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Higher-order Euler-type polynomials and their applications

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Abstract

In this paper, we construct generating functions for higher-order Euler-type polynomials and numbers. By using the generating functions, we obtain functional equations related to a generalized partial Hecke operator and Euler-type polynomials and numbers. A special case of higher-order Euler-type polynomials is eigenfunctions for the generalized partial Hecke operators. Moreover, we give not only some properties, but also applications for these polynomials and numbers.

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1 Introduction

In this section, we define generalized partial Hecke operators and we give some notation for these operators. Also, we define generalized Euler-type polynomials, Apostol-Bernoulli polynomials and Frobenius-Euler polynomials.

Throughout this paper, we use the following notations:

$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. We assume that $\ln(z)$ denotes the principal branch of the multi-valued function $\ln(z)$ with an imaginary part $\Im(\ln(z))$ constrained by $-\pi < \Im(\ln(z)) \leq \pi$. Furthermore, $0^n = 1$ if $n = 0$, and $0^n = 0$ if $n \in \mathbb{N}$.

$$N(M) = (N_1, N_2, \ldots, N_M),$$

where $M \in \mathbb{N}$ and $N_1, N_2, \ldots, N_M \in \mathbb{N}$.

Let $a \in \mathbb{N}$ and $\chi_{a,N(M)}$ be a function depending on $a, N_1, N_2, \ldots, N_M$ such that

$$\chi_{a,N(M)} : \mathbb{N}_0 \to \mathbb{C}.$$

$\chi_{a,N(M)}$ is defined by

$$\chi_{a,N(M)}(k) = \prod_{j=1}^{M} \xi^k(N_j),$$

where $0 \leq k \leq a-1$, $j \in \{1,2,\ldots,M\}$ and

$$\xi(N_j) = e^{\frac{2\pi i}{N_j}}.$$

$\chi_{a,N(M)}$ satisfies the following properties:

(i) $\chi_{a,N(M)}$ is a periodic function with $N_1N_2\cdots N_M$.

(ii) If we take $N_1 \geq 2$ and $N_2 = N_3 = \cdots = N_M = 1$, we have

$$\chi_{a,(N_1,1,1,\ldots,1)}(k) = \xi^k(N_1)\xi^k(1)\xi^k(1)\cdots\xi^k(1) = \xi^k(N_1).$$

We note that replacing $N(M)$ by $(N_1,1,1,\ldots,1)$, $\chi_{a,N(M)}$ is reduced to $\xi^k(N_1)$ (*cf.* [1]).

Let $\mathbb{C}[x]$ be a ring of polynomials with complex coefficients. By using $\chi_{a,N(M)}$, we give the following definition.

Definition 1.1 [2] Let $P \in \mathbb{C}[x]$. The generalized partial Hecke operator of $T_{\chi_{a,N(M)}}$ is defined by

$$T_{\chi_{a,N(M)}}\bigl(P(x)\bigr) = \sum_{k=0}^{a-1}\chi_{a,N(M)}(k)P\left(\frac{x+k}{a}\right).$$

The operator $T_{\chi_{a,N(M)}}$ satisfies the following properties:

(i) $T_{\chi_{a,N(M)}}$ is linear on $\mathbb{C}[x]$ and

$$T_{\chi_{a,N(M)}} : \mathbb{C}[x] \to \mathbb{C}[x].$$

(ii) $T_{\chi_{a,N(M)}}$ preserves the degree of the polynomials on $\mathbb{C}[x]$.

(iii) If we take $N_1 \geq 2$ and $N_2 = N_3 = \cdots = N_M = 1$, we have

$$T_{\chi_{a,N_1}}\bigl(P(x)\bigr) = \sum_{k=0}^{a-1}\xi^k(N_1)P\left(\frac{x+k}{a}\right).$$

Remark 1.2 Setting $N(M) = (N_1,1,1,\ldots,1)$, $T_{\chi_{a,(N_1,1,1,\ldots,1)}}$ is reduced to $T_{\chi_{a,N_1}}$ (*cf.* [1]).

