Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for Φ_p operator in cone Banach spaces

Ali Mutlu* and Nermin Yolcu

*Correspondence:
ali.mutlu@cbu.edu.tr
Department of Mathematics, Faculty of Science and Arts, Celal Bayar University, Muradiye Campus, Manisa, 45047, Turkey

Abstract
In this paper a class of self-mappings on cone Banach spaces which have at least one fixed point is considered. More precisely, for a closed and convex subset C of a cone Banach space with the norm $\|x\|_C = d(x,0)$, if there exist a, b, c, r and $T : C \to C$ satisfies the conditions $0 \le \Phi_q(r) < \Phi_q(a) + 2(\Phi_q(b) + \Phi_q(c))$ and $a\Phi_p(d(Tx,Ty)) + b\Phi_p(d(x,Tx)) + c\Phi_p(d(y,Ty)) \le r\Phi_p(d(x,y))$ for all $x, y \in C$, then T has at least one fixed point.

MSC: 47H10; 54H25

Keywords: cone metric space; complete cone metric space; fixed point

1 Introduction

Lin [1] considered the notion of K-metric spaces by replacing real numbers with a cone K in the metric function, that is, $d : X \times X \to K$. Without mentioning the papers of Lin and Rzepecki, in 2007, Huang and Zhang [2] announced the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach space. In that paper, they also discussed some properties of the convergence of sequences and proved the fixed point theorems of a contractive mapping for cone metric spaces: Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \le k < 1$, the inequality

$$d(Tx,Ty) \le kd(x,y) \tag{1}$$

for all $x, y \in X$ has a unique fixed point. Recently, many results on fixed point theorems have been extended to cone metric spaces in [3, 4]. Karapınar [5] extended some of well known results in the fixed point theory to cone Banach spaces which were defined and used in [6] where the existence of fixed points for self-mappings on cone Banach spaces were investigated.

In this study, we prove the fixed point theorems of Φ_p operator for cone Banach spaces.

Throughout this paper E means Banach algebra, $E := (E, \|\cdot\|)$ stands for real Banach space. Let $P := P_E$ always be a closed nonempty subset of E. P is called a cone if $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b where $P \cap (-P) = \{0\}$ and $P \neq 0$. Given a cone $P \subset E$, we define a partial ordering $\le$ with respect to P by $x \le y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \le y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \le x \le y$ implies $\|x\| \le K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

From now on, if we suppose that E is a real Banach space, then P is a cone in E with $\operatorname{int} P \neq \emptyset$, and $\leq$ is partial ordering with respect to P. Let X be a nonempty set, a function $d : X \times X \to E$. is called a cone metric on X if it satisfies the following conditions with respect to [2]:

(i) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then (X,d) is called a cone metric space (CMS).

Definition 1 (see [5]) Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|\cdot\|_C : X \to E$ satisfies

(N1) $\|x\|_C \geq 0$ for all $x \in X$,
(N2) $\|x\|_C = 0$ if and only if $x = 0$,
(N3) $\|x + y\|_C \leq \|x\|_C + \|y\|_C$ for all $x, y \in X$,
(N4) $\|kx\|_C = |k|\|x\|_C$ for all $k \in \mathbb{R}$ and for all $x \in X$,

then $\|\cdot\|_C$ is called a cone norm on X, and the pair $(X, \|\cdot\|_C)$ is called a cone normed space (CNS).

Definition 2 (see [5]) Let $(X, \|\cdot\|_C)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $\|x_n - x\|_C \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;
(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N, such that $\|x_n - x_m\|_C \ll c$ for all $n, m \geq N$;
(iii) $(X, \|\cdot\|_C)$ is a complete cone normed space if every Cauchy sequence is convergent.

Complete cone normed spaces will be called cone Banach spaces.

Lemma 3 *Let $(X, \|\cdot\|_C)$ be a CNS, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then*

(i) *the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_C \to 0$ as $n \to \infty$,*
(ii) *the sequence $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\|_C \to 0$ as $n, m \to \infty$,*
(iii) *the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y, then $\|x_n - y_n\|_C \to \|x - y\|_C$.*

Proof See [2] Lemmas 1, 4 and 5. □

Definition 4 (see [5]) Let C be a closed and convex subset of a cone Banach space with the norm $\|\cdot\|_C$ and $T : C \to C$ be a mapping. Consider the condition

$$\|Tx - Ty\|_C \leq \|x - y\|_C \quad \text{for all } x, y \in C, \tag{2}$$

then T is called nonexpansive if it satisfies the condition (2).

