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On residual algebraic torsion extensions of a valuation of a field K to $K(x_1,\ldots,x_n)$

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Abstract

Let v be a valuation of a field K with a value group G_v and a residue field k_v, w be an extension of v to $K(x)$. Then w is called a residual algebraic torsion extension of v to $K(x)$ if k_w/k_v is an algebraic extension and G_w/G_v is a torsion group. In this paper, a residual algebraic torsion extension of v to $K(x_1,\ldots,x_n)$ is described and its certain properties are investigated. Also, the existence of a residual algebraic torsion extension of a valuation on K to $K(x_1,\ldots,x_n)$ with given residue field and value group is studied.

MSC: 12J10; 12J20; 12F20

Keywords: extensions of valuations; residual algebraic torsion extensions; valued fields; value group; residue field

1 Introduction

Let K be a field, v be a valuation on K with a value group G_v and a residue field k_v. The big target is to define all extensions of v to $K(x_1,\ldots,x_n)$. Residual transcendental extensions of v to $K(x)$ are described by Popescu, Alexandru and Zaharescu in [1, 2]. Residual algebraic torsion extensions of v to $K(x)$ are studied for the first time in [3]. A residual transcendental extension of v to $K(x_1,\ldots,x_n)$ is defined in [4] by Öke. These studies are summarized in the second section. The paper is aimed to study residual algebraic torsion extensions of v to $K(x_1,\ldots,x_n)$. In the third section, a residual algebraic torsion extension of v to $K(x_1,\ldots,x_n)$ is defined and certain properties of the residual algebraic torsion extensions given in [3] are generalized. In the last section, the existence of an r.a.t. extension of v to $K(x_1,\ldots,x_n)$ with given residue field and value group is demonstrated.

2 Preliminaries and some notations

Throughout this paper, v is a valuation of a field K with a value group G_v, a valuation ring O_v and a residue field k_v, $\overline{K}$ is an algebraic closure of K, $\overline{v}$ is a fixed extension of v to $\overline{K}$. The value group of $\overline{v}$ is the divisible closure of G_v and its residue field is the algebraic closure of k_v. $K(x)$ and $K(x_1,\ldots,x_n)$ are rational function fields over K with one and n variables respectively. For any α in O_v, α^* denotes its natural image in k_v. If $a_1,\ldots,a_n \in \overline{K}$, then the restriction of $\overline{v}$ to $K(a_1,\ldots,a_n)$ will be denoted by $v_{a_1\cdots a_n}$.

Let w be an extension of v to $K(x)$. Then w is called a residual transcendental (r.t.) extension of v if k_w/k_v is a transcendental extension.

The valuation w, which is defined for each $F = \sum_t a_t x^t \in K[x]$ as $w(F) = \inf_t(\nu(a_t))$ is called Gauss extension of ν to $K(x)$, its residue field is $k_w = k_\nu(x^*)$, is the simple transcendental extension of k_ν and $G_w = G_\nu$ [5].

The valuation $\overline{w}$, which is defined for each $F = \sum_t c_t(x-a)^t \in \overline{K}[x]$ as

$$\overline{w}(F) = \inf_t\bigl(\bar{\nu}(c_t) + t\delta\bigr) \tag{1}$$

is called a valuation defined by the pair $(a,\delta) \in \overline{K} \times G_{\bar{\nu}}$ or $(a,\delta) \in \overline{K} \times G_{\bar{\nu}}$ is called a pair of definitions of w. Also, w is an r.t. extension of ν. If $[K(a):K] \leq [K(b):K]$ for every $b \in \overline{K}$ such that $\bar{\nu}(b-a) \geq \delta$, then (a,δ) is called a minimal pair with respect to K [2].