The generating function of generalized Euler-type numbers $P_{n,N(M)}$ is given by

$$\mathcal{F}_{N(M)}(t) = \sum_{n=0}^{\infty}P_{n,N(M)}\frac{t^n}{n!} = \frac{\prod_{j=1}^{M}\xi(N_j)-1}{-1+e^t\prod_{j=1}^{M}\xi(N_j)}$$

[2].

Now, we give the definition of Euler-type polynomials as follows.

Definition 1.3 [2] The polynomial $P_{n,N(M)}$ is defined by means of the following generating function:

$$\mathcal{F}_{N(M)}(t,x) = \sum_{n=0}^{\infty}P_{n,N(M)}(x)\frac{t^n}{n!} = \frac{((\prod_{j=1}^{M}\xi(N_j))-1)e^{tx}}{(\prod_{j=1}^{M}\xi(N_j))e^t-1}, \tag{1}$$

where

$$\left| t + \sum_{j=1}^{M} \frac{2\pi i}{N_j} \right| < 2\pi .$$

The polynomial $P_{n,N(M)}$ satisfies the following properties:

(i) $P_{n,N(M)} \in \mathbb{C}[x]$.

(ii) $P_{n,N(M)}$ is a polynomial with degree n and depends on $N_1, N_2, \ldots, N_M$.

(iii) If we take $N_1 \geq 2$ and $N_2 = N_3 = \cdots = N_M = 1$, we have

$$\sum_{n=0}^{\infty} P_{n,N_1}(x) \frac{t^n}{n!} = \frac{(\xi_{N_1} - 1)e^{tx}}{\xi_{N_1} e^t - 1},$$

where

$$\left| t + \frac{2\pi i}{N_1} \right| < 2\pi .$$

(iv) We derive the following functional equation:

$$\mathcal{F}_{N(M)}(t,x) = \mathcal{F}_{N(M)}(t) e^{tx}, \tag{2}$$

so that, obviously,

$$P_{n,N(M)}(0) = P_{n,N(M)}.$$

We now are ready to define Euler-type numbers and polynomials with order k.

Definition 1.4 Euler-type numbers with order k, $P^{(k)}_{n,N(M)}$, are defined by means of the following generating functions:

$$\mathcal{F}^{(k)}_{N(M)}(t) = \sum_{n=0}^{\infty} P^{(k)}_{n,N(M)} \frac{t^n}{n!}, \tag{3}$$

where $k \in \mathbb{N}$ and

$$\left| t + \sum_{j=1}^{M} \frac{2\pi i}{N_j} \right| < 2\pi .$$

Euler-type polynomials with order k are given by the following functional equation:

$$\mathcal{F}^{(k)}_{N(M)}(t,x) = \mathcal{F}^{(k)}_{N(M)}(t) e^{tx} = \sum_{n=0}^{\infty} P^{(k)}_{n,N(M)}(x) \frac{t^n}{n!}. \tag{4}$$

We see that

$$\mathcal{F}^{(0)}_{N(M)}(t,x) = e^{tx}.$$

Thus we obtain

$$P^{(0)}_{n,N(M)}(x) = x^n.$$

Remark 1.5 Substituting $k = 1$ into (4), we get (2). Therefore, (3) reduces to (1); that is,

$$P^{(1)}_{n,N(M)}(x) = P_{n,N(M)}(x)$$

so that, obviously,

$$P^{(1)}_{n,N(M)}(0) = P_{n,N(M)}.$$

By using (4) and (3), we obtain

$$\sum_{n=0}^{\infty} P^{(k)}_{n,N(M)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} P^{(k)}_{n,N(M)}\frac{t^n}{n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}.$$

Therefore, we get the following theorem.

Theorem 1.6

$$P^{(k)}_{n,N(M)}(x) = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}P^{(k)}_{j,N(M)}. \tag{5}$$

Hence, we arrive at the following definition.

