Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Common fixed points of a generalized ordered $g$-quasicontraction in partially ordered metric spaces

Xiaolan Liu[1,2*] and Siniša Ješić[3]

*Correspondence: stellalwp@163.com
[1]Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan 610068, China
[2]School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China
Full list of author information is available at the end of the article

**Abstract**
The concept of a generalized ordered $g$-quasicontraction is introduced, and some fixed and common fixed point theorems for a $g$-nondecreasing generalized ordered $g$-quasicontraction mapping in partially ordered complete metric spaces are proved. We also show the uniqueness of the common fixed point in the case of a generalized ordered $g$-quasicontraction mapping. Finally, we prove fixed point theorems for mappings satisfying the so-called weak contractive conditions in the setting of a partially ordered metric space. Presented theorems are generalizations of very recent fixed point theorems due to Golubović *et al.* (Fixed Point Theory Appl. 2012:20, 2012).

**MSC:** 47H10; 47N10

**Keywords:** $G$-nondecreasing; generalized ordered $g$-quasicontraction; coincidence point; common fixed point; comparable mappings

## 1 Introduction

The Banach fixed point theorem for contraction mappings has been extended in many directions (*cf.* [1–15]). Very recently Golubović *et al.* [16] presented some new results for ordered quasicontractions and $g$-quasicontractions in partially ordered metric spaces.

Recall that if $(X,\preceq)$ is a partially ordered set and $f : X \to X$ is such that for $x,y \in X$, $x \preceq y$ implies $fx \preceq fy$, then a mapping $F$ is said to be non-decreasing. The main result of Golubović *et al.* [16] is the following fixed point theorem.

**Theorem 1.1** (See [16], Theorem 1) *Let $(X,d,\preceq)$ be a partially ordered metric space and let $f,g : X \to X$ be two self-maps on $X$ satisfying the following conditions*:

(i) $fX \subset gX$;
(ii) *$gX$ is complete*;
(iii) *$f$ is $g$-nondecreasing*;
(iv) *$f$ is an ordered $g$-quasicontraction*;
(v) *there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$*;
(vi) *if $\{gx_n\}$ is a nondecreasing sequence that converges to some $gz \in gX$, then $gx_n \preceq gz$ for each $n \in \mathbb{N}$ and $gz \preceq g(gz)$.*

*Then $f$ and $g$ have a coincidence point, i.e., there exists $z \in X$ such that $fz = gz$. If, in addition,*

(vii) *$f$ and $g$ are weakly compatible* [17, 18], *i.e., $fx = gx$ implies $fgx = gfx$ for each $x \in X$, then they have a common fixed point.*

© 2013 Liu and Ješić; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

An open problem is to find sufficient conditions for the uniqueness of the common fixed point in the case of an ordered $g$-quasicontraction in Theorem 1.1.

In Section 2 of this article, we introduce generalized ordered $g$-quasicontractions in partially ordered metric spaces and prove the respective (common) fixed point theorems which generalize the results of Theorem 1.1.

In Section 3 of this article, the uniqueness of a common fixed point theorem is obtained when for all $x, u \in X$, there exists $a \in X$ such that $fa$ is comparable to $fx$ and $fu$ in addition to the hypotheses in Theorem 2.1 of Section 2. Our results are an answer to finding sufficient conditions for the uniqueness of a common fixed point in the case of an ordered $g$-quasicontraction in Theorem 1.1. Finally, two examples show that the comparability is a sufficient condition for the uniqueness of a common fixed point in the case of an ordered $g$-quasicontraction, so our results are extensions of known ones.

In Section 4 of this article, we consider weak contractive conditions in the setting of a partially ordered metric space and prove respective common fixed point theorems.

## 2 Common fixed points of a generalized ordered *g*-quasicontraction

We start this section with the following definitions. Consider a partially ordered set $(X, \preceq)$ and two mappings $f : X \to X$ and $g : X \to X$ such that $f(X) \subset g(X)$.

**Definition 2.1** (See [19]) We will say that the mapping $f$ is $g$-nondecreasing (resp., $g$-nonincreasing) if

$$gx \preceq gy \quad \Rightarrow \quad fx \preceq fy \tag{1}$$

(resp., $gx \preceq gy \Rightarrow fx \preceq fy$) holds for each $x, y \in X$.

**Definition 2.2** (See [16]) We will say that the mapping $f$ is an ordered $g$-quasicontraction if there exists $\alpha \in (0,1)$ such that for each $x, y \in X$ satisfying $gy \preceq gx$, the inequality

$$d(fx, fy) \le \alpha \cdot M(x, y)$$

holds, where

$$M(x, y) = \max\bigl\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\bigr\}.$$

**Definition 2.3** We will say that the mapping $f$ is a generalized ordered $g$-quasicontraction if there is a continuous and non-decreasing function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(s+t) \le \psi(s) + \psi(t)$ for each $s, t > 0$, $\psi(t) \ge t$ for $t \ge 0$ and there exists $\alpha \in (0,1)$

$$\begin{aligned}\psi\bigl(d(fx, fy)\bigr) \le \alpha \max\bigl\{&\psi\bigl(d(gx, gy)\bigr), \psi\bigl(d(gx, fx)\bigr), \psi\bigl(d(gy, fy)\bigr),\\ &\psi\bigl(d(gx, fy)\bigr), \psi\bigl(d(gy, fx)\bigr)\bigr\}\end{aligned} \tag{2}$$

for all $x, y \in X$ for which $gx \succeq gy$;

It is obvious that if $\psi = I$, then a generalized ordered $g$-quasicontraction reduces to an ordered $g$-quasicontraction.

For arbitrary $x_0 \in X$, one can construct the so-called Jungck sequence $\{y_n\}$ in the following way: Denote $y_0 = fx_0 \in f(X) \subset g(X)$; there exists $x_1 \in X$ such that $gx_1 = y_0 = fx_0$; now $y_1 = fx_1 \in f(X) \subset g(X)$ and there exists $x_2 \in X$ such that $gx_2 = y_1 = fx_1$ and the procedure can be continued.

**Theorem 2.1** *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $f, g : X \to X$ be two self-maps on $X$ satisfying the following conditions*:

(i) $f(X) \subset g(X)$;
(ii) $g(X)$ *is closed*;
(iii) $f$ *is a $g$-nondecreasing mapping*;
(iv) $f$ *is a generalized ordered $g$-quasicontraction*;
(v) *there exists an* $x_0 \in X$ *with* $gx_0 \preceq fx_0$;
(vi) $\{g(x_n)\} \subset X$ *is a non-decreasing sequence with* $g(x_n) \to gz$ *in* $g(X)$, *then* $gx_n \preceq gz$, $gz \preceq g(gz)$, $\forall n$ *hold*.

