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Krasnoselskii-type algorithm for fixed points of multi-valued strictly pseudo-contractive mappings

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Abstract

Let $q > 1$ and let K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued strictly pseudo-contractive map with a nonempty fixed point set. A Krasnoselskii-type iteration sequence $\{x_n\}$ is constructed and proved to be an approximate fixed point sequence of T, *i.e.*, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. This result is then applied to prove strong convergence theorems for a fixed point of T under additional appropriate conditions. Our theorems improve several important well-known results.

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1 Introduction

For decades, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well-known mathematicians (see, for example, Brouwer [1], Kakutani [2], Nash [3, 4], Geanakoplos [5], Nadla [6], Downing and Kirk [7]).

Interest in the study of fixed point theory for multi-valued maps stems, perhaps, mainly from the fact that many problems in some areas of mathematics such as in *Game Theory and Market Economy* and in *Non-Smooth Differential Equations* can be written as fixed point problems for multi-valued maps. We describe briefly the connection of fixed point theory for multi-valued mappings and these applications.

Game theory and market economy

In game theory and market economy, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [3, 4] showed the existence of equilibria for non-cooperative static games as a direct consequence of Brouwer [1] or Kakutani [2] fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued map* whose fixed points coincide with the equilibrium points of the game. A model example of such an application is the *Nash equilibrium theorem* (see, *e.g.*, [3]).

Consider a game $G = (u_n, K_n)$ with N players denoted by n, $n = 1, \dots, N$, where $K_n \subset \mathbb{R}^{m_n}$ is the set of possible strategies of the nth player and is assumed to be nonempty, compact and convex and $u_n : K := K_1 \times K_2 \cdots \times K_N \to \mathbb{R}$ is the payoff (or gain function) of the player n and is assumed to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n \in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$. For each n, $\sigma \in K$ and $z_n \in K_n$, we will use the following standard notations:

$$K_{-n} := K_1 \times \cdots \times K_{n-1} \times K_{n+1} \times \cdots \times K_N,$$

$$\sigma_{-n} := (\sigma_1, \dots, \sigma_{n-1}, \sigma_{n+1}, \dots, \sigma_N),$$

$$(z_n, \sigma_{-n}) := (\sigma_1, \dots, \sigma_{n-1}, z_n, \sigma_{n+1}, \dots, \sigma_N).$$

A strategy $\bar{\sigma}_n \in K_n$ permits the nth player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} if and only if

$$u_n(\bar{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}).$$

Now, let $T_n : K_{-n} \to 2^{K_n}$ be the multi-valued map defined by

$$T_n(\sigma_{-n}) := \operatorname*{Arg\,max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \quad \forall \sigma_{-n} \in K_{-n}.$$

Definition A collective action $\bar{\sigma} = (\bar{\sigma}_1, \dots, \bar{\sigma}_N) \in K$ is called a *Nash equilibrium point* if, for each n, $\bar{\sigma}_n$ is the best response for the nth player to the action $\bar{\sigma}_{-n}$ made by the remaining players. That is, for each n,

$$u_n(\bar{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \bar{\sigma}_{-n}) \tag{1.1}$$

or, equivalently,

$$\bar{\sigma}_n \in T_n(\bar{\sigma}_{-n}). \tag{1.2}$$

This is equivalent to $\bar{\sigma}$ is a fixed point of the multi-valued map $T : K \to 2^K$ defined by

$$T(\sigma) := \bigl[T_1(\sigma_{-1}), T_2(\sigma_{-2}), \dots, T_N(\sigma_{-N})\bigr].$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of fixed point theory of multi-valued mappings. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding of conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to an equilibrium solution. This is part of the problem that is being addressed by iterative methods for a fixed point of multi-valued mappings.

Non-smooth differential equations

The mainstream of applications of fixed point theory for multi-valued maps has been initially motivated by the problem of differential equations (DEs) with discontinuous right-hand sides which gave birth to the existence theory of differential inclusion (DI). Here is a simple model for this type of application.

Consider the initial value problem

$$\frac{du}{dt} = f(t,u), \quad \text{a.e. } t \in I := [-a,a], u(0) = u_0. \tag{1.3}$$

If $f : I \times \mathbb{R} \to \mathbb{R}$ is discontinuous with bounded jumps, measurable in t, one looks for *solutions* in the sense of Filippov [8] which are solutions of the differential inclusion

$$\frac{du}{dt} \in F(t,u), \quad \text{a.e. } t \in I, u(0) = u_0, \tag{1.4}$$

where

$$F(t,x) = \Big[\liminf_{y\to x} f(t,y), \limsup_{y\to x} f(t,y)\Big]. \tag{1.5}$$

Now, set $H := L^2(I)$ and let $N_F : H \to 2^H$ be the *multi-valued Nemystkii operator* defined by

$$N_F(u) := \big\{v \in H : v(t) \in F\big(t,u(t)\big) \text{ a.e. } t \in I\big\}.$$

Finally, let $T : H \to 2^H$ be a multi-valued map defined by $T := N_F \circ L^{-1}$, where L^{-1} is the inverse of the derivative operator $Lu = u'$ given by

$$L^{-1}v(t) := u_0 + \int_0^t v(s)\,ds.$$

One can see that problem (1.4) reduces to the fixed point problem: $u \in Tu$.