If w is an r.t. extension of ν to $K(x)$, there exists a minimal pair $(a,\delta) \in \overline{K} \times G_{\bar{\nu}}$ such that a is separable over K. Two pairs (a_1,δ_1) and (a_2,δ_2) define the same valuation w if and only if $\delta_1 = \delta_2$ and $\bar{\nu}(a_1 - a_2) \geq \delta_1$ [2]. Let $f = \operatorname{Irr}(a,K)$ be the minimal polynomial of a with respect to K and $\gamma = w(f)$. For each $F \in K[x]$, let $F = F_1 + F_2 f + \cdots + F_n f^n$, where $F_t \in K[x]$, $\deg F_t < \deg f$, $t = 1,\ldots,n$, be the f-expansion of F. Then w is defined as follows:

$$w(F) = \inf_t\bigl(\nu_a\bigl(F_t(a)\bigr) + t\gamma\bigr). \tag{2}$$

Then $G_w = G_{\nu_a} + Z\gamma$. Let e be the smallest non-zero positive integer such that $e\gamma \in G_{\nu_a}$. Then there exists $h \in K[x]$ such that $\deg h < \deg f$, $\nu_a(h(a)) = e\gamma$ and $r = f^e/h$ is an element of O_w and $r^* \in k_w$ is transcendental over k_ν. k_{ν_a} can be identified canonically with the algebraic closure of k_ν in k_w and $k_w = k_{\nu_a}(r^*)$ [2].

Let w be an extension of ν to $K(x)$. w is called a residual algebraic (r.a.) extension of ν if k_w/k_ν is an algebraic extension. If w is an r.a. extension of ν to $K(x)$ and G_w/G_ν is not a torsion group, then w is called a residual algebraic free (r.a.f.) extension of ν. In this case, the quotient group G_w/G_ν is a free abelian group. More precisely, G_w/G_ν is isomorphic to Z [3].

w is called a residual algebraic torsion (r.a.t) extension of ν if w is an r.a. extension of ν and G_w/G_ν is a torsion group. In this case, $G_\nu \subseteq G_w \subseteq G_{\bar{\nu}}$ is satisfied [3].

The order relation on the set of all r.t. extensions of ν to $K(x)$ is defined as follows: $w_1 \leq w_2 \Leftrightarrow w_1(f) \leq w_2(f)$ for all polynomials $f \in K[x]$. If $w_1 \leq w_2$ and there exists $f \in K[x]$ such that $w_1(f) < w_2(f)$, then it is written $w_1 < w_2$. Let $(a_1,\delta_1), (a_2,\delta_2) \in \overline{K} \times G_{\bar{\nu}}$ be minimal pairs of the definition of the r.t. extensions w_1 and w_2 of ν to $K(x)$, respectively. Then $w_1 \leq w_2$ if and only if $\delta_1 \leq \delta_2$ and $\bar{\nu}(a_1 - a_2) \geq \delta_1$; moreover, $w_1 < w_2$ if and only if $\delta_1 \leq \delta_2$ and $\nu(a_1 - a_2) > \delta_1$ [3].

Let I be a well-ordered set without the last element and $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of ν to $K(x)$, where w_i is defined by a minimal pair $(a_i,\delta_i) \in \overline{K} \times G_{\bar{\nu}}$ for all $i \in I$. If $w_i \leq w_j$ for all $i < j$, then $(w_i)_{i\in I}$ is called an ordered system of r.t. extensions of ν to $K(x)$.

Then the valuation of $K(x)$ defined as

$$w(f) = \sup_i\bigl(w_i(f)\bigr) \tag{3}$$

for all $f \in K[x]$ is an extension of ν to $K(x)$ and it is called a limit of the ordered system $(w_i)_{i\in I}$. w may not be an r.t. extension of ν to $K(x)$ [3].

Using the above studies an r.a.t extension of v to $K(x_1,\dots,x_n)$ can be defined. For this reason the r.t. extension of v to $K(x_1,\dots,x_n)$ defined in [4] can be used. An r.t. extension of v to $K(x_1,\dots,x_n)$ is defined by using r.t. extensions of v to $K(x_m)$ for $m = 1,\dots,n$ in [4].

Let u_m be an r.t. extension of v to $K(x_m)$ defined by a minimal pair $(a_m,\delta_m) \in \overline{K} \times G_{\bar{v}}$ for $m = 1,\dots,n$, where $[K(a_1,\dots,a_n) : K] = \prod_{m=1}^{n}[K(a_m) : K]$ and $f_m = \operatorname{Irr}(a_m,K)$, $\gamma_m = u_m(f_m)$ for $m = 1,\dots,n$. Each polynomial $F \in K[x_1,\dots,x_n]$ can be uniquely written as $F = \sum_{t_1,\dots,t_n} F_{t_1\cdots t_n} f_1^{t_1}\cdots f_n^{t_n}$, where $F_{t_1\cdots t_n} \in K[x_1,\dots,x_n]$, $\deg_{x_m} F_{t_1\cdots t_n} < \deg f_m$ for $m = 1,\dots,n$.