*Then $f$ and $g$ have a coincidence point. Further, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a common fixed point.*

*Proof* Let $x_0 \in X$ be such that $gx_0 \preceq fx_0$. Since $f(X) \subset g(X)$, we can choose $x_1 \in X$ so that $gx_1 = fx_0$. Again from $f(X) \subset g(X)$, we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process, we can construct a Jungck sequence $\{y_n\}$ in X such that

$$gx_{n+1} = fx_n = y_n, \quad \forall n \geq 0. \tag{3}$$

Since $gx_0 \preceq fx_0$ and $gx_1 = fx_0$, we have $gx_0 \preceq gx_1$. Then by (1),

$$fx_0 \preceq fx_1. \tag{4}$$

Thus, by (3), $gx_1 \preceq gx_2$. Again by (1),

$$fx_1 \preceq fx_2, \tag{5}$$

that is, $gx_2 \preceq gx_3$. Continuing this process, we obtain

$$fx_0 \preceq fx_1 \preceq fx_2 \preceq fx_3 \preceq \cdots \preceq fx_n \preceq fx_{n+1}. \tag{6}$$

Let $O(y_k, n) = \{y_k, y_{k+1}, \ldots, y_{k+n}\}$. We will claim that $\{y_n\}$ is a Cauchy sequence. To prove our claim, we follow the arguments of Das and Naik [20]. Fix $k \geq 0$ and $n \in \{1, 2, \ldots\}$. If $\operatorname{diam}[O(y_k; n)] = 0$, then $y_k = y_{k+1}$, which yields that $\{y_n\}$ is a constant sequence and also a Cauchy sequence. Then our claims holds. Thus we suppose that $\operatorname{diam}[O(y_k; n)] > 0$. Now, for $i, j$ with $1 \leq i < j$, by (2), we have

$$\begin{aligned}
&\psi\bigl(d(y_i, y_j)\bigr) \\
&\quad = \psi\bigl(d(fx_i, fx_j)\bigr) \\
&\quad \leq \alpha \max\bigl\{\psi\bigl(d(gx_i, gx_j)\bigr), \psi\bigl(d(gx_i, fx_i)\bigr), \psi\bigl(d(gx_j, fx_j)\bigr), \psi\bigl(d(x_i, fx_j)\bigr), \psi\bigl(d(gx_j, fx_i)\bigr)\bigr\}
\end{aligned}$$

$$\begin{aligned}&= \alpha \max\{\psi(d(y_{i-1}, y_{j-1})), \psi(d(y_{i-1}, y_i)), \psi(d(y_{j-1}, y_j)), \psi(d(y_{i-1}, y_j)), \psi(d(y_{j-1}, y_i))\} \\ &\leq \alpha\psi(\operatorname{diam}[O(y_{i-1}; j-i+1)]),\end{aligned}$$

and so

$$\psi(d(y_i, y_j)) \leq \alpha\psi(\operatorname{diam}[O(y_{i-1}; j-i+1)]). \tag{7}$$

Now, for some $i$, $j$ with $k \leq i < j \leq k+n$, $\operatorname{diam}[O(y_k; n)] = d(y_i, y_j)$. If $i > k$ by (2) and (7), then we have

$$\begin{aligned}\psi(\operatorname{diam}[O(y_k; n)]) &\leq \alpha\psi(\operatorname{diam}[O(y_{i-1}; j-i+1)]) \\ &\leq \alpha\psi(\operatorname{diam}[O(y_k; n)]).\end{aligned} \tag{8}$$

It follows that $\psi(\operatorname{diam}[O(y_k; n)]) = 0$, as $\operatorname{diam}[O(y_k; n)] \leq \psi(\operatorname{diam}[O(y_k; n)]) = 0$, then $\operatorname{diam}[O(y_k; n)] = 0$. It is a contradiction! Thus,

$$\operatorname{diam}[O(y_k; n)] = d(y_k, y_j) \quad \text{for } j \text{ with } k < j \leq k+n. \tag{9}$$

Also, by (7) and (9), we have

$$\begin{aligned}\psi(\operatorname{diam}[O(y_k; n)]) &= \psi(d(y_k, y_j)) \\ &\leq \alpha\psi(\operatorname{diam}[O(y_{k-1}; j-k+1)]) \\ &\leq \alpha\psi(\operatorname{diam}[O(y_{k-1}; n+1)]).\end{aligned} \tag{10}$$

Using the triangle inequality, by (7), (9) and (10), we obtain that

$$\begin{aligned}\psi(\operatorname{diam}[O(y_l; m)]) &= \psi(d(y_l, y_j)) \\ &\leq \psi(d(y_l, y_{l+1}) + d(y_{l+1}, y_j)) \\ &\leq \psi(d(y_l, y_{l+1})) + \psi(d(y_{l+1}, y_j)) \\ &\leq \psi(d(y_l, y_{l+1})) + \alpha\psi(\operatorname{diam}[O(y_{l+1}; m-1)]) \\ &\leq \psi(d(y_l, y_{l+1})) + \alpha\psi(\operatorname{diam}[O(y_l; m)]),\end{aligned} \tag{11}$$

and so

$$\psi(\operatorname{diam}[O(y_l; m)]) \leq \frac{1}{1-\alpha}\psi(d(y_l, y_{l+1})). \tag{12}$$

As a result, we have

$$\begin{aligned}\psi(\operatorname{diam}[O(y_k; n)]) &\leq \alpha\psi(\operatorname{diam}[O(y_{k-1}; n+1)]) \\ &\leq \alpha \cdot \alpha\psi(\operatorname{diam}[O(y_{k-2}; n+2)]) \\ &\leq \alpha^k\psi(\operatorname{diam}[O(y_0; n+k)]) \\ &\leq \frac{\alpha^k}{1-\alpha}\psi(d(y_0, y_1)).\end{aligned} \tag{13}$$

Now let $\epsilon > 0$, there exists an integer $n_0$ such that

$$\alpha^k \psi\big(d(y_0,y_1)\big) < (1-\alpha)\epsilon \quad \text{for all } k > n_0. \tag{14}$$

For $m > n > n_0$, we have

$$\begin{aligned} \psi\big(d(y_m,y_n)\big) &\le \psi\big(\operatorname{diam}\big[O(y_{n_0};m-n_0)\big]\big) \\ &\le \frac{\alpha^{n_0}}{1-\alpha}\psi\big(d(y_0,y_1)\big) \\ &< \epsilon. \end{aligned} \tag{15}$$

Since $\psi(t) \ge t$ as $t > 0$, then $d(y_m,y_n) \le \psi(d(y_m,y_n)) < \epsilon$. Therefore, $\{y_n\}$ is a Cauchy sequence.

Since by (3) we have $\{fx_n = gx_{n+1}\} \subseteq g(X)$ and $g(X)$ is closed, then there exists $z \in X$ such that

$$\lim_{n\to\infty} gx_n = gz. \tag{16}$$

Now we show that $z$ is a coincidence point of $f$ and $g$. Since from condition (iv) and (9) we have $gx_n \preceq gz$ for all $n$, then by the triangle inequality and (2), we have that

$$\begin{aligned} \psi\big(d(fz,gz)\big) &\le \psi\big(d(gz,fx_n) + d(fx_n,fz)\big) \\ &\le \psi\big(d(gz,fx_n)\big) + \psi\big(d(fx_n,fz)\big) \\ &\le \psi\big(d(gz,fx_n)\big) + \alpha \max\big\{\psi\big(d(gx_n,gz)\big), \psi\big(d(gx_n,fx_n)\big), \\ &\quad \psi\big(d(gz,fz)\big), \psi\big(d(gx_n,fz)\big), \psi\big(d(gz,fx_n)\big)\big\}. \end{aligned} \tag{17}$$

So, letting $n \to \infty$ yields $\psi(d(fz,gz)) \le \alpha\psi(d(fz,gz))$. Hence $\psi(d(fz,gz)) = 0$, hence $d(fz,gz) = 0$, which yields $fz = gz$. Thus we have proved that $f$ and $g$ have a coincidence point.

Suppose now that $f$ and $g$ commute at $z$. Set $w = fz = gz$. Then

$$fw = f(gz) = g(fz) = gw. \tag{18}$$

Since from (vi) we have that $gz \preceq g(gz) = gw$ and as $fz = gz$ and $fw = gw$, from (2) we have that

$$\begin{aligned} \psi\big(d(fz,fw)\big) &\le \alpha \max\big\{\psi\big(d(gz,gw)\big), \psi\big(d(gz,fz)\big), \psi\big(d(gw,fw)\big), \\ &\quad \psi\big(d(gz,fw)\big), \psi\big(d(gw,fz)\big)\big\} \\ &= \alpha\psi\big(d(gz,gw)\big). \end{aligned} \tag{19}$$

Hence, $\psi(d(fz,fw)) = 0$, that is, $d(w,fw) = 0$. Therefore,

$$fw = gw = w. \tag{20}$$

Thus, we have proved that $f$ and $g$ have a common fixed point. □

Accordingly, we can also obtain the results similar to Theorem 2 in [16].