Definition 1.7 Euler-type polynomials with order k, $P^{(k)}_{n,N(M)}$, are defined by means of the following generating functions:

$$\mathcal{F}^{(k)}_{N(M)}(t,x) = \sum_{n=0}^{\infty} P^{(k)}_{n,N(M)}(x)\frac{t^n}{n!}, \tag{6}$$

where

$$\left| t + \sum_{j=1}^{M}\frac{2\pi i}{N_j}\right| < 2\pi.$$

Note that there is one generating function for each value of k. These are given explicitly as follows:

$$\begin{aligned}\mathcal{F}^{(k)}_{N(M)}(t,x) &= \left(\frac{-1+\prod_{j=1}^{M}\xi(N_j)}{-1+e^t\prod_{j=1}^{M}\xi(N_j)}\right)^k e^{tx}\\ &= \sum_{n=0}^{\infty} P^{(k)}_{n,N(M)}(x)\frac{t^n}{n!}.\end{aligned}$$

We derive the following functional equation:

$$\mathcal{F}^{(k+l)}_{N(M)}(t,x) = \mathcal{F}^{(k)}_{N(M)}(t,x)\mathcal{F}^{(l)}_{N(M)}(t). \tag{7}$$

By using the above functional equation, we arrive at the following theorem.

Theorem 1.8

$$P^{(k+l)}_{n,N(M)}(x) = \sum_{j=0}^{n} \binom{n}{j} P^{(k)}_{j,N(M)}(x) P^{(l)}_{n-j,N(M)}. \tag{8}$$

Proof By using (3), (6) and (7), we get

$$\sum_{n=0}^{\infty} P^{(k+l)}_{n,N(M)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} P^{(k)}_{j,N(M)}(x) P^{(l)}_{n-j,N(M)} \right) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

Substituting $x = 0$ into (8), we obtain a convolution formula for the numbers by the following corollary.

Corollary 1.9

$$P^{(k+l)}_{n,N(M)} = \sum_{j=0}^{n} \binom{n}{j} P^{(k)}_{j,N(M)} P^{(l)}_{n-j,N(M)}.$$

By differentiating both sides of equation (2) with respect to the variable x, we obtain the following higher-order differential equation:

$$\frac{\partial^j}{\partial x^j} \mathcal{F}_{N(M)}(t,x) = t^j \mathcal{F}_{N(M)}(t,x). \tag{9}$$

Remark 1.10 Setting $N(M) = (N_1, 1, 1, \ldots, 1)$, $P_{n,(N_1,1,1,\ldots,1)}$ is reduced $P_{n,N_1}(x)$ (*cf.* [1]). Therefore $P_{n,N}(x)$ was defined by generalized Bernoulli-Euler polynomials in [1] as follows:

$$\sum_{n=0}^{\infty} P_{n,N}(x) \frac{t^n}{n!} = \begin{cases} \frac{te^{tx}}{e^t-1}, & N = 1, \\ \frac{(\xi_N-1)e^{tx}}{\xi_N e^t-1}, & N \geq 2, \end{cases}$$

so that, obviously,

$$P_{n,1}(x) = B_n(x)$$

and

$$P_{n,2}(x) = E_n(x).$$

Here $B_n(x)$ and $E_n(x)$ are Bernoulli polynomials and Euler polynomials, respectively (*cf.* [1–19]).

The Frobenius-Euler polynomial is defined as follows:

Let u be an algebraic number such that $1 \neq u \in \mathbb{C}$. Then the Frobenius-Euler polynomial $H_n(x,u)$ is defined by

$$\frac{1-u}{e^t-u}e^{tx} = \sum_{n=0}^{\infty} H_n(x,u)\frac{t^n}{n!},$$

where

$$\left|t + \ln\frac{1}{u}\right| < 2\pi$$

(*cf.* [1–19]).

Remark 1.11 Frobenius-Euler number is denoted by $H_n(u)$ such that $H_n(0,u) = H_n(u)$. Also, $H_n(x,-1) = E_n(x)$ (*cf.* [1–19]).

By using Frobenius-Euler numbers, one can obtain the Frobenius-Euler polynomials as follows:

$$H_n(x,u) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} H_j(u)$$

(*cf.* [1–19]).

The Apostol-Bernoulli polynomial is defined as follows.