The valuation w defined as

$$u(F) = \inf_{t_1,\dots,t_n}\bigl(v_{a_1\cdots a_n}\bigl(F_{t_1t_2\cdots t_n}(a_1,\dots,a_n)\bigr) + t_1\gamma_1 + \cdots + t_n\gamma_n\bigr) \tag{4}$$

is an extension of v to $K(x_1,\dots,x_n)$. u is an r.t. extension of v which is a common extension of $u_1,\dots,u_n$ to $K(x_1,\dots,x_n)$. Then $G_u = G_{v_{a_1\cdots a_n}} + Z\gamma_1 + \cdots + Z\gamma_n$. Let e_m be the smallest positive integer such that $e_m\gamma_m \in G_{v_{a_m}}$, where v_{a_m} is the restriction of $\bar{v}$ to $K(a_m)$. Then there exists $h_m \in K[x_m]$ such that $\deg h_m < \deg f_m$, $v_{a_m}(h(a_m)) = e_m\gamma_m$, $r_m = f_m^{e_m}/h_m \in O_{u_m}$ and r_m^* is transcendental over k_v for $m = 1,\dots,n$. $k_{v_{a_1\cdots a_n}}$ can be canonically identified with the algebraic closure of k_v in k_w and $k_u = k_{v_{a_1\cdots a_n}}(r_1^*,\dots,r_n^*)$ [4].

In the next section, an r.a.t extension of v to $K(x_1,\dots,x_n)$ will be defined by using that r.t. extension.

3 A residual algebraic torsion extension of v to $K(x_1,\dots,x_n)$

Let u_m be an r.t. extension of v to $K(x_m)$ defined by a minimal pair $(a_m,\delta_m) \in \overline{K} \times G_{\bar{v}}$ for $m = 1,\dots,n$, where $[K(a_1,\dots,a_n) : K] = \prod_{m=1}^{n}[K(a_m) : K]$ and let u be the r.t. extension of v to $K(x_1,\dots,x_n)$ defined as in (4). Let u'_m be an r.t. extension of v to $K(x_m)$ defined by a minimal pair $(a'_m,\delta'_m) \in \overline{K} \times G_{\bar{v}}$ for $m = 1,\dots,n$, where $[K(a'_1,\dots,a'_n) : K] = \prod_{m=1}^{n}[K(a'_m) : K]$ and let u' be the r.t. extension of v to $K(x_1,\dots,x_n)$ defined as in (4). A relation between such kind of r.t. extensions of v to $K(x_1,\dots,x_n)$ can be defined so that $u \le u'$ if and only if $u_m \le u'_m$ for $m = 1,\dots,n$. This is an order relation, and if $u \le u'$, then for each polynomial $F \in K[x_1,\dots,x_n]$, $u(F) \le u'(F)$ is satisfied. Because, for $F = \sum_{t_1,\dots,t_n} d_{t_1\cdots t_n} x_1^{t_1}\cdots x_n^{t_n} \in K[x_1,\dots,x_n]$,

$$\begin{aligned} u(F) &= \inf_{t_1,\dots,t_n}\bigl(v(d_{t_1\cdots t_n}) + t_1u_1(x_1) + \cdots + t_nu_n(x_n)\bigr) \\ &\le \inf_{t_1,\dots,t_n}\bigl(v(d_{t_1\cdots t_n}) + t_1u'_1(x_1) + \cdots + t_nu'_n(x_n)\bigr) = u'(F). \end{aligned}$$