**Theorem 2.2** *Let the conditions of Theorem* 2.1 *be satisfied, except that* (iii), (v) *and* (vi) *are, respectively, replaced by*:

(iii′) *$f$ is a $g$-nonincreasing mapping*;
(v′) *there exists $x_0 \in X$ such that $fx_0$ and $gx_0$ are comparable*;
(vi′) *if $\{gx_n\}$ is a sequence in $g(X)$ which has comparable adjacent terms and that converges to some $gz \in gX$, then there exists a subsequence $gx_{n_k}$ of $\{gx_n\}$ having all the terms comparable with $gz$ and $gz$ is comparable with $ggz$. Then all the conclusions of Theorem* 2.1 *hold.*

*Proof* Regardless of whether $fx_0 \preceq gx_0$ or $gx_0 \preceq fx_0$ (condition (v′)), Lemma 1 of [16] implies that the adjacent terms of the Jungck sequence $\{y_n\}$ are comparable. This is again sufficient to imply that $\{y_n\}$ is a Cauchy sequence. Hence, it converges to some $gz \in gX$.

By (vi′), there exists a subsequence $y_{n_k} = fx_{n_k} = gx_{n_k+1}$, $k \in \mathbb{N}$, having all the terms comparable with $gz$. Hence, we can apply the contractive condition to obtain

$$\begin{aligned}\psi\big(d(fz,gz)\big) &\le \psi\big(d(gz,fx_{n_k})\big) + \psi\big(d(fz,fx_{n_k})\big)\\ &\le \psi\big(d(gz,fx_{n_k})\big) + \alpha\max\big\{\psi\big(d(gz,gx_{n_k})\big), \psi\big(d(gz,fz)\big),\\ &\quad \psi\big(d(gx_{n_k},fx_{n_k})\big), \psi\big(d(gz,fx_{n_k})\big), \psi\big(d(gx_{n_k},fz)\big)\big\}.\end{aligned}$$

Letting $k \to \infty$, it yields that $\psi(d(fz,gz)) \le \alpha\psi(d(gz,fz))$, then $\psi(d(fz,gz)) = 0$. Thus $d(fz,gz) = 0$. It follows that $fz = gz = w$. The rest of conclusions follow in the same way as in Theorem 2.1. □

**Corollary 2.1** (a) *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X,d)$ is a complete metric space. Let $f : X \to X$ be a nondecreasing self-map such that for some $\alpha \in (0,1)$*

$$d(fx,fy) \le \alpha\max\big\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\big\}$$

*for all $x, y \in X$ for which $x \succeq y$. Suppose also that either*

(i) *$\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to u$ in $X$, then $x_n \preceq u$, $\forall n$ hold, or*
(ii) *$f$ is continuous.*

*If there exists an $x_0 \in X$ with $x_0 \preceq fx_0$, then $f$ has a fixed point.*

(b) *The same holds if $f$ is nonincreasing, there exists $x_0$ comparable with $fx_0$ and* (i) *is replaced by*

(i′) *if a sequence $\{x_n\}$ converging to some $u \in X$ has every two adjacent terms comparable, then there exists a subsequence $\{x_{n_k}\}$ having each term comparable with $x$.*

*Proof* (a) If (i) holds, then take $\psi = I$ and $g = I$ ($I$ = the identity mapping) in Theorem 2.1.

If (ii) holds, then from (16) with $g = I$, we get

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\Big(\lim_{n\to\infty} x_n\Big) = fz. \tag{21}$$

(b) Let $u$ be the limit of the Picard sequence $\{f^n x_0\}$ and let $f^{n_k} x_0$ be a subsequence having all the terms comparable with $u$. Then we can apply the contractivity condition to obtain

$$\begin{aligned}\psi\big(d(fu,u)\big) &\le \psi\big(d\big(u,f^{n_k+1}x_0\big) + d\big(fu,f^{n_k+1}x_0\big)\big)\\ &\le \psi\big(d\big(u,f^{n_k+1}x_0\big)\big) + \psi\big(d\big(fu,f^{n_k+1}x_0\big)\big)\\ &\le \psi\big(d\big(u,f^{n_k+1}x_0\big)\big) + \alpha\max\big\{\psi\big(d\big(u,f^{n_k}x_0\big)\big), \psi\big(d(u,fu)\big),\\ &\quad \psi\big(d\big(f^{n_k}x_0,f^{n_k+1}x_0\big)\big), \psi\big(d\big(u,f^{n_k+1}x_0\big)\big), \psi\big(d\big(fu,f^{n_k}x_0\big)\big)\big\}.\end{aligned}$$

Letting $k \to \infty$, we have that

$$\begin{aligned}\psi\big(d(fu,u)\big) &\le \alpha\max\big\{0, \psi\big(d(u,fu)\big), 0, 0, \psi\big(d(u,fu)\big)\big\}\\ &= \alpha\psi\big(d(u,fu)\big).\end{aligned}$$

It follows that $\psi(d(fu,u)) = 0$. Thus $d(fu,u) = 0$ as $d(fu,u) \le \psi(d(fu,u)) = 0$. Therefore, $fu = u$.

Note also that instead of the completeness of $X$, its $f$-orbitally completeness is sufficient to obtain the conclusion of the corollary. □

## 3 Uniqueness of a common fixed point of *f* and *g*

The following theorem gives the sufficient condition for the uniqueness of a common fixed point of $f$ and $g$.

**Theorem 3.1** *In addition to the hypotheses of Theorem* 2.1, *suppose that for all* $x, u \in X$, *there exists* $a \in X$ *such that*

$$fa \quad \text{is comparable to} \quad fx \quad \text{and} \quad fu. \tag{22}$$

*Then* $f$ *and* $g$ *have a unique common fixed point.*

*Proof* Since a set of common fixed points of $f$ and $g$ is not empty due to Theorem 2.1, assume now that $x$ and $u$ are two common fixed points of $f$ and $g$, *i.e.*,

$$fx = gx = x, \qquad fu = gu = u. \tag{23}$$

We claim that $gx = gu$.

By assumption, there exists $a \in X$ such that $fa$ is comparable to $fx$ and $fu$. Define a sequence $\{ga_n\}$ such that $a_0 = a$ and

$$ga_n = fa_{n-1} \quad \text{for all } n. \tag{24}$$

Further, set $x_0 = x$ and $u_0 = u$ and in the same way define $\{gx_n\}$ and $\{gu_n\}$ such that

$$gx_n = fx_{n-1}, \qquad gu_n = fu_{n-1} \quad \text{for all } n. \tag{25}$$

Since $fx$ $(= gx_1 = gx)$ is comparable to $fa$ $(= fa_0 = ga_1)$ and $f$ is $g$-nondecreasing, it is easy to show

$$gx \succeq ga_1. \tag{26}$$

Recursively, we can get that

$$ga_n \preceq gx \quad \text{for all } n. \tag{27}$$

By (27), we have that

$$\begin{aligned} \psi\bigl(d(ga_{n+1}, gx)\bigr) &= \psi\bigl(d(fa_n, fx)\bigr) \\ &\le \alpha \max\bigl\{\psi\bigl(d(ga_n, gx)\bigr), \psi\bigl(d(ga_n, fa_n)\bigr), \psi\bigl(d(gx, fx)\bigr), \\ &\quad \psi\bigl(d(ga_n, fx)\bigr), \psi\bigl(d(gx, fa_n)\bigr)\bigr\}. \end{aligned} \tag{28}$$