Definition 1.12 [3, 16] The Apostol-Bernoulli polynomial $\mathcal{B}_n(x,\lambda)$ is defined by

$$\frac{t}{\lambda e^t - 1}e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n(x,\lambda)\frac{t^n}{n!},$$

where λ is the arbitrary real or complex parameter and

$$|t| < |\ln\lambda|.$$

Remark 1.13 For $\lambda = 1$, we obtain that $\mathcal{B}_n(x,1) = B_n(x)$ (*cf.* [1–19]).

2 A functional equation of generalized Euler-type polynomials

Bayad, Aygunes and Simsek showed that for $a \equiv 1 \bmod(N)$, there exists a unique sequence of monic polynomials $(P_{n,N})_{n\in\mathbb{N}_0}$ in $\mathbb{Q}(\xi_N)[x]$ with $\deg P_{n,N} = n$ such that

$$T_{\chi_{a,N}}\big(P_{n,N}(x)\big) = a^{-n}P_{n,N}(x),$$

where $a, N \in \mathbb{N}$ (*cf.* [1]).

In this section, we give the following theorem.

Theorem 2.1 *Let* $a, N_1, N_2, \ldots, N_M \in \mathbb{N}$ *and* $a \equiv 1(\bmod N_1N_2\cdots N_M)$. *Then there exists a sequence* $(P_{n,N(M)})_{n\in\mathbb{N}_0}$ *in*

$$\mathbb{Q}\big(\xi(N_1)\xi(N_2)\cdots\xi(N_M)\big)[x]$$

with

$$\deg P_{n,N(M)} = n$$

such that

$$T_{\chi_{a,N(M)}}\big(P_{n,N(M)}(x)\big) = a^{-n}P_{n,N(M)}(x). \tag{10}$$

Proof Since $P_{n,N(M)} \in \mathbb{C}[x]$ and $T_{\chi_{a,N(M)}} : \mathbb{C}[x] \to \mathbb{C}[x]$, we get

$$T_{\chi_{a,N(M)}}\big(P_{n,N(M)}(x)\big) = \sum_{k=0}^{a-1} \chi_{a,N(M)}(k)P_{n,N(M)}\left(\frac{x+k}{a}\right).$$

From the definition of $\chi_{a,N(M)}(k)$, we have

$$T_{\chi_{a,N(M)}}\big(P_{n,N(M)}(x)\big) = \sum_{k=0}^{a-1}\left(\prod_{j=1}^{M} e^{\frac{2\pi ik}{N_j}}\right)P_{n,N(M)}\left(\frac{x+k}{a}\right).$$

By using the generating function of $P_{n,N(M)}(x)$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{a-1}\left(\prod_{j=1}^{M} e^{\frac{2\pi ik}{N_j}}\right)P_{n,N(M)}\left(\frac{x+k}{a}\right)\frac{t^n}{n!} \\
&\quad= \sum_{k=0}^{a-1}\left(\prod_{j=1}^{M} e^{\frac{2\pi ik}{N_j}}\right)\sum_{n=0}^{\infty}P_{n,N(M)}\left(\frac{x+k}{a}\right)\frac{t^n}{n!} \\
&\quad= \sum_{k=0}^{a-1}\left(\prod_{j=1}^{M} e^{\frac{2\pi ik}{N_j}}\right)\frac{((\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}}) - 1)e^{t(\frac{x+k}{a})}}{(\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}})e^t - 1} \\
&\quad= \frac{((\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}}) - 1)e^{\frac{tx}{a}}}{(\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}})e^t - 1}\sum_{k=0}^{a-1}\left(\exp\left(\sum_{j=1}^{M} e^{\frac{2\pi ik}{N_j}}\right)\right)\exp\left(\frac{tk}{a}\right) \\
&\quad= \frac{((\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}}) - 1)e^{\frac{tx}{a}}}{(\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}})e^t - 1}\sum_{k=0}^{a-1}\left(\exp\left(\frac{t}{a} + \sum_{j=1}^{M}\frac{2\pi i}{N_j}\right)\right)^k \\
&\quad= \frac{((\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}}) - 1)e^{\frac{tx}{a}}}{(\prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}})e^t - 1}\,\frac{e^t(\exp(\sum_{j=1}^{M}\frac{2\pi i}{N_j}))^a - 1}{e^{\frac{t}{a}}(\exp(\sum_{j=1}^{M}\frac{2\pi i}{N_j})) - 1}.
\end{aligned}$$