Now, let I be a well-ordered set without the last element and $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of v to $K(x_1,\dots,x_n)$, where w_i is defined as in (4), *i.e.*, w_i is the common extension of w_{i_m}, where w_{i_m} is the r.t. extension of v to $K(x_m)$ defined by the minimal pair $(a_{i_m},\delta_{i_m}) \in \overline{K} \times G_{\bar{v}}$, where $[K(a_{i_1},\dots,a_{i_n}) : K] = \prod_{m=1}^{n}[K(a_{i_m}) : K]$ for all $i \in I$. If $w_i \le w_j$ for all $i < j$, then $(w_i)_{i\in I}$ is an ordered system of r.t. extensions of v to $K(x_1,\dots,x_n)$. Then the valuation w of $K(x_1,\dots,x_n)$ defined as

$$w(F) = \sup_i\bigl(w_i(F)\bigr) \tag{5}$$

for all $F \in K[x_1,\dots,x_n]$ is an extension of v to $K(x_1,\dots,x_n)$ and it is called a limit of the ordered system $(w_i)_{i\in I}$.

If w_m is the restriction of w to $K(x_m)$ for $m = 1, \dots, n$, then w_m is the limit of the ordered system $(w_{i_m})_{i \in I}$ of r.t. extensions of v to $K(x_m)$. Also, w is the common extension of $w_1, \dots, w_n$ to $K(x_1, \dots, x_n)$. Since w_m may not be an r.t. extension of v to $K(x_m)$, then w may not be an r.t. extension of v to $K(x_1, \dots, x_n)$.

If $w = \sup_i w_i$ is a residual algebraic torsion extension of v to $K(x_1, \dots, x_n)$, then $G_v \subseteq G_w \subseteq G_{\bar{v}}$ is satisfied. Some other properties of w are studied below.

Denote the extension of w_i to $\overline{K}(x_1, \dots, x_n)$ by $\overline{w}_i$ and the extension of w_{i_m} to $\overline{K}(x_m)$ by $\overline{w}_{i_m}$ for $m = 1, \dots, n$ and for all $i \in I$.

Theorem 3.1 *Let $(\overline{w}_i)_{i \in I}$ be an ordered system of r.t. extensions of $\bar{v}$ to $\overline{K}(x_1, \dots, x_n)$, where $\overline{w}_i$ is defined as in* (4), *i.e., w_i is the r.t. extension of v to $\overline{K}(x_1, \dots, x_n)$ which is the common extension of w_{i_m} for $m = 1, \dots, n$ and for all $i \in I$. Denote the restriction of $\overline{w}_i$ to $K(x_1, \dots, x_n)$ by w_i and the restriction of $\bar{v}$ to $K(a_{i_1}, \dots, a_{i_n})$ by $v_i = v_{a_{i_1} \cdots a_{i_n}}$. Then*

1. *For all $i, j \in I$, $i < j$, one has $w_i < w_j$, i.e., $(w_i)_{i \in I}$ is an ordered system of r.t. extensions of v to $K(x_1, \dots, x_n)$.*
2. *For all $i, j \in I$, $i < j$, one has $k_{v_i} \subseteq k_{v_j}$ and $G_{v_i} \subseteq G_{v_j}$.*
3. *Suppose that $\overline{w} = \sup_i \overline{w}_i$ and $\overline{w}$ is not an r.t. extension of $\bar{v}$ to $\overline{K}(x_1, \dots, x_n)$ and denote that w is the restriction of $\overline{w}$ to $K(x_1, \dots, x_n)$. Then $w = \sup_i w_i$ and $k_w = \bigcup_i k_{v_i}$ and $G_w = \bigcup_i G_{v_i}$.*

Proof For every $i \in I$ and $m = 1, \dots, n$, denote that $f_{i_m} = \operatorname{Irr}(a_{i_m}, K)$ and $\deg_{x_m} f_{i_m} = n_{i_m}$.