By the proof of Theorem 2.1, we obtain that $\{ga_n\}$ is a convergent sequence, and there exists $g\bar{a}$ such that $ga_n \to g\bar{a}$. Letting $n \to \infty$ in (28) and $\psi$ is continuous, we can obtain that

$$\begin{aligned} \lim_{n\to\infty} \psi\bigl(d(ga_{n+1}, gx)\bigr) &= \psi\bigl(d(g\bar{a}, gx)\bigr) \\ &\le \alpha \max\bigl\{\psi\bigl(d(g\bar{a}, gx)\bigr), 0, 0, \psi\bigl(d(g\bar{a}, fx)\bigr), \psi\bigl(d(gx, g\bar{a})\bigr)\bigr\} \\ &= \alpha\psi\bigl(d(g\bar{a}, gx)\bigr). \end{aligned}$$

Therefore, we obtain

$$\psi\bigl(d(g\bar{a}, gx)\bigr) = 0.$$

Since $\psi(t) \ge t$ as $t \ge 0$, then $d(g\bar{a}, gx) = 0$ and hence

$$g\bar{a} = gx. \tag{29}$$

Similarly, we can show that

$$\begin{aligned} \lim_{n\to\infty} \psi\bigl(d(ga_{n+1}, gu)\bigr) &= \psi\bigl(d(g\bar{a}, gu)\bigr) \\ &\le \alpha \max\bigl\{\psi\bigl(d(g\bar{a}, gu)\bigr), 0, 0, \psi\bigl(d(g\bar{a}, fu)\bigr), \psi\bigl(d(gu, g\bar{a})\bigr)\bigr\} \\ &= \alpha\psi\bigl(d(g\bar{a}, gu)\bigr). \end{aligned}$$

Therefore, we obtain

$$\psi\bigl(d(g\bar{a}, gu)\bigr) = 0.$$

Since $\psi(t) \ge t$ as $t \ge 0$, then $d(g\bar{a}, gu) = 0$ and hence

$$g\bar{a} = gu. \tag{30}$$

Thus, from (29) and (30), we have $gx = gu$. It follows that

$$x = fx = gx = gu = fu = u. \tag{31}$$

It means that $x$ is the unique common fixed point of $f$ and $g$. □

**Remark 3.1** Theorem 3.1 can be considered as an answer to Theorem 3 in [16]. We find the sufficient conditions for the uniqueness of the common fixed point in the case of an ordered $g$-quasicontraction. In this paper, condition (vi) in Theorem 2.1 is weaker than the ordered $g$-quasicontraction in [16]. When $\psi = I$ ($I$ = the identity mapping), our condition (vi) reduces to the ordered $g$-quasicontraction in [16].

**Example 3.1** Let $X = \{(0,2),(2,3)\}$, let $(a,b) \preceq (c,d)$ if and only if $a \leq c$ and $b \geq d$, and let $d$ be the Euclidean metric. We define the functions as follows:

$$f\bigl((x,y)\bigr) = \bigl(x^2, 5y-8\bigr), \qquad g\bigl((x,y)\bigr) = \bigl(2x, y^2-2\bigr) \quad \text{for all } (x,y) \in X.$$

Let $\phi, \psi : [0,\infty) \to [0,\infty)$ be given by

$$\psi(t) = \frac{2}{5}t \quad \text{for all } t \in [0,\infty).$$

Obviously, for $(0,2)$ and $(2,3) \in X$, but $f((0,2)) = (0,2)$ is not comparable to $g((2,3)) = (2,3)$. However, $f$ and $g$ have two common fixed points $(0,2)$ and $(2,3)$ since

$$f\bigl((0,2)\bigr) = g(0,2) = (0,2), \qquad f\bigl((2,3)\bigr) = g\bigl((2,3)\bigr) = (2,3).$$

**Example 3.2** Let $X = [-\infty,+\infty)$ with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Let $f : X \to X$ and $g : X \to X$ be given by

$$f(x) = \frac{x}{16}, \qquad g(x) = \frac{3}{4}x$$

for all $x,y,z,w \in X$. Let $\phi, \psi : [0,\infty) \to [0,\infty)$ be given by

$$\psi(t) = 3t \quad \text{for all } t \in [0,\infty).$$

It is easy to check that all the conditions of Theorem 2.1 are satisfied.

$$\begin{aligned}
\psi\bigl(d(fx,fy)\bigr) &= \frac{3}{16}|x-y| \\
&\leq 3 \cdot \alpha \cdot \frac{3}{4}|x-y| \\
&\leq \max\biggl\{3 \cdot \frac{3}{4}|x-y|, 3 \cdot \left|\frac{3}{4}x - \frac{x}{16}\right|, 3 \cdot \left|\frac{3}{4}y - \frac{y}{16}\right|, \\
&\quad 3 \cdot \left|\frac{3}{4}x - \frac{y}{16}\right|, 3 \cdot \left|\frac{3}{4}y - \frac{x}{16}\right|\biggr\} \\
&= \max\bigl\{\psi\bigl(d(gx,gy)\bigr), \psi\bigl(d(gx,fx)\bigr), \psi\bigl(d(gy,fy)\bigr), \psi\bigl(d(gx,fy)\bigr), \psi\bigl(d(gy,fx)\bigr)\bigr\}.
\end{aligned}$$

It holds when $\alpha = \frac{1}{12}$ and $gx \geq gy$, *i.e.*, $\frac{3}{4}x \geq \frac{3}{4}y$, *i.e.*, $x \geq y$.

In addition, $\forall x, u \in X$, there exists $a \in X$ such that $fa = \frac{a}{16}$ is comparable to $fx = \frac{x}{16}$ and $fu = \frac{u}{16}$. So, all the conditions of Theorem 3.1 are satisfied.

By applying Theorem 3.1, we conclude that $f$ and $g$ have a unique common fixed point. In fact, $f$ and $g$ have only one common fixed point. It is $x = 0$.

## 4 Weak ordered contractions

We denote by $\Psi$ the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

($\psi_1$) $\psi$ is continuous and nondecreasing,
($\psi_2$) $\psi(t) = 0$ if and only if $t = 0$.

We denote by $\Phi$ the set of functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

($\phi_1$) $\lim_{s\to t+} \phi(s) > 0$ for all $t > 0$,
($\phi_2$) $\phi(t) = 0$ if and only if $t = 0$.

Let $(X, d)$ be a metric space and let $f, g : X \to X$. In the article [16] (in the setting of partially ordered metric spaces), the authors obtained contractive conditions of the form

$$\psi\big(d(fx,fy)\big) \leq \psi\big(M(x,y)\big) - \phi\big(M(x,y)\big), \tag{32}$$

where

$$M(x;y) = \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy) + d(gy,fx)}{2}\right\}. \tag{33}$$

We will use here the following more general contractive condition:

$$M(x,y) = \max\big\{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\big\}. \tag{34}$$

We begin with the following result.

**Theorem 4.1** *Let $(X, d, \preceq)$ be a partially ordered metric space and let $f$ and $g$ be self-mappings of $X$ satisfying the following conditions*:

(i) $f(X) \subset g(X)$;
(ii) *$g(X)$ is complete*;
(iii) *$f$ is $g$-nondecreasing*;
(iv) *$f$ and $g$ satisfy the following condition*:

$$\psi\big(d(fx,fy)\big) \leq \psi\big(M(x,y)\big) - \phi\big(M(x,y)\big) \tag{35}$$

*for all $x, y \in X$ such that $gy \preceq gx$, where $\psi \in \Psi$, $\phi \in \Phi$ and*

$$M(x,y) = \max\big\{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\big\}. \tag{36}$$

*Suppose that, in addition*,

(v) *$\psi(t) - \phi(t)$ is nondecreasing*;

(vi) $\psi(s+t) \le \psi(s) + \psi(t)$ *for each* $s, t > 0$;
(vii) $\lim_{t\to+\infty} \phi(t) = \infty$;
(viii) *there exists* $x_0 \in X$ *such that* $gx_0 \preceq fx_0$;
(ix) *if* $\{gx_n\}$ *is a nondecreasing sequence that converges to some* $gz \in gX$, *then* $gx_n \preceq gz$ *for each* $n \in N$ *and* $gz \preceq g(gz)$.

