

(i) *if* $G(x,y,z) = 0$, *then* $x = y = z$;
(ii) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$;
(iii) $G(x,y,y) \leq 2G(y,x,x)$;
(iv) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$;
(v) $G(x,y,z) \leq \frac{2}{3}[G(x,y,a) + G(x,a,z) + G(a,y,z)]$;
(vi) $G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$.

Proposition 8 (see [44]) *Let* (X,G) *be a G-metric space. Then the function* $G(x,y,z)$ *is jointly continuous in all three of its variables.*

Now, we introduce the following notion of a G^m-Meir-Keeler contractive mapping.

Definition 9 Let (X,G) be a G-metric space. Suppose that $f : X \to X$ is a self-mapping satisfying the following condition:

For each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x,y \in X$ and for all $m \in \mathbb{N}$, we have

$$\varepsilon \leq G\big(x,f^{(m)}x,y\big) < \varepsilon + \delta \quad \text{implies} \quad G\big(fx,f^{(m+1)}x,fy\big) < \varepsilon. \tag{2.1}$$

Then f is called a G^m-Meir-Keeler contractive mapping.

Remark 10 If $f : X \to X$ is a G^m-Meir-Keeler contractive mapping on a G-metric space X, then

$$G\big(fx,f^{(m+1)}x,fy\big) < G\big(x,f^{(m)}x,y\big) \tag{2.2}$$

holds for all $x,y \in X$ and for all $m \in \mathbb{N}$ when $G(x,f^{(m)}x,y) > 0$. On the other hand, if $G(x,f^{(m)}x,y) = 0$, by Proposition 7, $x = f^{(m)}x = y$, and so $G(fx,f^{(m+1)}x,fy) = 0$. Hence, for all $x,y \in X$ and for all $m \in \mathbb{N}$, we have

$$G\big(fx,f^{(m+1)}x,fy\big) \leq G\big(x,f^{(m)}x,y\big). \tag{2.3}$$

3 Fixed point result for G^m-Meir-Keeler contractive mappings

Now, we are ready to state and prove our main result.

Theorem 11 *Let* (X,G) *be a G-complete G-metric space and let* f *be a* G^m*-Meir-Keeler contractive mapping on* X. *Then* f *has a unique fixed point.*

Proof Define the sequence $\{x_n\}$ in X as follows:

$$x_n = fx_{n-1} \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Suppose that there exists n_0 such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = fx_{n_0}$, then x_{n_0} is the fixed point of f. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, and so

$$G(x_n,x_{n+1},x_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

By Remark 10 with $m = 1$, we get

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G\bigl(fx_n, f^2x_n, fx_{n+1}\bigr) \\ &< G(x_n, fx_n, x_{n+1}) \\ &= G(x_n, x_{n+1}, x_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Define $s_n = G(x_n, x_{n+1}, x_{n+1})$. Then $\{s_n\}$ is a strictly decreasing sequence in $\mathbb{R}_+$ and so it is convergent, say, to $s \in \mathbb{R}_+$. Now, we show that s must be equal to 0. Suppose, to the contrary, that $s > 0$. Clearly, we have

$$0 < s < G(x_n, x_{n+1}, x_{n+1}) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Assume $\varepsilon = s > 0$. Then by hypothesis, there exists a convenient $\delta(\varepsilon) > 0$ such that (2.1) holds. On the other hand, by the definition of ε, there exists $n_0 \in \mathbb{N}$ such that

$$\varepsilon < s_{n_0} = G(x_{n_0}, x_{n_0+1}, x_{n_0+1}) < \varepsilon + \delta. \tag{3.4}$$

Now, by condition (2.1) with $m = 1$ and (3.4), we get

$$s_{n_0+1} = G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2}) = G\bigl(fx_0, f^2x_0, fx_{n_0+1}\bigr) < \varepsilon = s, \tag{3.5}$$

which contradicts (3.3). Hence $s = 0$, that is, $\lim_{n\to+\infty} s_n = 0$.