1. Since $\overline{w}_i < \overline{w}_j$ for all $i, j \in I$, $i < j$, we have $w_i \leq w_j$. We show that $w_i < w_j$. Assume that $w_i = w_j$. Since w_i is the common extension of w_{i_m} and w_j is the common extension of w_{j_m} for $m = 1, \dots, n$, we have $w_{i_m} = w_{j_m}$ for $m = 1, \dots, n$. Since (a_{i_m}, δ_{i_m}) is a minimal pair of the definition of w_{i_m}, we have $\delta_{i_m} = \delta_{j_m}$ for $m = 1, \dots, n$. But it is a contradiction, because $(\overline{w}_{i_m})_{i \in I}$ is an ordered system of r.t. extensions of $\bar{v}$ to $\overline{K}(x_m)$ and so $\overline{w}_{i_m} < \overline{w}_{j_m}$, *i.e.*, $\delta_{i_m} < \delta_{j_m}$ [3]. Hence $w_{i_m} < w_{j_m}$ for $m = 1, \dots, n$. Since w_i and w_j are common extensions of w_{i_m} and w_{j_m} respectively for $m = 1, \dots, n$ and for all $i \in I$, it is concluded that $w_i < w_j$ for all $i < j$.
2. It is enough to study for $B = F(a_{i_1}, \dots, a_{i_n}) \in K[a_{i_1}, \dots, a_{i_n}]$, where $F(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$, $\deg_{x_m} F(x_1, \dots, x_n) < n_{i_m}$.

   It is seen that $v_i(B) = v_i(F(a_{i_1}, \dots, a_{i_n})) = \bar{v}(F(a_{i_1}, \dots, a_{i_n})) = v_j(F(a_{j_1}, \dots, a_{j_n})) = v_j(B)$ by using the [3, Th. 2.3] and this gives $G_{v_i} \subseteq G_{v_j}$.

   Assume that $v_i(B) = 0$. Then $v_j(B) = 0$. Since (a_{i_m}, δ_{i_m}) is a minimal pair of the definition of w_{i_m} for $m = 1, \dots, n$, we have $B^* = F(a_{i_1}, \dots, a_{i_{m-1}}, x_m, a_{i_{m+1}}, \dots, a_{i_n})^* = F(a_{i_1}, \dots, a_{i_{m-1}}, a_{i_m}, a_{i_{m+1}}, \dots, a_{i_n})^* \in k_{v_i}$ coincides with the $F(a_{j_1}, \dots, a_{j_{m-1}}, a_{j_m}, a_{j_{m+1}}, \dots, a_{j_n})^*$ which is the residue of B in k_{v_j}. Hence $k_{v_i} \subseteq k_{v_j}$ for all $i, j \in I$, $i < j$.
3. Since $\overline{w} = \sup_i \overline{w}_i$, we have $w = \sup_i w_i$ and w is not an r.t. extension of v to $K(x_1, \dots, x_n)$. Using [3, Th. 2.3] and the definition of w_{i_m}, the proof can be completed. Take $F(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ such that $\deg_{x_m} F < n_{i_m}$. Since (a_{i_m}, δ_{i_m}) is a minimal pair of the definition of w_{i_m}, $\overline{w}(F(x_1, \dots, x_n)) = w(F(x_1, \dots, x_n)) = \bar{v}(F(a_{i_1}, \dots, a_{i_n})) = v_i(F(a_{i_1}, \dots, a_{i_n}))$. This means that $G_{v_i} \subseteq G_w$ for all $i \in I$ and so $\bigcup_i G_{v_i} \subseteq G_w$.

   Conversely, let v_i^m be the restriction of $\overline{w}_i$ to $K(a_{i_1}, \dots, a_{i_{m-1}}, x_m, a_{i_{m+1}}, \dots, a_{i_n})$ for $m = 1, \dots, n$ and for all $i \in I$. Since

$v_i^m(P(a_{i_1},\dots,a_{i_{m-1}},x_m,a_{i_{m+1}},\dots,a_{i_n})) = v_i(P(a_{i_1},\dots,a_{i_m},\dots,a_{i_n}))$, then $w(P(x_1,\dots,x_n)) = v_i(P(a_{i_1},\dots,a_{i_n})) \in G_{v_i}$ for every $P(x_1,\dots,x_n) \in K[x_1,\dots,x_n]$. This gives $G_w \subseteq \bigcup_i G_{v_i}$.