*Then* $f$ *and* $g$ *have a coincidence point. If, in addition,*

(x) $f$ *and* $g$ *are weakly compatible, then they have a common fixed point.*

*Further, if*

(xi) *for arbitrary* $v, w \in X$, *there exists* $y_0 \in X$ *such that* $fy_0$ *is comparable to* $fv$ *and* $fw$, *then* $f$ *and* $g$ *have a unique common fixed point.*

*Proof* As in the proof of Theorem 2.1, we can construct a nondecreasing Jungck sequence $\{y_n\}$ with

$$y_n = fx_n = gx_{n+1}$$

for all $n \ge 0$. Denote

$$O(y_k, n) = \{y_k, y_{k+1}, y_{k+2}, \dots, y_{k+n}\}, \tag{37}$$

$$O(y_k) = \{y_k, y_{k+1}, y_{k+2}, \dots, y_{k+n}, \dots\}. \tag{38}$$

We will prove that the Jungck sequence $\{y_n\}$ is bounded, that is,

$$\operatorname{diam}\bigl(O(y_0)\bigr) = \operatorname{diam}\bigl(\{y_0, y_1, y_2, \dots, y_n, \dots\}\bigr) \le K \tag{39}$$

for some $K \in \mathbb{R}$. Let $k < n$ be any fixed positive integer and let $\operatorname{diam}(O(y_k, n)) = d(y_i, y_j)$ for some $i, j$ with $k \le i < j \le k + n$. We will show that

$$\psi\bigl(\operatorname{diam}\bigl(O(y_k,n)\bigr)\bigr) \le \psi\bigl(\operatorname{diam}\bigl(O(y_{i-1}, j-i+1)\bigr)\bigr) - \phi\bigl(\operatorname{diam}\bigl(O(y_{i-1}, j-i+1)\bigr)\bigr). \tag{40}$$

Since $\operatorname{diam}(O(y_k, n)) = d(y_i, y_j)$, $y_i = fx_i$, $y_j = fx_j$ and $gx_i \preceq gx_j$, then from (35) we have

$$\psi\bigl(\operatorname{diam}\bigl(O(y_k,n)\bigr)\bigr) = \psi\bigl(d(fx_i, fx_j)\bigr) \le \psi\bigl(M(x_i, x_j)\bigr) - \phi\bigl(M(x_i, x_j)\bigr), \tag{41}$$

where

$$\begin{aligned} M(x_i, x_j) &= \max\bigl\{d(gx_i, gx_j), d(gx_i, fx_i), d(gx_j, fx_j), d(gx_i, fx_j), d(gx_j, fx_i)\bigr\} \\ &= \max\bigl\{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{j-1}, y_j), d(y_{i-1}, y_j), d(y_{j-1}, y_i)\bigr\}. \end{aligned}$$

Since $y_{i-1}, y_i, y_{j-1}, y_j \in O(y_{i-1}, j-i+1)$, then

$$M(x_i, x_j) \le \operatorname{diam}\bigl(\{y_{i-1}, y_i, y_{j-1}, y_j\}\bigr) \le \operatorname{diam}\bigl(O(y_{i-1}, j-i+1)\bigr).$$

So, from (v),

$$\psi\bigl(M(x_i,x_j)\bigr) - \phi\bigl(M(x_i,x_j)\bigr) \le \psi\bigl(\operatorname{diam}\bigl(O(y_{i-1}, j-i+1)\bigr)\bigr) - \phi\bigl(\operatorname{diam}\bigl(O(y_{i-1}, j-i+1)\bigr)\bigr).$$

Hence from (41) we obtain (40).

Note that $\phi(\operatorname{diam}(O(y_{i-1}, j-i+1))) > 0$, and so from (40),

$$\operatorname{diam}\bigl(O(y_k, n)\bigr) < \operatorname{diam}\bigl(O(y_{i-1}, j-i+1)\bigr). \tag{42}$$

Now we will show that if $\operatorname{diam}(O(y_k, n)) = d(y_i, y_j)$, then $i = k$, that is,

$$\operatorname{diam}\bigl(O(y_k, n)\bigr) = d(y_k, y_j) \quad \text{for some } k < j \le k+n. \tag{43}$$

Suppose, to the contrary, that $i > k$. Then $\{y_{i-1}, y_i, \ldots, y_j\} \subseteq \{y_k, y_{k+1}, \ldots, y_i, \ldots, y_j\}$ and hence we conclude that

$$\begin{aligned}\operatorname{diam}\bigl(O(y_k, n)\bigr) &= d(y_i, y_j) = \operatorname{diam}\bigl(O(y_{i-1}, j-i+1)\bigr) \\ &= \operatorname{diam}\bigl(O(y_i, j-i)\bigr) = \operatorname{diam}\bigl(O(y_k, j-k)\bigr).\end{aligned}$$

This contradicts (42). Therefore, $i = k$ and so we have proved (43).

We will prove that the Jungck sequence $\{y_n\}$ is bounded. From (43) it follows that $\operatorname{diam}(O(y_0, n)) = d(y_0, y_j)$ for some $y_j \in \{y_1, y_2, \ldots, y_n\}$. By the triangle inequality,

$$\operatorname{diam}\bigl(O(y_0, n)\bigr) = d(y_0, y_j) \le d(y_0, y_1) + d(y_1, y_j).$$

Now, from $(\psi_1)$ and $(\psi_3)$, we get

$$\begin{aligned}\psi\bigl(\operatorname{diam}\bigl(O(y_0, n)\bigr)\bigr) &\le \psi\bigl[d(y_0, y_1) + d(y_1, y_j)\bigr] \\ &\le \psi\bigl(d(y_0, y_1)\bigr) + \psi\bigl(d(y_1, y_j)\bigr).\end{aligned} \tag{44}$$

Since $d(y_1, y_j) = d(fx_1, fx_j)$ and as $gx_1 \preceq gx_j$, from (35) we have

$$\psi\bigl(d(y_1, y_j)\bigr) \le \psi\bigl(M(x_1, x_j)\bigr) - \phi\bigl(M(x_1, x_j)\bigr),$$

where

$$M(x_1, x_j) = \max\bigl\{d(y_0, y_{j-1}), d(y_0, y_1), d(y_{j-1}, y_j), d(y_0, y_j), d(y_{j-1}, y_1)\bigr\}.$$

Clearly, $M(x_1, x_j) \le \operatorname{diam}\{y_0, y_1, y_{j-1}, y_j\} \le \operatorname{diam}(O(y_0, n))$. Thus by (v), we get

$$\psi\bigl(M(x_1, x_j)\bigr) - \phi\bigl(M(x_1, x_j)\bigr) \le \psi\bigl(\operatorname{diam}\bigl(O(y_0, n)\bigr)\bigr) - \phi\bigl(\operatorname{diam}\bigl(O(y_0, n)\bigr)\bigr).$$

Now, by (44),

$$\psi\bigl(\operatorname{diam}\bigl(O(y_0, n)\bigr)\bigr) \le \psi\bigl(d(y_0, y_1)\bigr) + \psi\bigl(\operatorname{diam}\bigl(O(y_0, n)\bigr)\bigr) - \phi\bigl(\operatorname{diam}\bigl(O(y_0, n)\bigr)\bigr).$$

Hence

$$\phi\bigl(\operatorname{diam}\bigl(O(y_0, n)\bigr)\bigr) \le \psi\bigl(d(y_0, y_1)\bigr). \tag{45}$$

Since $\operatorname{diam}(\{y_0, y_1, \ldots, y_n\}) \leq \operatorname{diam}(\{y_0, y_1, \ldots, y_{n+1}\})$, the sequence $\{\operatorname{diam}(O(y_0, n))\}_{n=1}^{\infty}$ is nondecreasing, and so there exists its limit $\operatorname{diam}(O(y_0))$, which is finite or infinite. Suppose that $\lim_{n\to\infty} \operatorname{diam}(O(y_0, n)) = +\infty$. Then (vii) implies that the left-hand side of (45) becomes unbounded when $n$ tends to infinity, but the right-hand side is bounded, a contradiction. Therefore, $\lim_{n\to\infty} \operatorname{diam}(O(y_0, n)) = \operatorname{diam}(O(y_0)) < +\infty$. Thus we have proved (39).