Since $a \equiv 1(\operatorname{mod} N_1N_2\cdots N_M)$, the following relation holds:

$$\left(\exp\left(\sum_{j=1}^{M}\frac{2\pi i}{N_j}\right)\right)^a = \exp\left(\sum_{j=1}^{M}\frac{2\pi i}{N_j}\right) = \prod_{j=1}^{M} e^{\frac{2\pi i}{N_j}}.$$

Therefore, we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{a-1}\left(\prod_{j=1}^{M} e^{\frac{2\pi ik}{N_j}}\right)P_{n,N(M)}\left(\frac{x+k}{a}\right)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty} a^{-n}P_{n,N(M)}(x)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

Remark 2.2 A different proof of (10) is given in [2]. If we take $N_1 \geq 2$ and $N_2 = N_3 = \cdots = N_M = 1$, we have the following functional equation:

$$T_{\chi_{a,N_1}}\left(P_{n,N_1}(x)\right) = a^{-n}P_{n,N_1}(x)$$

which is satisfied for generalized Bernoulli-Euler polynomials in [1].

3 Some properties of generalized Euler-type polynomials

In this section, we obtain some relations between generalized Euler-type polynomials, Apostol-Bernoulli polynomials and Frobenius-Euler polynomials. Also, we give a formula to obtain the generalized Euler-type polynomials.

Theorem 3.1 *Let $n \in \mathbb{N}$. Then we have*

$$P_{n+1,N(M)}(x) = P_{n,N(M)}(x) + \frac{\prod_{j=1}^{M}\xi(N_j)}{1-\prod_{j=1}^{M}\xi(N_j)}\sum_{k=0}^{n}\binom{n}{k}P_{k,N(M)}^{(2)}(x).$$

Proof By differentiating both sides of equation (2) with respect to the variable t, we have

$$\begin{aligned}\sum_{n=0}^{\infty}P_{n+1,N(M)}(x)\frac{t^n}{n!} &= \frac{\partial}{\partial t}\mathcal{F}_{N(M)}(t,x)\\ &= \mathcal{F}_{N(M)}(t,x) + \left(\frac{\prod_{j=1}^{M}\xi(N_j)}{1-\prod_{j=1}^{M}\xi(N_j)}\right)e^t e^{tx}\left(\mathcal{F}_{N(M)}(t)\right)^2\\ &= \sum_{n=0}^{\infty}P_{n,N(M)}(x)\frac{t^n}{n!} + \left(\frac{\prod_{j=1}^{M}\xi(N_j)}{1-\prod_{j=1}^{M}\xi(N_j)}\right)e^t\left(\sum_{n=0}^{\infty}P_{n,N(M)}^{(2)}(x)\frac{t^n}{n!}\right).\end{aligned}$$

Therefore, we obtain

$$\sum_{n=0}^{\infty}P_{n+1,N(M)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(P_{n,N(M)}(x) + \frac{\prod_{j=1}^{M}\xi(N_j)}{1-\prod_{j=1}^{M}\xi(N_j)}\sum_{k=0}^{n}\binom{n}{k}P_{k,N(M)}^{(2)}(x)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

In the following theorem, we give a relation between the polynomials $P_{n,N(M)}(x)$ and Frobenius-Euler polynomials.

Theorem 3.2 [2] *Let $n \in \mathbb{N}_0$. Then we have*

$$P_{n,N(M)}(x) = H_n\left(x, \prod_{j=1}^{M}\frac{1}{\xi(N_j)}\right).$$

Proof By using the generating function of $P_{n,N(M)}(x)$, we have

$$\sum_{n=0}^{\infty} P_{n,N(M)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n\left(x, \prod_{j=1}^{M} \frac{1}{\xi(N_j)}\right)\frac{t^n}{n!}.$$

In the above equation, if we compare the coefficients of $\frac{t^n}{n!}$, we get the desired result. □

In the following theorem, we give a relation between $P_{n,N(M)}(x)$ and Apostol-Bernoulli polynomials.