We will show that $\{x_n\}$ is a G-Cauchy sequence. For all $\varepsilon > 0$, by the hypothesis, there exists a suitable $\delta(\varepsilon) > 0$ such that (2.1) holds. Without loss of generality, we assume $\delta < \varepsilon$. Since $s = 0$, there exists $N \in \mathbb{N}$ such that

$$s_{n-1} = G(x_{n-1}, x_n, x_n) < \delta \quad \text{for all } n \geq N. \tag{3.6}$$

We assert that for any fixed $k \geq N$, the condition

$$G(x_k, x_{k+l}, x_{k+l}) \leq \varepsilon \quad \text{for all } l \in \mathbb{N} \tag{3.7}$$

holds. To prove it, we use the method of induction. By Remark 10 and (3.6), assertion (3.7) is satisfied for $l = 1$. Suppose that (3.7) is satisfied for $l = 1, 2, \ldots, m$ for some $m \in \mathbb{N}$. Now, for $l = m + 1$, using (3.6), we obtain

$$\begin{aligned} G\bigl(x_{k-1}, f^{(m+1)}x_{k-1}, x_{k+m}\bigr) &= G(x_{k-1}, x_{k+m}, x_{k+m}) \\ &\leq G(x_{k-1}, x_k, x_k) + G(x_k, x_{k+m}, x_{k+m}) \\ &< \varepsilon + \delta. \end{aligned} \tag{3.8}$$

If $G(x_{k-1}, x_{k+m}, x_{k+m}) \geq \varepsilon$, then by (2.1) we get

$$G(x_k, x_{k+m+1}, x_{k+m+1}) = G\bigl(fx_{k-1}, f^{(m+2)}x_{k-1}, fx_{k+m}\bigr) < \varepsilon$$

and hence (3.7) is satisfied.

If $G(x_{k-1}, x_{k+m}, x_{k+m}) = 0$, then $x_{k-1} = x_{k+m}$ and hence $x_k = fx_{k-1} = fx_{k+m} = x_{k+m+1}$. This implies

$$G(x_k, x_{k+m+1}, x_{k+m+1}) = G(x_k, x_k, x_k) = 0 < \varepsilon$$

and (3.7) is satisfied.

If $0 < G(x_{k-1}, x_{k+m}, x_{k+m}) < \varepsilon$, by Remark 10, we obtain

$$\begin{aligned} G(x_k, x_{k+m+1}, x_{k+m+1}) &= G\bigl(fx_{k-1}, f^{(m+2)}x_{k-1}, fx_{k+m}\bigr) \\ &< G(x_{k-1}, x_{k+m}, x_{k+m}) < \varepsilon. \end{aligned}$$

Consequently, (3.7) is satisfied for $l = m + 1$ and hence

$$G(x_n, x_m, x_m) < \varepsilon \quad \text{for all } m \geq n \geq N. \tag{3.9}$$

Now, if $n > m \geq N$, by (3.9) and Proposition 7, we have

$$G(x_n, x_m, x_m) \leq 2G(x_m, x_n, x_n) < 2\varepsilon.$$

Hence, for all $m, n \geq N$, the following holds:

$$G(x_n, x_m, x_m) < 2\varepsilon.$$

Thus $\{x_n\}$ is a G-Cauchy sequence. Since (X, G) is G-complete, there exists $z \in X$ such that $\{x_n\}$ is G-convergent to z. Now, by Remark 10 with $m = 1$, we have

$$G(x_{n+1}, x_{n+2}, fz) = G\bigl(fx_n, f^{(2)}x_n, fz\bigr) \leq G(x_n, fx_n, z) = G(x_n, x_{n+1}, z). \tag{3.10}$$