Now we show that $\{y_n\}$ is a Cauchy sequence. For all $n \geq 1$, set similarly as in (38),

$$O(y_n) = \{y_n, y_{n+1}, \ldots\}.$$

Clearly, $O(y_{n+1}) \subset O(y_n)$ and so $\operatorname{diam}(O(y_{n+1})) \leq \operatorname{diam}(O(y_n))$. Therefore, $\{\operatorname{diam}(O(y_n))\}_{n=0}^{\infty}$ is the monotone decreasing sequence of finite nonnegative numbers and converges to some $\delta \geq 0$.

We will prove that $\delta = 0$. Let $n \geq 1$ and $s \geq n + 2$. Since $gx_{n+1} \preceq gx_s$, from (35),

$$\psi\bigl(d(y_{n+1}, y_s)\bigr) = \psi\bigl(d(fx_{n+1}, fx_s)\bigr) \leq \psi\bigl(M(x_{n+1}, x_s)\bigr) - \phi\bigl(M(x_{n+1}, x_s)\bigr),$$

where

$$M(x_{n+1}, x_s) = \max\bigl\{d(y_n, y_{s-1}), d(y_n, y_{n+1}), d(y_{s-1}, y_s), d(y_n, y_s), d(y_{s-1}, y_{n+1})\bigr\}.$$

Since $y_n, y_{n+1}, y_{s-1}, y_s \in \{y_n, y_{n+1}, \ldots\} = O(y_n)$, we conclude that $M(x_{n+1}, x_s) \leq \operatorname{diam}(O(y_n))$, and so by (v), we get

$$\psi\bigl(d(y_{n+1}, y_s)\bigr) \leq \psi\bigl(\operatorname{diam}\bigl(O(y_n)\bigr)\bigr) - \phi\bigl(\operatorname{diam}\bigl(O(y_n)\bigr)\bigr). \tag{46}$$

Since $\lim_{s\to+\infty} d(y_{n+1}, y_s) = \operatorname{diam}(O(y_{n+1}))$ and $\psi$ is continuous, we have $\lim_{s\to+\infty} \psi(d(y_{n+1}, y_s)) = \psi(\operatorname{diam}(O(y_{n+1})))$. Thus, taking the limit in (46) when $s \to +\infty$, we get

$$\psi\bigl(\operatorname{diam}\bigl(O(y_{n+1})\bigr)\bigr) \leq \psi\bigl(\operatorname{diam}\bigl(O(y_n)\bigr)\bigr) - \phi\bigl(\operatorname{diam}\bigl(O(y_n)\bigr)\bigr). \tag{47}$$

Suppose that $\lim_{n\to\infty} \operatorname{diam}(O(y_n)) = \delta > 0$. Since $\operatorname{diam}(O(y_n)) \to \delta+$ as $n \to \infty$, then from $(\phi_1)$, we have $\lim_{n\to\infty} \phi(\operatorname{diam}(O(y_n))) = q > 0$. Therefore, taking the limits as $n \to +\infty$ in (47) and using the continuity of $\psi$, we get

$$\psi(\delta) \leq \psi(\delta) - q < \psi(\delta),$$

a contradiction. Therefore, $\delta = 0$ and so we have proved that

$$\lim_{n\to\infty} \operatorname{diam}\bigl(\{y_n, y_{n+1}, \ldots\}\bigr) = 0.$$

Hence we conclude that $\{y_n\}$ is a Cauchy sequence.

Since $y_n = fx_n = gx_{n+1}$, by the assumption (ii) that $g(X)$ is complete, there is some $z \in X$ such that

$$\lim_{n\to\infty} gx_n = gz.$$

We show that $fz = gz$. Suppose, to the contrary, that $d(fz, gz) > 0$. Condition (ix) implies that $gx_n \preceq gz$ and we can apply the contractive condition (35) to obtain

$$\psi\left(d(fz, fx_{n+1})\right) \le \psi\left(M(z, x_{n+1})\right) - \phi\left(M(z, x_{n+1})\right), \tag{48}$$

where

$$\begin{aligned} M(z, x_{n+1}) &= \max\left\{d(gz, gx_{n+1}), d(gz, fz), d(gx_{n+1}, fx_{n+1}), d(gz, fx_{n+1}), d(gx_{n+1}, fz)\right\} \\ &= \max\left\{d(gz, fx_n), d(gz, fz), d(fx_n, fx_{n+1}), d(gz, fx_{n+1}), d(fx_n, fz)\right\}. \end{aligned}$$

By the triangle inequality,

$$d(gz, fz) \le d(gz, fx_{n+1}) + d(fz, fx_{n+1}).$$

Now, from $(\psi_1)$ and $(\psi_3)$,

$$\begin{aligned} \psi\left(d(gz, fz)\right) &\le \psi\left[d(gz, fx_{n+1}) + d(fz, fx_{n+1})\right] \\ &\le \psi\left(d(gz, fx_{n+1})\right) + \psi\left(d(fz, fx_{n+1})\right). \end{aligned}$$

Hence from (48) we have

$$\psi\left(d(gz, fz)\right) \le \psi\left(d(gz, fx_{n+1})\right) + \psi\left(M(z, x_{n+1})\right) - \phi\left(M(z, x_{n+1})\right). \tag{49}$$

Since $\lim_{n\to\infty} fx_n = gz$, for large enough $n$, we have

$$M(z, x_{n+1}) = \max\left\{d(gz, fz), d(fx_n, fz)\right\}.$$

If $M(z, x_{n+1}) = d(gz, fz)$, then from (49)

$$\psi\left(d(gz, fz)\right) \le \psi\left(d(gz, fx_{n+1})\right) + \psi\left(d(gz, fz)\right) - \phi\left(d(gz, fz)\right).$$

Letting $n$ tend to infinity and using the continuity of $\psi$, we get

$$\psi\left(d(gz, fz)\right) \le \psi\left(d(gz, fz)\right) - \phi\left(d(gz, fz)\right).$$

Hence $\phi(d(gz, fz)) = 0$, a contradiction with $(\phi_2)$ and the assumption $d(gz, fz) > 0$.

Similarly, if $M(z, x_{n+1}) = d(fx_n, fz)$, then from (48)

$$\psi\left(d(gz, fz)\right) \le \psi\left(d(gz, fx_n)\right) + \psi\left(d(fx_n, fz)\right) - \phi\left(d(fx_n, fz)\right).$$

Letting $n$ tend to infinity and having in mind that $d(fx_n, fz) \to d(gz, fz)+$, we obtain

$$\psi\left(d(gz, fz)\right) \le \psi\left(d(gz, fz)\right) - \lim_{d(fx_n, fz)\to d(gz, fz)+} \phi\left(d(fx_n, fz)\right)$$

and hence we get

$$\lim_{d(fx_n, fz)\to d(gz, fz)+} \phi\left(d(fx_n, fz)\right) \le 0,$$

a contradiction with ($\phi_1$). Thus our assumption $d(gz,fz) > 0$ is wrong. Therefore, $d(gz, fz) = 0$. Hence $gz = fz$, that is, $z$ is a coincidence point of $f$ and $g$.