Now, assume that $\bar{v}(F(a_{i_1},\dots,a_{i_n})) = v_i(F(a_{i_1},\dots,a_{i_m})) = 0$. Then $w(F(x_1,\dots,x_n)) = 0$ and since $\deg_{x_m} F(x_1,\dots,x_n) < n_{i_m}$, $F(a_1,\dots,a_n)^*$, which is v_i-residue of $F(x_1,\dots,x_n)$, coincides with the residue of $F(x_1,\dots,x_n)$ in k_w. This shows $k_{v_i} \subseteq k_w$ for all $i \in I$, and then $\bigcup_i k_{v_i} \subseteq k_w$.

For the reverse inclusion, let $P(x_1,\dots,x_n) \in K[x_1,\dots,x_n]$ and $w(P(x_1,\dots,x_n)) = 0$. For $m = 1,\dots,n$, $P(a_{i_1},\dots,a_{i_{m-1}},x_m,a_{i_{m+1}},\dots,a_{i_n})^*$ is equal to $P(a_{i_1},\dots,a_{i_{m-1}},a_{i_m},a_{i_{m+1}},\dots,a_{i_n})^* \in k_{v_i}$ and so $P(x_1,\dots,x_n)^* = P(a_{i_1},\dots,a_{i_n})^* \in k_{v_i}$. Hence $k_w \subseteq \bigcup_i k_{v_i}$.

□

The following theorem can be obtained as a result of Theorem 3.1.

Corollary 3.2 *Under the above notations, let w be an r.a.t. extension of v to $K(x_1,\dots,x_n)$. Then the following are satisfied*:

1. $G_{v_i} \subseteq G_{v_j}$ *and* $k_{v_i} \subseteq k_{v_j}$ *for all* $i,j \in I$, $i < j$.
2. $(w_i)_{i\in I}$ *is an ordered system of r.t. extensions of v to $K(x_1,\dots,x_n)$ and* $w = \sup_i w_i$. *Moreover, we have* $k_w = \bigcup_i k_{v_i}$ *and* $G_w = \bigcup_i G_{v_i}$.

Proof If w is an r.a.t. extension of v to $K(x_1,\dots,x_n)$, then $\overline{w}$ is an r.a.t. extension $\bar{v}$ to $\overline{K}(x_1,\dots,x_n)$ and so $\overline{w}_m$ is an r.a.t. extension of $\bar{v}$ to $\overline{K}(x_m)$ for $m = 1,\dots,n$. We can take $\{\delta_{i_m}\}_{i\in I}$ for $m = 1,\dots,n$ as co-final well-ordered subsets of $M_{\overline{w}_m} = \{\overline{w}(x_m - a) | a \in \overline{K}\}$. I has no last element because $\overline{w}_m$ is not an r.t. extension of $\bar{v}$. For every $i \in I$, choose the element $(a_{i_m},\delta_{i_m}) \in \overline{K} \times G_{\bar{v}}$ such that for $m = 1,\dots,n$, $\overline{w}(x_m - a_{i_m}) = \delta_{i_m}$ and $[K(a_{i_m}) : K]$ is the smallest possible for δ_{i_m}. This means that if $c_m \in \overline{K}$ such that $\overline{w}(x_m - c_m) = \delta_{i_m}$, then $[K(c_m) : K] \geq [K(a_{i_m}) : K]$. Then (a_{i_m},δ_{i_m}) is a minimal pair of the definition of $\overline{w}_{i_m}$ with respect to K for $m = 1,\dots,n$. According to [3, Th. 4.1], $\overline{w}_{i_m} < \overline{w}_{j_m}$ if $i < j$, which means that $(\overline{w}_{i_m})_{i\in I}$ is an ordered system of r.t. extensions of $\bar{v}$ to $\overline{K}(x_m)$ for $m = 1,\dots,n$ and $(\overline{w}_{i_m})_{i\in I}$ has a limit $\overline{w}_m = \sup_i \overline{w}_{i_m}$ which is an r.a.t extension of v to $\overline{K}(x_m)$. For all $i \in I$, take $\overline{w}_i$ as the common extension of $\overline{w}_{i_m}$ to $K(x_1,\dots,x_n)$ and $\overline{w}$ as the common extension of $\overline{w}_m$ to $\overline{K}(x_1,\dots,x_n)$ for $m = 1,\dots,n$. Denote the restriction of $\overline{w}_i$ to $K(x_1,\dots,x_n)$ by w_i and denote the restriction of $\overline{w}$ to $K(x_1,\dots,x_n)$ by w. In the same way as that in the proof of Theorem 3.1, it is seen that $w_i < w_j$ for $i,j \in I$, $i < j$ and $w_i < w$ for all $i \in I$ and $w = \sup_i w_i$. Moreover, $k_w = \bigcup_i k_{v_i}$ and $G_w = \bigcup_i G_{v_i}$ are satisfied. □