By taking the limit as $n \to +\infty$ in the above inequality and using the continuity of the function G, we get

$$G(z, z, fz) = \lim_{n \to +\infty} G(x_{n+1}, x_{n+2}, fz) = 0$$

and hence, $z = fz$, that is, z is a fixed point of f. To prove the uniqueness, we assume that $w \in X$ is another fixed point of f such that $z \neq w$. Then $G(z, f^{(m)}z, w) = G(z, z, w) > 0$. Now, by Remark 10, we get

$$G(z, z, w) = G\bigl(fz, f^{(m+1)}z, fw\bigr) < G\bigl(z, f^{(m)}z, w\bigr) = G(z, z, w),$$

which is a contradiction and hence $z = w$. □

Example 12 Let $X = [0, \infty)$ and

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y\} + \max\{y, z\} + \max\{x, z\}, & \text{otherwise} \end{cases}$$

be a G-metric on X. Define $f : X \to X$ by $fx = \frac{1}{2}x$. Then $f^m x = \frac{1}{2^m}x$. Assume that $x \le y$. Then

$$G(x, f^m x, y) = \max\{x, f^m x\} + \max\{f^m x, y\} + \max\{x, y\} = x + 2y$$

and

$$\begin{aligned} G(fx, f^{m+1}x, fy) &= \max\{fx, f^{m+1}x\} + \max\{f^{m+1}x, fy\} + \max\{fx, fy\} \\ &= fx + 2fy = \frac{1}{2}(x + 2y). \end{aligned}$$

Let, $\epsilon > 0$. Then, for any $\delta = \epsilon$, condition (2.1) holds. Similarly, condition (2.1) holds when $y \le x$. That is, f is a G^m-Meir-Keeler contractive mapping. The condition of Theorem 11 holds, and so f has a unique fixed point.

4 Fixed point for G-(α, ψ)-Meir-Keeler contractive mappings

In this section we introduce a notion of a G-(α, ψ)-Meir-Keeler contractive mapping and establish some results of a fixed point for such class of mappings.

Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ continuous in $t = 0$ such that

- $\psi(t) = 0$ if and only if $t = 0$;
- $\psi(t + s) \le \psi(t) + \psi(s)$.

Samet, Vetro and Vetro [19] introduced the following concept.

Definition 13 Let $f : X \to X$ and $\alpha : X \times X \to \mathbb{R}_+$. We say that f is an α-admissible mapping if

$$\alpha(x, y) \ge 1 \quad \text{implies} \quad \alpha(fx, fy) \ge 1, x, y \in X.$$

Now, we apply this concept in the following definition.

Definition 14 Let (X, G) be a G-metric space and $\psi \in \Psi$. Suppose that $f : X \to X$ is an α-admissible mapping satisfying the following condition:

For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \le \psi\big(G(x, y, z)\big) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x, x)\alpha(y, y)\alpha(z, z)\psi\big(G(fx, fy, fz)\big) < \varepsilon \tag{4.1}$$

for all $x, y, z \in X$. Then f is called a G-(α, ψ)-Meir-Keeler contractive mapping.

Remark 15 Let f be a G-(α, ψ)-Meir-Keeler contractive mapping. Then

$$\alpha(x, x)\alpha(y, y)\alpha(z, z)\psi\big(G(fx, fy, fz)\big) < \psi\big(G(x, y, z)\big)$$

for all $x, y \in X$ when $G(x, y, z) > 0$. Also, if $G(x, y, z) = 0$, then $x = y = z$, which implies $G(fx, fy, fz) = 0$, *i.e.*,

$$\alpha(x, x)\alpha(y, y)\alpha(z, z)\psi\big(G(fx, fy, fz)\big) \le \psi\big(G(x, y, z)\big)$$

for all $x, y, z \in X$.

Theorem 16 *Let (X,G) be a G-complete G-metric space. Suppose that f is a continuous G-(α,ψ)-Meir-Keeler contractive mapping and that there exists $x_0 \in X$ such that $\alpha(x_0,x_0) \geq 1$. Then f has a fixed point.*

Proof Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is an α-admissible mapping and $\alpha(x_0,x_0) \geq 1$, we deduce that $\alpha(x_1,x_1) = \alpha(fx_0,fx_0) \geq 1$. By continuing this process, we get $\alpha(x_n,x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then obviously f has a fixed point. Hence, we suppose that

$$x_n \neq x_{n+1} \tag{4.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. By (G2), we have