If the condition (x) is fulfilled, put $w = fz = gz$. We will show that $w$ is a common fixed point of $f$ and $g$. Since $fz = gz$ and $f$ and $g$ are weakly compatible, we obtain, by the definition of weak compatibility, that $fgz = gfz$. Thus we have $fw = gw$. Using the condition (ix) that $gz \preceq ggz = gw$, we can apply the contractive condition (35) to obtain

$$\psi\big(d(fw,fz)\big) \le \psi\big(M(w,z)\big) - \phi\big(M(w,z)\big),$$

where

$$M(w,z) = \max\big\{d(gw,gz), d(gw,fw), d(gz,fz), d(gw,fz), d(gz,fw)\big\} = d(fw,fz).$$

Thus

$$\psi\big(d(fw,fz)\big) \le \psi\big(d(fw,fz)\big) - \phi\big(d(fw,fz)\big).$$

Hence $\phi(d(fw,fz)) = 0$, and so by ($\phi_2$), $d(fw,fz) = 0$. Hence $fw = fz$. Therefore

$$w = fz = fw = ffz = gfz = gw.$$

Thus we showed that $w$ is a common fixed point of $f$ and $g$.

Suppose now that the condition (xi) is fulfilled. Since a set of common fixed points of $f$ and $g$ is not empty, assume that $w$ and $v$ are two common fixed points of $f$ and $g$, *i.e.*,

$$fw = gw = w, \qquad fv = gv = v. \tag{50}$$

We claim that $gw = gv$.

By assumption, there exists $y_0 \in X$ such that $fy_0$ is comparable to $fw$ and $fv$. Define a sequence $\{gy_n\}$ such that

$$gy_n = fy_{n-1} \quad \text{for all } n. \tag{51}$$

Further, set $w_0 = w$ and $v_0 = v$ and, in the same way, define $\{gw_n\}$ and $\{gv_n\}$ such that

$$gw_n = fw_{n-1}, \qquad gv_n = fv_{n-1} \quad \text{for all } n. \tag{52}$$

From (50) and (52), we have $fw_0 = gw_1 = gw_0$ and $fv_0 = gv_1 = gv_0$. Since $fy_0$ is comparable to $fw$ and $fv$, and $f$ is $g$-nondecreasing, it is easy to show

$$gw \succeq gy_1. \tag{53}$$

Recursively, we can get that

$$gy_n \preceq gw \quad \text{for all } n. \tag{54}$$

By (35), we have that

$$\begin{aligned}
&\psi\big(d(gy_{n+1},gw)\big)\\
&\quad= \psi\big(d(fy_n,fw)\big)\\
&\quad\le \psi\big(\max\{d(gy_n,gw),d(gy_n,fy_n),d(gw,fw),d(gy_n,fw),d(gw,fy_n)\}\big)\\
&\qquad - \phi\big(\max\{d(gy_n,gw),d(gy_n,fy_n),d(gw,fw),d(gy_n,fw),d(gw,fy_n)\}\big). \qquad (55)
\end{aligned}$$

Similarly as in the proof of Theorem 2.1, we can prove that $\{gy_n\}$ is a convergent sequence. Thus there exists $\bar{y} \in X$ such that $gy_n \to g\bar{y}$. Since also $\lim_{n\to\infty} fy_n = g\bar{y}$, for large enough $n$, we have

$$\max\{d(gy_n,gw),d(gy_n,fy_n),d(gw,fw),d(gy_n,fw),d(gw,fy_n)\} = d(g\bar{y},gw).$$

Thus from (55), for large enough $n$,

$$\psi\big(d(gy_{n+1},gw)\big) \le \psi\big(d(g\bar{y},gw)\big) - \phi\big(d(g\bar{y},gw)\big). \qquad (56)$$

Letting $n \to \infty$ in (56), by $(\psi_1)$ we get

$$\lim_{n\to\infty} \psi\big(d(gy_{n+1},gw)\big) = \psi\big(d(g\bar{y},gw)\big) \le \psi\big(d(g\bar{y},gw)\big) - \phi\big(d(g\bar{y},gw)\big).$$

Hence we obtain

$$\psi\big(d(g\bar{y},gw)\big) = 0.$$

Then by $(\psi_2)$, $d(g\bar{y},gw) = 0$ and hence

$$g\bar{y} = gw. \qquad (57)$$

Similarly, we can show that

$$\lim_{n\to\infty} \psi\big(d(gy_{n+1},gv)\big) = \psi\big(d(g\bar{y},gv)\big) \le \psi\big(d(g\bar{y},gv)\big) - \phi\big(d(g\bar{y},gv)\big),$$

and hence we obtain

$$g\bar{y} = gv. \qquad (58)$$

Therefore, from (57) and (58), we have $gw = gv$. It follows that

$$w = fw = gw = gv = fv = v. \qquad (59)$$

It means that $w$ is the unique common fixed point of $f$ and $g$. □

**Corollary 4.1** *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let $f$ be a self-mapping of $X$ satisfying the following condition*:

$$d(fx, fy) \leq m(x, y) - \phi\bigl(m(x, y)\bigr)$$

*for all $x, y \in X$ such that $gy \preceq gx$, where*

$$m(x, y) = \max\bigl\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\bigr\}$$

*and $\phi \in \Phi$. Suppose that, in addition, $t - \phi(t)$ is non-decreasing, $\lim_{t\to+\infty} \phi(t) = \infty$, there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ and if $\{fx_n\}$ is a nondecreasing sequence such that it converges to some $z \in X$, then $fx_n \preceq z$. Then $f$ has a unique fixed point.*

*Proof* Taking $\psi(t) = t$ and $g(t) = t$ in the proof of Theorem 4.1, we obtain Corollary 4.1. □

**Remark 4.1** Theorem 4.1 extends Theorem 1 due to Berinde [21], Theorems 2.1 and 2.5 due to Beg and Abbas [22] and Theorem 3.1 due to Song [23].

We present an example to show that our result is a real generalization of the recent result of Golubović *et al.* [16] as well as of the existing results in the literature.

**Example 4.1** Let $X = [0, \frac{1}{2}]$ be the closed interval with the usual metric and let $f, g : X \to X$ and $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be mappings defined as follows:

$$\begin{aligned}
&f(x) = x^2 - x^4 \quad \text{for all } x \in X, \\
&g(x) = x^2 \quad \text{for all } x \in X, \\
&\psi(t) = t \quad \text{for all } x \in X, \\
&\phi(t) = t^2 \quad \text{for } 0 \leq t \leq \frac{1}{2}, \\
&\phi(t) = \frac{1}{2}t \quad \text{for } t > \frac{1}{2}.
\end{aligned}$$

Let $x$, $y$ in $X$ be arbitrary. We say that $x \preceq y$ if $x \leq y$. For any $x, y \in X$ such that $x \preceq y$, we have

$$\begin{aligned}
M(x, y) &= \max\bigl\{d\bigl(g(x), g(y)\bigr), d\bigl(g(x), f(x)\bigr), d\bigl(g(y), f(y)\bigr), d\bigl(g(x), f(y)\bigr), d\bigl(g(y), f(x)\bigr)\bigr\} \\
&= d\bigl(g(y), f(x)\bigr), \\
\psi\bigl(d\bigl(g(y), f(x)\bigr)\bigr) &= d\bigl(g(y), f(x)\bigr) = \bigl|y^2 - x^2\bigl(1 - x^2\bigr)\bigr| \\
&= y^2 - x^2\bigl(1 - x^2\bigr).
\end{aligned}$$

Since $y^2 \geq y^2 - x^2(1 - x^2)$ for all $x \in [0, \frac{1}{2}]$, it follows that

$$-y^4 \leq -\bigl(y^2 - x^2\bigl(1 - x^2\bigr)\bigr)^2.$$

Thus we have

$$\begin{aligned}\psi\left(d\big(f(x),f(y)\big)\right) &= \left|y^2-y^4-x^2+x^4\right| = \big(y^2-x^2\big(1-x^2\big)\big)-y^4\\ &\le \big(y^2-x^2\big(1-x^2\big)\big)-\big(y^2-x^2\big(1-x^2\big)\big)^2\\ &= d\big(g(y),f(x)\big)-\big[d\big(g(y),f(x)\big)\big]^2\\ &= \psi\big(M(x,y)\big)-\phi\big(M(x,y)\big).\end{aligned}$$

Therefore, $f$ and $g$ satisfy (35). Also, it is easy to see that the mappings $\psi(t)$ and $\phi(t)$ possess all properties $(\psi_1)$, $(\psi_2)$ and $(\phi_1)$, $(\phi_2)$ respectively, as well as hypotheses (v), (vi) and (vii) in Theorem 4.1. Thus we can apply our Theorem 4.1 and Corollary 4.1.