4 Existence of r.a.t. extensions of valuations of K to $K(x_1,\dots,x_n)$ with given residue field and value group

It can be concluded from section three and from [3] that if w is an r.a.t. extension of v to $K(x_1,\dots,x_n)$, then k_w/k_v is a countable generated infinite algebraic extension and G_w/G_v is a countable infinite torsion group. In this section, the converse is studied.

Theorem 4.1 *Let k/k_v be a countably generated infinite algebraic extension and G be an ordered group such that $G_v \subset G$ and G/G_v is a countably infinite torsion group. Then there exists an r.a.t. extension w of v to $K(x_1,\dots,x_n)$ such that $k_w \cong k$ and $G_w \cong G$.*

Proof Since $k_{\bar{v}}$ is the algebraic closure of k_v, we have $k_v \subseteq k \subseteq k_{\bar{v}}$. Since k/k_v is countably generated, there exists a tower of fields $k_v \subseteq k_1 \subseteq k_2 \subseteq \cdots$ such that $\bigcup_s k_s = k$, and since G/G_v is a countable torsion group, there exists a sequence of subgroups of G such that $G_v = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_s \cdots \subset G$, $G_s \neq G_{s+1}$, G_s/G_v is finite for all s and that $\bigcup_s G_s = G$. According to [6, Th. 3.2], there exists an r.t. extension u_s of v to $K(x_1,\dots,x_n)$ such that $\operatorname{trans\,deg} k_{u_s}/k_v = n$, the algebraic closure of k_v in k_{u_s} is k_s, $G_{u_s} = G_s$ and if $m \neq m'$, then the restriction of u_s to $K(x_m,x_{m'})$ is not the Gauss extension of the restriction of u_s to $K(x_m)$ for $m,m' = 1,\dots,n$ and for all s. $k_{u_s} = k_s(z_1,\dots,z_n)$, where z_m is transcendental over k_s for $m = 1,\dots,n$ and for all s. Denote the restriction of u_s to $K(x_m)$ by u_{s_m} and the algebraic closure of k_v in $k_{u_{s_m}}$ by k_{s_m} for $m = 1,\dots,n$ and for all s. Then $k_{u_{s_m}} = k_{s_m}(z_m)$, z_m is transcendental over k_{s_m} for $m = 1,\dots,n$ and for all s.

Then $k_v \subseteq k_{1_m} \subseteq k_{2_m} \subseteq \cdots \subseteq k_{s_m} \subseteq \cdots$ is the tower of finite extensions of k_v for $m = 1,\dots,n$. Denote $G_{u_{s_m}} = G_{s_m}$. $G_v \subset G_{1_m} \subset G_{2_m} \subset \cdots \subset G_{s_m} \subset \cdots \subset G$ is the sequence of subgroups of G such that $G_{s_m} \neq G_{(s+1)_m}$ and G_{s_m}/G_v is finite for all s and for $m = 1,\dots,n$. Then there exists an r.a.t. extension w_m of v to $K(x_m)$ such that $k_{w_m} \cong \bigcup_s k_{s_m}$ and $G_{w_m} \cong \bigcup_s G_{s_m}$ [3].

It means that $w_m = \sup_s(u_{s_m})$. Since $x_1,x_2,\dots,x_n$ are algebraic independent over K, $k_{w_1}k_{w_2}/k_{w_1}$ is a countable generated infinite algebraic extension and $\langle G_{w_1} \cup G_{w_2}\rangle/G_{w_1}$ is a countable torsion group. Hence there exists an r.a.t. extension v_2 of $w_1 = v_1$ to $K(x_1,x_2)$ such that $k_{v_2} \cong k_{w_1}k_{w_2}$ and $G_{v_2} \cong \langle G_{w_1} \cup G_{w_2}\rangle$. Using the induction on n, it is obtained that there exits an r.a.t. extension $v_n = w$ of v_{n-1} of $K(x_1,\dots,x_{n-1})$ to $K(x_1,\dots,x_n)$ such that