Finally, a variety of fixed point theorems for multi-valued maps with nonempty and convex values is available to conclude the existence of a solution. We used a first-order differential equation as a model for simplicity of presentation, but this approach is most commonly used with respect to second-order boundary value problems for ordinary differential equations or partial differential equations. For more details about these topics, one can consult [9–12] and references therein as examples. Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E, ρ_E, is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}, \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

It is well known (see, *e.g.*, [13], p.16, [14]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$, where

$$L_p \text{ (or } l_p) \text{ or } W_p^m \text{ is } \begin{cases} 2\text{-uniformly smooth} & \text{if } 2 \leq p < \infty; \\ p\text{-uniformly smooth} & \text{if } 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\},$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J. It is well known that if E is smooth, J_q is single-valued.

Every uniformly smooth space has a uniformly Gâteaux differentiable norm (see, *e.g.*, [13], p.17).

Let K be a nonempty subset of E. The set K is called *proximinal* (see, *e.g.*, [15–17]) if for each $x \in E$, there exists $u \in K$ such that

$$d(x,u) = \inf\{\|x-y\| : y \in K\} = d(x,K),$$

where $d(x,y) = \|x-y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(K)$ and $P(K)$ denote the families of nonempty, closed and bounded subsets and nonempty, proximinal and bounded subsets of K, respectively. The *Hausdorff metric* on $CB(K)$ is defined by

$$D(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\}$$

for all $A, B \in CB(K)$. Let $T : D(T) \subseteq E \to CB(E)$ be a *multi-valued mapping* on E. A point $x \in D(T)$ is called a *fixed point of* T if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$D(Tx, Ty) \le L\|x-y\| \quad \forall x, y \in D(T). \tag{1.6}$$

When $L \in (0,1)$ in (1.6), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Definition 1.1 Let K be a nonempty subset of a real Hilbert space H. A map $T : K \to H$ is called *k-strictly pseudo-contractive* if there exists $k \in (0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\big\|x - y - (Tx - Ty)\big\|^2 \quad \forall x, y \in K. \tag{1.7}$$

Browder and Petryshyn [18] introduced and studied the class of strictly pseudo-contractive maps as an important generalization of the class of nonexpansive maps (mappings $T : K \to K$ satisfying $\|Tx - Ty\| \le \|x-y\|$ $\forall x, y \in K$). It is trivial to see that every nonexpansive map is strictly pseudo-contractive.

Motivated by this, Chidume *et al.* [19] introduced the class of *multi-valued strictly pseudo-contractive* maps defined on a real Hilbert space H as follows.

Definition 1.2 A multi-valued map $T : D(T) \subset H \to CB(H)$ is called *k-strictly pseudo-contractive* if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$,

$$\big(D(Tx, Ty)\big)^2 \le \|x-y\|^2 + k\big\|x - y - (u - v)\big\|^2 \quad \forall u \in Tx, v \in Ty. \tag{1.8}$$

They then proved convergence theorems for approximating fixed points of multi-valued strictly pseudo-contractive maps (see [19]) which extend recent results from the class of multi-valued *nonexpansive maps* to the more general and important class of multi-valued *strictly pseudo-contractive maps*.

Single-valued strictly pseudo-contractive maps have also been defined and studied *in real Banach spaces, which are much more general than Hilbert spaces*.

Definition 1.3 Let K be a nonempty subset of a real normed space E. A map $T : K \to E$ is called *k-strictly pseudo-contractive* (see, *e.g.*, [13], p.145, [18]) if there exists $k \in (0,1)$ such that for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\bigl\langle Tx - Ty, j(x-y)\bigr\rangle \le \|x-y\|^2 - k\bigl\|x - y - (Tx - Ty)\bigr\|^2. \tag{1.9}$$

In this paper, we define multi-valued strictly pseudo-contractive maps in *arbitrary normed space E* as follows.

Definition 1.4 A multi-valued map $T : D(T) \subset E \to CB(E)$ is called *k-strictly pseudo-contractive* if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$,

$$k\bigl(D(Ax, Ay)\bigr)^2 \le \bigl\langle u - v, j(x-y)\bigr\rangle \quad \forall u \in Ax, v \in Ay, \tag{1.10}$$

where $A := I - T$ and I is the identity map on E.