$$G(x_n,x_{n+1},x_{n+1}) > 0 \tag{4.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, define $s_n = \psi(G(x_n,x_{n+1},x_{n+1}))$. By Remark 15, we deduce that for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\alpha(x_{n+1},x_{n+1})\alpha(x_{n+2},x_{n+2})\alpha(x_{n+2},x_{n+2})\psi\bigl(G(x_{n+1},x_{n+2},x_{n+2})\bigr) \\
&\quad = \alpha(x_{n+1},x_{n+1})\alpha(x_{n+2},x_{n+2})\alpha(x_{n+2},x_{n+2})\psi\bigl(G(fx_n,fx_{n+1},fx_{n+1})\bigr) \\
&\quad < \psi\bigl(G(x_n,x_{n+1},x_{n+1})\bigr),
\end{aligned}$$

which implies

$$\psi\bigl(G(x_{n+1},x_{n+2},x_{n+2})\bigr) < \psi\bigl(G(x_n,x_{n+1},x_{n+1})\bigr).$$

Hence, the sequence $\{s_n\}$ is decreasing in $\mathbb{R}_+$ and so it is convergent to $s \in \mathbb{R}_+$. We will show that $s = 0$. Suppose, to the contrary, that $s > 0$. Hence, we have

$$0 < s < \psi\bigl(G(x_n,x_{n+1},x_{n+1})\bigr) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4.4}$$

Let $\varepsilon = s > 0$. Then by hypothesis, there exists a $\delta(\varepsilon) > 0$ such that (4.10) holds. On the other hand, by the definition of ε, there exists $n_0 \in \mathbb{N}$ such that

$$\varepsilon < s_{n_0} = \psi\bigl(G(x_{n_0},x_{n_0+1},x_{n_0+1})\bigr) < \varepsilon + \delta.$$

Now, by (4.10) we have

$$\begin{aligned}
s_{n_0+1} &= \psi\bigl(G(x_{n_0+1},x_{n_0+2},x_{n_0+2})\bigr) \\
&\leq \alpha(x_{n_0+1},x_{n_0+1})\alpha(x_{n_0+2},x_{n_0+2})\alpha(x_{n_0+2},x_{n_0+2})\psi\bigl(G(x_{n_0+1},x_{n_0+2},x_{n_0+2})\bigr) \\
&= \alpha(x_{n_0+1},x_{n_0+1})\alpha(x_{n_0+2},x_{n_0+2})\alpha(x_{n_0+2},x_{n_0+2})\psi\bigl(G(fx_{n_0},fx_{n_0+1},fx_{n_0+1})\bigr) \\
&< \varepsilon = s,
\end{aligned}$$

which is a contradiction. Hence $s = 0$, that is, $\lim_{n\to+\infty} s_n = 0$. Now, by the continuity of ψ in $t = 0$, we have

$$\lim_{n\to+\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

For given $\varepsilon > 0$, by the hypothesis, there exists a $\delta = \delta(\varepsilon) > 0$ such that (4.10) holds. Without loss of generality, we assume $\delta < \varepsilon$. Since $s = 0$, then there exists $N \in \mathbb{N}$ such that

$$s_{n-1} = \psi\big(G(x_{n-1}, x_n, x_n)\big) < \delta \quad \text{for all } n \geq N. \tag{4.5}$$

We will prove that for any fixed $k \geq N_0$,

$$\psi\big(G(x_k, x_{k+l}, x_{k+l})\big) \leq \varepsilon \quad \text{for all } l \in \mathbb{N} \tag{4.6}$$

holds. Note that (4.6), by (4.5), holds for $l = 1$. Suppose condition (4.10) is satisfied for some $m \in \mathbb{N}$. For $l = m + 1$, by (G5) and (4.5), we get

$$\begin{aligned}\psi\big(G(x_{k-1}, x_{k+m}, x_{k+m})\big) &\leq \psi\big(G(x_{k-1}, x_k, x_k) + G(x_k, x_{k+m}, x_{k+m})\big)\\ &\leq \psi\big(G(x_{k-1}, x_k, x_k)\big) + \psi\big(G(x_k, x_{k+m}, x_{k+m})\big)\\ &< \varepsilon + \delta.\end{aligned} \tag{4.7}$$