On the other hand, for $x = 0$ and each $y > 0$, the contractive condition in Theorems 1 and 2 of Golubović *et al.* [16]:

$$d(fx,fy) \le \lambda \cdot M(x,y), \tag{60}$$

where $0 < \lambda < 1$ and

$$M(x;y) = \max\big\{d(gx;gy);d(gx;fx);d(gy;fy);d(gx;fy),d(gy;fx)\big\},$$

is not satisfied. Indeed,

$$\begin{aligned}M(0;y) &= \max\big\{d\big(g(0);g(y)\big);d\big(g(0);f(0)\big);d\big(g(y);f(y)\big);d\big(g(0);f(y)\big),d\big(g(y);f(0)\big)\big\}\\ &= \max\big\{y^2;0;y^4;\big(y^2-y^4\big),y^2\big\} = y^2.\end{aligned}$$

Thus, for any fixed $\lambda$; $0 < \lambda < 1$, we have, for $x = 0$ and each $y \in X$ with $0 < y < \sqrt{1-\lambda}$,

$$\begin{aligned}d\big(f(0),f(y)\big) &= y^2-y^4 = \big(1-y^2\big)y^2 > \lambda \cdot y^2\\ &= \lambda \cdot d\big(g(y),g(0)\big) = \lambda \cdot M(0,y).\end{aligned}$$

Thus, $f$ does not satisfy (60). Therefore, the theorems of Jungck and Hussain [24], Al-Thagafi and Shahzad [25] and Das and Naik [26] also cannot be applied.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

Both authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

**Author details**

[1]Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan 610068, China. [2]School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China. [3]Department of Applied Mathematics, Faculty of Electrical Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, Belgrade, Serbia.

**Acknowledgements**

Siniša Ješić was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, Project grant number 174032. Xiaolan Liu was supported by Scientific Research Fund of Sichuan Provincial Education Department (12ZA098), Scientific Research Fund of Sichuan University of Science and Engineering (2012KY08), and Scientific Research Fund of School of Science SUSE (10LXYB03).

Received: 9 October 2012 Accepted: 21 February 2013 Published: 12 March 2013

**References**

1. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**(1), 109-116 (2008)
2. Agarwal, RP, O'Regan, D, Sambandham, M: Random and deterministic fixed point theory for generalized contractive maps. Appl. Anal. **83**(7), 711-725 (2004)
3. Ahmad, B, Nieto, JJ: The monotone iterative technique for three-point second-order integrodifferential boundary value problems with $p$-Laplacian. Bound. Value Probl. **2007**, Article ID 57481 (2007)
4. Boyd, DW, Wong, JSW: On nonlinear contractions. Proc. Am. Math. Soc. **20**(2), 458-464 (1969)
5. Cabada, A, Nieto, JJ: Fixed points and approximate solutions for nonlinear operator equations. J. Comput. Appl. Math. **113**(1-2), 17-25 (2000)
6. Ćirić, LB: Generalized contractions and fixed-point theorems. Publ. Inst. Math. (Belgr.) **12**(26), 19-26 (1971)
7. Ćirić, LB: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45**(2), 267-273 (1974)
8. Ćirić, LB: Fixed points of weakly contraction mappings. Publ. Inst. Math. (Belgr.) **20**(34), 79-84 (1976)
9. Ćirić, LB: Coincidence and fixed points for maps on topological spaces. Topol. Appl. **154**(17), 3100-3106 (2007)
10. Ćirić, LB, Ume, JS: Nonlinear quasi-contractions on metric spaces. Prakt. Akad. Athēnōn **76**(A), 132-141 (2001)
11. Ćirić, LB: Common fixed points of nonlinear contractions. Acta Math. Hung. **80**(1-2), 31-38 (1998)
12. Drici, Z, McRae, FA, Vasundhara Devi, J: Fixed-point theorems in partially ordered metric spaces for operators with PPF dependence. Nonlinear Anal., Theory Methods Appl. **67**(2), 641-647 (2007)
13. Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal., Theory Methods Appl. **65**(7), 1379-1393 (2006)
14. Gajić, L, Rakočević, V: Quasicontraction nonself-mappings on convex metric spaces and common fixed point theorems. Fixed Point Theory Appl. **3**, 365-375 (2005)
15. Hussain, N: Common fixed points in best approximation for Banach operator pairs with Ćrić type $I$-contractions. J. Math. Anal. Appl. **338**(2), 1351-1363 (2008)
16. Golubović, Z, Kadelburg, Z, Radenović, S: Common fixed points of ordered $g$-quasicontractions and weak contractions in ordered metric spaces. Fixed Point Theory Appl. **2012**, 20 (2012)
17. Jungck, G: Commuting mappings and fixed points. Am. Math. Mon. **83**, 261-263 (1976). doi:10.2307/2318216
18. Jungck, G: Compatible mappings and common fixed points. Int. J. Math. Math. Sci. **9**, 771-779 (1986). doi:10.1155/S0161171286000935
19. Ćirić, LB, Cakić, N, Rajović, M, Ume, JS: Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2008**, Article ID 131294 (2008). doi:10.1155/2008/131294
20. Das, KM, Naik, KV: Common fixed point theorems for commuting maps on a metric space. Proc. Am. Math. Soc. **77**, 369-373 (1979)
21. Berinde, V: A common fixed point theorem for quasi contractive type mappings. Ann. Univ. Sci. Bp. **46**, 81-90 (2003)
22. Beg, I, Abbas, M: Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition. Fixed Point Theory Appl. **2006**, Article ID 74503 (2006)
23. Song, Y: Coincidence points for noncommuting $f$-weakly contractive mappings. Int. J. Comput. Appl. Math. **2**, 51-57 (2007)
24. Jungck, G, Hussain, N: Compatible maps and invariant approximations. J. Math. Anal. Appl. **325**, 1003-1012 (2007)
25. Al-Thagafi, MA, Shahzad, N: Banach operator pairs, common fixed points, invariant approximations and $*$-nonexpansive multimaps. Nonlinear Anal. **69**, 2733-2739 (2008)
26. Das, KM, Naik, KV: Common fixed point theorems for commuting maps on a metric space. Proc. Am. Math. Soc. **77**, 369-373 (1979)

doi:10.1186/1687-1812-2013-53
**Cite this article as:** Liu and Ješić: **Common fixed points of a generalized ordered $g$-quasicontraction in partially ordered metric spaces.** *Fixed Point Theory and Applications* 2013 **2013**:53.

**Submit your manuscript to a SpringerOpen journal and benefit from:**

- ► Convenient online submission
- ► Rigorous peer review
- ► Immediate publication on acceptance
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

**Submit your next manuscript at ► springeropen.com**