Theorem 3.3 [2] *Let $n \in \mathbb{N}$. Then we have*

$$P_{n-1,N(M)}(x) = \left(\prod_{j=1}^{M} \xi(N_j) - 1\right)\frac{1}{n}\mathcal{B}_n\left(x, \prod_{j=1}^{M} \xi(N_j)\right).$$

Proof We arrange the generating function of generalized Euler-type polynomials as follows:

$$\sum_{n=1}^{\infty} P_{n-1,N(M)}\frac{t^{n-1}}{(n-1)!} = \frac{\prod_{j=1}^{M} \xi(N_j) - 1}{e^t \prod_{j=1}^{M} \xi(N_j) - 1} e^{xt}.$$

Therefore, we have

$$\sum_{n=1}^{\infty} P_{n-1,N(M)}\frac{t^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\left(\prod_{j=1}^{M} \xi(N_j) - 1\right)\mathcal{B}_n\left(x, \prod_{j=1}^{M} \xi(N_j)\right)\right)\frac{t^{n-1}}{(n-1)!}.$$

In the above equation, if we compare the coefficients of $\frac{t^{n-1}}{(n-1)!}$, we get the desired result. □

In the following theorem, it is possible to find the generalized Euler-type polynomials.

Theorem 3.4 *Let $n \in \mathbb{N}_0$. Then we have*

$$P_{n,N(M)}(x) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} P_{j,N(M)}. \tag{11}$$

Proof of (11) is the same as that of (5), so we omit it [2].

$$P_{1,N(M)} = \frac{1}{\chi_{a,N(M)}^{-1} - 1},$$

$$P_{2,N(M)} = \frac{2}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{1}{\chi_{a,N(M)}^{-1} - 1},$$

$$P_{3,N(M)} = \frac{6}{(\chi_{a,N(M)}^{-1} - 1)^3} + \frac{6}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{1}{\chi_{a,N(M)}^{-1} - 1}$$

and

$$P_{4,N(M)} = \frac{24}{(\chi_{a,N(M)}^{-1} - 1)^4} + \frac{36}{(\chi_{a,N(M)}^{-1} - 1)^3} + \frac{14}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{1}{\chi_{a,N(M)}^{-1} - 1}.$$

By using (11), we have the following list for the generalized Euler-type polynomials:

$$\begin{aligned}
P_{0,N(M)}(x) &= 1, \\
P_{1,N(M)}(x) &= x + \frac{1}{\chi_{a,N(M)}^{-1} - 1}, \\
P_{2,N(M)}(x) &= x^2 + x\left(\frac{2}{\chi_{a,N(M)}^{-1} - 1}\right) + \left(\frac{2}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{1}{\chi_{a,N(M)}^{-1} - 1}\right), \\
P_{3,N(M)}(x) &= x^3 + x^2\left(\frac{3}{\chi_{a,N(M)}^{-1} - 1}\right) + x\left(\frac{6}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{3}{\chi_{a,N(M)}^{-1} - 1}\right) \\
&\quad + \left(\frac{6}{(\chi_{a,N(M)}^{-1} - 1)^3} + \frac{6}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{1}{\chi_{a,N(M)}^{-1} - 1}\right)
\end{aligned}$$

and

$$\begin{aligned}
P_{4,N(M)}(x) &= x^4 + x^3\left(\frac{4}{\chi_{a,N(M)}^{-1} - 1}\right) + x^2\left(\frac{12}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{6}{\chi_{a,N(M)}^{-1} - 1}\right) \\
&\quad + x\left(\frac{24}{(\chi_{a,N(M)}^{-1} - 1)^3} + \frac{24}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{4}{\chi_{a,N(M)}^{-1} - 1}\right) \\
&\quad + \left(\frac{24}{(\chi_{a,N(M)}^{-1} - 1)^4} + \frac{36}{(\chi_{a,N(M)}^{-1} - 1)^3} + \frac{14}{(\chi_{a,N(M)}^{-1} - 1)^2} + \frac{1}{\chi_{a,N(M)}^{-1} - 1}\right).
\end{aligned}$$

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author completed the paper himself. The author read and approved the final manuscript.

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