If $\psi(G(x_{k-1}, x_{k+m}, x_{k+m})) \geq \varepsilon$, then by (4.10) we get

$$\begin{aligned}&\psi\big(G(x_k, x_{k+m+1}, x_{k+m+1})\big)\\ &\quad\leq \alpha(x_k, x_k)\alpha(x_{k+m+1}, x_{k+m+1})\alpha(x_{k+m+1}, x_{k+m+1})\psi\big(G(x_k, x_{k+m+1}, x_{k+m+1})\big)\\ &\quad= \alpha(x_k, x_k)\alpha(x_{k+m+1}, x_{k+m+1})\alpha(x_{k+m+1}, x_{k+m+1})\psi\big(G(fx_{k-1}, fx_{k+m}, fx_{k+m})\big)\\ &\quad< \varepsilon\end{aligned}$$

and hence (4.6) holds.

If $\psi(G(x_{k-1}, x_{k+m}, x_{k+m})) < \varepsilon$, by Remark 15, we get

$$\begin{aligned}&\psi\big(G(x_k, x_{k+m+1}, x_{k+m+1})\big)\\ &\quad\leq \alpha(x_k, x_k)\alpha(x_{k+m+1}, x_{k+m+1})\alpha(x_{k+m+1}, x_{k+m+1})\psi\big(G(x_k, x_{k+m+1}, x_{k+m+1})\big)\\ &\quad\leq \psi\big(G(x_{k-1}, x_{k+m}, x_{k+m})\big) < \varepsilon.\end{aligned}$$

Consequently, (4.6) holds for $l = m + 1$. Hence, $\psi(G(x_k, x_{k+l}, x_{k+l})) \leq \varepsilon$ for all $k \geq N_0$ and $l \geq 1$, which means

$$G(x_n, x_m, x_m) < \varepsilon \quad \text{for all } m \geq n \geq N_0. \tag{4.8}$$

Then, for all $n > m \geq N_0$, by (4.8) and Proposition 7, we have

$$\psi\big(G(x_n, x_m, x_m)\big) \leq \psi\big(2G(x_m, x_n, x_n)\big) = 2\psi\big(G(x_m, x_n, x_n)\big) < 2\varepsilon.$$

That is, for all $m, n \geq N_0$, the following condition holds:

$$\psi\big(G(x_n, x_m, x_m)\big) < 2\varepsilon.$$

Consequently, $\lim_{m,n\to+\infty} \psi(G(x_n, x_m, x_m)) = 0$. By the continuity of ψ in $t = 0$, we get $\lim_{n\to+\infty} G(x_n, x_m, x_m) = 0$. Hence $\{x_n\}$ is a G-Cauchy sequence. Since (X, G) is G-complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} G(x_n, z, z) = \lim_{n\to+\infty} G(x_n, x_n, z) = 0. \tag{4.9}$$

Also, by the continuity of f, we have

$$\lim_{n\to+\infty} G(x_n, fz, fz) = 0$$

and hence

$$G(z, fz, fz) \leq \lim_{n\to+\infty} G(z, x_n, x_n) + \lim_{n\to+\infty} G(x_n, fz, fz) = 0,$$

that is, $z = fz$. □

Theorem 17 *Let* (X, G) *be a G-complete G-metric space and let* f *be a G-*(α, ψ)*-Meir-Keeler contractive mapping. If the following conditions hold*:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, x_0) \geq 1$;

(ii) *if* $\{x_n\}$ *is a sequence in X such that* $\alpha(x_n, x_n) \geq 1$ *for all n and* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x, x) \geq 1$,

then f has a fixed point.

Proof Let $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Following the proof of Theorem 16, we say that $\alpha(x_n, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and that there exists $z \in X$ such that $x_n \to z$ as $n \to +\infty$. Hence, from (ii) $\alpha(z, z) \geq 1$. By Remark 15, we have

$$\begin{aligned}
\psi\big(G(fz, z, z)\big) &\leq \psi\big(G(fz, fx_n, fx_n) + G(fx_n, z, z)\big) \\
&\leq \psi\big(G(fz, fx_n, fx_n)\big) + \psi\big(G(fx_n, z, z)\big) \\
&\leq \alpha(z, z)\alpha(x_n, x_n)\alpha(x_n, x_n)\psi\big(G(fz, fx_n, fx_n)\big) + \psi\big(G(fx_n, z, z)\big) \\
&\leq \psi\big(G(z, x_n, x_n)\big) + \psi\big(G(x_{n+1}, z, z)\big).
\end{aligned}$$