$$k_w = k_{v_n} \cong k_{w_1}\cdots k_{w_n} = \langle k_{w_1} \cup \cdots \cup k_{w_n}\rangle = \left\langle \bigcup_{m=1}^{n}\left(\bigcup_s k_{u_{s_m}}\right)\right\rangle = \bigcup_s k_{u_s}$$

and

$$G_w = G_{v_n} \cong \langle G_{w_1} \cup \cdots \cup G_{w_n}\rangle = \left\langle \bigcup_{m=1}^{n}\left(\bigcup_s G_{u_{s_m}}\right)\right\rangle = \bigcup_s G_{u_s}.$$

Since v_i is an r.a.t. extension of v_{i-1} for $i = 1,\dots,n$, then $v_n = w$ is an r.a.t. extension of v to $K(x_1,\dots,x_n)$. □

Theorem 4.2 *Let k/k_v be a finite extension, G be an ordered group such that $G_v \subset G$ and G/G_v is finite. Assume that* $\operatorname{tr\,deg}\tilde{K}/K > 0$. *Then there exists an r.a.t. extension of v to $K(x_1,\dots,x_n)$ such that $k_w \cong k$ and $G_w \cong G$.*

Proof Since k/k_v is a finite extension, it can be written that $k = k_v(b_1,\dots,b_t)$, where b_r is algebraic over k_v for $r = 1,\dots,t$. It can be taken $t \geq n$, because if $t < n$, $n - t$ elements can be chosen as equal. Since G/G_v is finite, there exists a sequence of subgroups of G such that $G_v = H_0 \subset H_1 \subset \cdots \subset H_n = G$ and H_{r+1}/H_r is finite for $r = 1,\dots,n-1$.

Hence there exists an r.a.t. extension w_1 of v to $K(x_1)$ such that $k_{w_1} \cong k_v(b_1)$ and $G_{w_1} \cong H_1$ [3]. Let $\tilde{K}$ be the completion of K with respect to v and $\tilde{v}$ be the extension of v to $\tilde{K}$. According to [7, Prop. 1], the completion of $K(x_1)$ with respect to w_1 is isomorphic to a field belonging to $F_c(\tilde{\Omega}/\tilde{K})$, where $\tilde{\Omega}$ is the completion of the algebraic closure Ω of $\tilde{K}$ with respect to the unique extension of $\bar{\tilde{v}}$ to Ω and $F_c(\tilde{\Omega}/\tilde{K})$ is the set of complete fields L such that $\tilde{K} \subseteq L \subseteq \tilde{\Omega}$. Moreover, since $\operatorname{tr\,deg}\tilde{K}/K > 0$, there exists an element $\tilde{a} \in \tilde{K}$

which is transcendental over K. That is, there exists a Cauchy sequence $\{a_i\}_{i\in I} \subseteq K$ which converges to $\tilde{a}$.

Therefore if we denote the completion of $K(x_1)$ with respect to w_1 by $\widetilde{K(x_1)}$, then $\operatorname{tr}\deg \widetilde{K(x_1)}/K(x_1) > 0$. Also, H_2/H_1 is finite, then there exists an r.a.t. extension w_2 of w_1 to $K(x_1, x_2)$ such that $k_{w_2} \cong k_v(b_1, b_2)$ and $G_{w_2} \cong H_2$. Using the induction, it is obtained that there exists an r.a.t. extension w_{n-1} of w_{n-2} on $K(x_1, \ldots, x_{n-2})$ to $K(x_1, \ldots, x_{n-1})$ such that its residue field is $k_{w_{n-1}} = k_v(b_1, \ldots, b_{n-1})$ and its value group is $G_{w_{n-1}} = H_{n-1}$. Finally, there exists an r.a.t. extension $w = w_n$ of w_{n-1} to $K(x_1, \ldots, x_n)$ such that $k_w \cong k_v(b_n, \ldots, b_t) = k$ and $G_w \cong G$. □

Competing interests
The authors declare that they have no competing interests.

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