2 Main result

From now on, $X = (X, \|\cdot\|_C)$ will be a cone Banach space, P will be a normal cone with a normal constant K and T, a self-mapping operator defined on a subset C of X. Let Φ_p be an increasing, positive and self-mapping operator defined on E, where E is a Banach algebras. In this paper, we give a generalization of Theorem 2.4 in [5] for Φ_p operator.

Definition 5 Let E be Banach algebra and $(E, \|\cdot\|_C)$ be a Banach space. $\Phi_p : E \to E$ is an increasing and positive mapping, *i.e.*, $\Phi_p(x) = \|x\|^{p-2}x$, where $\frac{1}{p} + \frac{1}{q} = 1$.

If $E = \mathbb{R}$, then $\Phi_p : \mathbb{R} \to \mathbb{R}$ is a p-Laplacian operator, *i.e.*, $\Phi_p(x) = |x|^{p-2}x$ for some $p > 1$.

By using this definition, we can show that the operator $\Phi_p : E \to E$ holds the following properties:

(1) if $x \leq y$, then $\Phi_p(x) \leq \Phi_p(y)$ for all $x, y \in E$,
(2) Φ_p is a continuous bijection and its inverse mapping is also continuous,
(3) $\Phi_p(xy) = \Phi_p(x)\Phi_p(y)$ for all $x, y \in E$,
(4) $\Phi_p(x + y) \leq \Phi_p(x) + \Phi_p(y)$ for all $x, y \in E$.

Theorem 6 *Let C be a closed and convex subset of a cone Banach space X with the norm $\|\cdot\|_C$. Let E be a Banach algebra and $\Phi_p : E \to E$ and $T : C \to C$ be mappings and T satisfy the following condition*:

$$\Phi_p\big(d(x, Tx)\big) + \Phi_p\big(d(y, Ty)\big) \leq k\Phi_p\big(d(x, y)\big) \tag{3}$$

for all $x, y \in C$, where $2^{p-1} \leq k < 4^{p-1}$ in E. Then T has at least one fixed point.

Proof Let $x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ in the following way:

$$x_{n+1} = \frac{x_n + Tx_n}{2}, \quad n = 0, 1, 2, \ldots \tag{4}$$

Then

$$x_n - Tx_n = 2(x_n - x_{n+1})$$

which yields that

$$d(x_n, Tx_n) = \|x_n - Tx_n\|_C = 2\|x_n - x_{n+1}\|_C = 2d(x_n, x_{n+1}). \tag{5}$$

Substitute $x = x_{n-1}$ and $y = x_n$ in (3). Then we obtain

$$\Phi_p\big(d(x_{n-1}, Tx_{n-1})\big) + \Phi_p\big(d(x_n, Tx_n)\big) \leq k\Phi_p\big(d(x_{n-1}, x_n)\big).$$

By (5), we can obtain

$$\Phi_p\big(2d(x_{n-1}, x_n)\big) + \Phi_p\big(2d(x_n, x_{n+1})\big) \leq k\Phi_p\big(d(x_{n-1}, x_n)\big).$$

From the property of Φ_p operator,

$$\Phi_p\big(2\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big]\big) \leq k\Phi_p\big(d(x_{n-1}, x_n)\big),$$

when the essential arrangement is applied, we can get

$$d(x_n, x_{n+1}) \leq \left(\frac{\Phi_q(k)}{2} - 1\right) d(x_{n-1}, x_n).$$

Repeating this relation, we get

$$d(x_n, x_{n+1}) \leq \left(\frac{\Phi_q(k)}{2} - 1\right)^n d(x_0, x_1). \tag{6}$$

Let $m > n$; then from (6), we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \left[\left(\frac{\Phi_q(k)}{2} - 1\right)^{m-1} + \cdots + \left(\frac{\Phi_q(k)}{2} - 1\right)^n\right] d(x_1, x_0) \\ &\leq \frac{(\frac{\Phi_q(k)}{2} - 1)^n}{2 - \frac{\Phi_q(k)}{2}} d(x_1, x_0). \end{aligned}$$

Since $2^{p-1} \leq k < 4^{p-1}$, $\{x_n\}$ is a Cauchy sequence in C. Because C is a closed and convex subset of a cone Banach space, thus $\{x_n\}$ sequence converges to some $z \in C$. That is, $x_n \to z$, $z \in C$.

Regarding the inequality,

$$d(z, Tx_n) \leq d(z, x_n) + d(x_n, Tx_n),$$

from (5),

$$d(z, Tx_n) \leq d(z, x_n) + 2d(x_n, x_{n+1}).$$

as $n \to \infty$, then $d(z, Tx_n) \leq 0$. Thus $Tx_n \to z$.