By taking limit as $n \to +\infty$, in the above inequality, we get $\psi(G(fz, z, z)) \leq 0$, that is, $G(fz, z, z) = 0$. Hence $fz = z$. □

Theorem 18 *Assume that all the hypotheses of Theorem* 16 (*and* 17) *hold. Adding the following conditions*:

(iii) $\alpha(z, z) \geq 1$ *for all* $z \in X$,

we obtain the uniqueness of the fixed point of f.

Proof Suppose that z and z^* are two fixed points of f such that $z \neq z^*$. Then $G(z^*, z, z) > 0$. Now, by Remark 15, we have

$$\psi\big(G(z^*, z, z)\big) \leq \alpha(z^*, z^*)\alpha(z, z)\alpha(z, z)\psi\big(G(fz^*, fz, fz)\big) < \psi\big(G(z^*, z, z)\big),$$

which is a contradiction. Hence, $z = z^*$. □

If in Theorems 17 and 18 we take $\alpha(x, y) = a$ and $\psi(t) = t$ where $a \geq 1$, then we have the following corollary.

Corollary 19 *Let* (X, G) *be a G-complete G-metric space. Suppose that* $f : X \to X$ *is a mapping satisfying the following condition*:

For each $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq G(x, y, z) < \varepsilon + \delta \quad \textit{implies} \quad aG(fx, fy, fz) < \varepsilon \tag{4.10}$$

for all $x, y, z \in X$ *where* $a \geq 1$. *Then* f *has a unique fixed point.*

5 Fixed point in *G*-cone metric spaces

In this section we recall the notion of a cone G-metric [36], we introduce the notion of a G_c^m-Meir-Keeler contractive mapping and establish the result of a fixed point for such class of mappings.

Definition 20 ([3]) Let E be a real Banach space with θ as the zero element and with the norm $\|\cdot\|$. A subset P of E is called a cone if and only if the following conditions are satisfied:

(i) P is closed, nonempty and $P \neq \{\theta\}$;
(ii) $a, b \geq 0$ and $x \in P$ implies $ax + by \in P$;
(iii) $x \in P$ and $-x \in P$ implies $x = \theta$.

Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$; we write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (the interior of P). The cone $P \subset E$ is called normal if there is a positive real number K such that for all $x, y \in E$, $\theta \preceq x \preceq y \Rightarrow \|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P. If $K = 1$, then the cone P is called monotone.

Definition 21 Let $(E, \|\cdot\|)$ be a real Banach space with a monotone solid cone P. A mapping $G_c : X \times X \times X \longrightarrow E$ satisfying the following conditions:

(F1) if $x = y = z$, then $G_c(x, y, z) = \theta$;
(F2) $\theta \ll G_c(x, y, y)$ for any $x, y \in X$ with $x \neq y$;
(F3) $G_c(x, x, y) \preceq G_c(x, y, z)$ for any points $x, y, z \in X$, with $y \neq z$;
(F4) $G_c(x, y, z) = G_c(x, z, y) = G_c(y, z, x) = \cdots$, symmetry in all three variables;
(F5) $G_c(x, y, z) \preceq G_c(x, a, a) + G_c(a, y, z)$ for any $x, y, z, a \in X$

is a cone G-metric on X and (X, G_c) is a cone G-metric space.

Lemma 22 ([8, 41]) *Let $(E,\|\cdot\|)$ be a real Banach space with a monotone solid cone P. Then*

$$\theta \preceq x \ll y \quad \Rightarrow \quad \|x\| < \|y\|.$$

Proposition 23 ([8]) *Let $(E,\|\cdot\|)$ be a real Banach space with a monotone solid cone P. If $G_c : X \times X \times X \longrightarrow E$ is a G-cone metric on X, then the function $G : X \times X \times X \longrightarrow [0,+\infty)$ defined by $G(x,y,z) = \|G_c(x,y,z)\|$ is a G-metric on X and (X,G) a G-metric space.*

Definition 24 Let $(E,\|\cdot\|)$ be a real Banach space with a monotone solid cone P and (X,G_c) be a cone G-metric space. Suppose that $f : X \to X$ is a self-mapping satisfying the following condition:

For each $\Upsilon \in \operatorname{int} P$, there exists $\Delta \in \operatorname{int} P$ such that for all $x,y \in X$ and for all $m \in \mathbb{N}$,

$$\begin{cases} \Upsilon - G_c(x,f^{(m)}x,y) \notin \operatorname{int} P, \\ G_c(x,f^{(m)}x,y) - (\Upsilon + \Delta) \notin P, \end{cases} \quad \Rightarrow \quad G_c\bigl(fx,f^{(m+1)}x,fy\bigr) \ll \Upsilon. \tag{5.1}$$

Then f is called a G_c^m-Meir-Keeler contractive mapping.

Theorem 25 *Let $(E,\|\cdot\|)$ be a real Banach space with a monotone solid cone P and (X,G_c) be a G-complete G-cone metric space and f be a G_c^m-Meir-Keeler contractive mapping on X. Then f has a unique fixed point.*

Proof For a given $\varepsilon > 0$, let $\varepsilon \le G(x,f^{(m)}x,y)$, where $G = \|G_c\|$. This implies

$$\frac{\varepsilon H}{\|H\|} - G_c\bigl(x,f^{(m)}x,y\bigr) \notin \operatorname{int} P \tag{5.2}$$

for given $H \in \operatorname{int} P$. Indeed, if $\frac{\varepsilon H}{\|H\|} - G_c(x,f^{(m)}x,y) \in \operatorname{int} P$, then

$$G_c\bigl(x,f^{(m)}x,y\bigr) \ll \frac{\varepsilon H}{\|H\|}$$

and so by Lemma 22, we get $G(x,f^{(m)}x,y) < \varepsilon$, which is a contradiction. Therefore (5.2) holds.

Now suppose that $G(x,f^{(m)}x,y) < \varepsilon + \delta$. This implies

$$G_c\bigl(x,f^{(m)}x,y\bigr) - \left(\frac{\varepsilon H}{\|H\|} + \frac{\delta H}{\|H\|}\right) \notin P. \tag{5.3}$$

Indeed if

$$G_c\bigl(x,f^{(m)}x,y\bigr) - \left(\frac{\varepsilon H}{\|H\|} + \frac{\delta H}{\|H\|}\right) \in P,$$

then

$$(\varepsilon + \delta)\frac{H}{\|H\|} = \frac{\varepsilon H}{\|H\|} + \frac{\delta H}{\|H\|} \preceq G_c\bigl(x,f^{(m)}x,y\bigr)$$

and so $\varepsilon + \delta \le G(x,f^{(m)}x,y)$, which is a contradiction. This implies that (5.3) holds.

Now, by (5.4), (5.2) and (5.3), we have

$$G_c\big(fx,f^{(m+1)}x,fy\big) \ll \frac{\varepsilon H}{\|H\|}.$$

Again, by Lemma 22, we get

$$G\big(fx,f^{(m+1)}x,fy\big) < \varepsilon.$$

Thus f is a G^m-Meir-Keeler contractive mapping, and by Theorem 11, f has a unique fixed point. □

Similarly, we have the following corollary.

Corollary 26 *Let $(E,\|\cdot\|)$ be a real Banach space with a monotone solid cone P and (X,G_c) be a G-complete G-cone metric space and f be a mapping such that for each $\Upsilon \in \operatorname{int} P$, there exists $\Delta \in \operatorname{int} P$ such that*

$$\begin{cases} \Upsilon - G_c(x,y,z) \notin \operatorname{int} P, \\ G_c(x,y,z) - (\Upsilon + \Delta) \notin P, \end{cases} \quad \Rightarrow \quad aG_c(fx,fy,fz) \ll \Upsilon \tag{5.4}$$

for all $x,y \in X$, where $a \geq 1$. Then f has a unique fixed point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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