Finally, we substitute $x = z$ and $y = x_n$ in (3). Then we can get

$$\Phi_p\big(d(z, Tz)\big) + \Phi_p\big(d(x_n, Tx_n)\big) \leq k\Phi_p\big(d(z, x_n)\big)$$

from the property of Φ_p mapping,

$$\Phi_p\big(d(z, Tz) + d(x_n, Tx_n)\big) \leq k\Phi_p\big(d(z, x_n)\big).$$

By using (5), we obtain

$$d(z, Tz) + 2d(x_n, x_{n+1}) \leq \Phi_q(k) d(z, x_n),$$

when $n \to \infty$, $d(z, Tz) = 0$. Then $Tz = z$. □

Theorem 7 *Let C be a closed and convex subset of a cone Banach space X with the norm $\|\cdot\|_C$. Let E be a Banach algebra, $\Phi_p : E \to E$ and $T : C \to C$ be mappings. If there exist a, b, c, r in E and T satisfies the following conditions*:

$$a\Phi_p\big(d(Tx, Ty)\big) + b\Phi_p\big(d(x, Tx)\big) + c\Phi_p\big(d(y, Ty)\big) \leq r\Phi_p\big(d(x, y)\big) \tag{7}$$

for all $x, y \in C$, where $0 \leq \Phi_q(r) < \Phi_q(a) + 2(\Phi_q(b) + \Phi_q(c))$, then T has at least one fixed point.

Proof Construct a sequence $\{x_n\}$ as in the proof of Theorem 6. Then

$$x_n - Tx_{n-1} = \frac{1}{2}(x_{n-1} - Tx_{n-1}).$$

Thus,

$$d(x_n, Tx_{n-1}) = \frac{1}{2}d(x_{n-1}, Tx_{n-1}). \tag{8}$$

In addition, we know $d(x_n, Tx_n) = 2d(x_n, x_{n+1})$. Thus the triangle inequality implies

$$d(x_n, Tx_n) - d(x_n, Tx_{n-1}) \le d(Tx_{n-1}, Tx_n).$$

Then, from (8) and (5), we can get

$$2d(x_n, x_{n+1}) - d(x_{n-1}, x_n) \le d(Tx_{n-1}, Tx_n).$$

By substituting $x = x_{n-1}$ and $y = x_n$ in (7), we obtain

$$a\Phi_p\big(d(Tx_{n-1}, Tx_n)\big) + b\Phi_p\big(d(x_{n-1}, Tx_{n-1})\big) + c\Phi_p\big(d(x_n, Tx_n)\big) \le r\Phi_p\big(d(x_{n-1}, x_n)\big).$$

As in the proof of Theorem 6, we can obtain

$$d(x_n, x_{n+1}) \le \left(\frac{\Phi_q(r) + \Phi_q(a) - 2\Phi_q(b)}{2\Phi_q(a) + 2\Phi_q(c)}\right)d(x_{n-1}, x_n)$$

for all $n \ge 1$. Repeating this relation, we get

$$d(x_n, x_{n+1}) \le h^n d(x_0, x_1), \tag{9}$$

where $h = (\frac{\Phi_q(r)+\Phi_q(a)-2\Phi_q(b)}{2\Phi_q(a)+2\Phi_q(c)}) < 1$. Let $m > n$; then from (9), we have

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\le \big[h^{m-1} + \cdots + h^n\big]d(x_1, x_0) \\ &\le \frac{h^n}{1-h}d(x_1, x_0). \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in C and thus it converges to some $z \in C$. As in the proof of Theorem 6, we can show $Tz = z$. □

Competing interests
The authors declare that they have no competing interests.

Authors' information
The authors did not provide this information

Received: 10 December 2012 Accepted: 25 February 2013 Published: 14 March 2013

References

1. Lin, SD: A common fixed point theorem in abstract spaces. Indian J. Pure Appl. Math. **18**, 685-690 (1987)
2. Huang, LG, Zhang, X: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**, 1468-1476 (2007)
3. Kirk, WA: A fixed point theorem for mappings which do not increase distance. Am. Math. Mon. **72**, 1004-1006 (1965)
4. Şahin, İ, Telci, M Fixed points of contractive mappings on complete cone metric spaces. Hacet. J. Math. Stat. **38**, 59-67 (2009)
5. Karapınar, E: Fixed point theorems in cone Banach spaces. Fixed Point Theory Appl. **2009**, 1-9 (2009)
6. Abdeljawad, T, Türkoğlu, D, Abuloha, M: Some theorems and examples of cone metric spaces. J. Comput. Anal. Appl. **12**(4), 739-753 (2010)

doi:10.1186/1687-1812-2013-56
Cite this article as: Mutlu and Yolcu: Fixed point theorems for Φ_p operator in cone Banach spaces. *Fixed Point Theory and Applications* 2013 **2013**:56.