We observe that if T is single-valued, then inequality (1.10) reduces to (1.9).

Several papers deal with the problem of approximating fixed points of *multi-valued nonexpansive mappings defined on Hilbert spaces* (see, for example, Sastry and Babu [15], Panyanak [16], Song and Wong [17], Khan *et al.* [20], Abbas *et al.* [21] and the references contained therein) and their generalizations (see, *e.g.*, Chidume *et al.* [19] and the references contained therein).

Chidume *et al.* [19] proved the following theorem for multi-valued k-strictly pseudo-contractive mappings defined on *real Hilbert spaces*.

Theorem CCDM (Theorem 3.2 [19]) *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined iteratively from $x_0 \in K$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{1.11}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Using Theorem CCDM, Chidume *et al.* proved several convergence theorems for the approximation of fixed points of strictly pseudo-contractive maps under various additional mild compactness-type conditions either on the operator T or on the domain of T. The theorems proved in [19] are significant generalizations of several important results on Hilbert spaces (see, *e.g.*, [19]).

Our purpose in this paper is to extend Theorem CCDM and other related results in [19], using Definition 1.4, from Hilbert spaces to the much more general class of *q-uniformly smooth real Banach spaces*. As we have noted, theses spaces include the L_p, l_p and $W^{m,p}$ spaces, $1 < p < \infty$ and $m \geq 1$. Finally, we give important examples of multi-valued maps satisfying the conditions of our theorems.

2 Preliminaries

In the sequel, we need the following definitions and results.

Definition 2.1 Let E be a real Banach space and T be a multi-valued mapping. The multi-valued map $(I - T)$ is said to be *strongly demiclosed* at 0 (see, *e.g.*, [22]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges strongly to x^* and $d(x_n, Tx_n)$ converges to 0, then $d(x^*, Tx^*) = 0$.

Lemma 2.2 [19] *Let E be a reflexive real Banach space and let $A, B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a - b\| \leq D(A, B). \tag{2.1}$$

Proposition 2.3 *Let K be a nonempty subset of a real Banach space E and let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping. Assume that for every $x \in K$, Tx is weakly closed. Then T is Lipschitzian.*

Proof We first observe that for any $x \in D(T)$, the set Tx is weakly closed if and only if the set Ax is weakly closed. Now, let $x, y \in D(T)$ and $u \in Ax$. From Lemma 2.2, there exists $v \in Ay$ such that

$$\|u - v\| \leq D(Ax, Ay). \tag{2.2}$$

Using the fact that T is k-strictly pseudo-contractive and inequality (2.2), we have

$$\begin{aligned} k\bigl(D(Ax, Ay)\bigr)^2 &\leq \bigl\langle u - v, j(x - y)\bigr\rangle \\ &\leq \|u - v\|\|x - y\| \\ &\leq D(Ax, Ay)\|x - y\|. \end{aligned}$$

So,

$$D(Ax, Ay) \leq \frac{1}{k}\|x - y\| \quad \forall x, y \in D(T). \tag{2.3}$$

From the definition of the Hausdorff distance, we have

$$D(Tx, Ty) \leq D(Ax, Ay) + \|x - y\| \quad \forall x, y \in D(T). \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$D(Tx, Ty) \leq L_k\|x - y\| \quad \forall x, y \in D(T), \text{ where } L_k := \frac{1 + k}{k}.$$

Therefore, T is L_k-Lipschitzian. □

Remark 1 We note that *for a single-valued map T*, for each $x \in D(T)$, the set Tx is always weakly closed.

Lemma 2.4 *Let $q > 1$, E be a q-uniformly smooth real Banach space, $k \in (0,1)$. Suppose $T : D(T) \subset E \to CB(E)$ is a multi-valued map with $F(T) \neq \emptyset$, and for all $x \in D(T)$, $x^* \in F(T)$,*

$$k\big(D\big(Ax, Ax^*\big)\big)^2 \leq \big\langle u - v^*, j\big(x - x^*\big)\big\rangle \quad \forall u \in Ax, v^* \in Ax^*, \tag{2.5}$$

where $A := I - T$, I is the identity map on E. If $Tx^* = \{x^*\}$ for all $x^* \in F(T)$, then the following inequality holds:

$$\big\langle x - y, j_q\big(x - x^*\big)\big\rangle \geq k^{q-1}\|x - y\|^q, \quad \forall x \in D(T), \forall y \in Tx.$$

Proof Let $x \in D(T)$, $u \in Ax$, $x^* \in F(T)$. Then, from inequality (2.5), the definition of the Hausdorff metric and the assumption that $Tx^* = \{x^*\}$, we have

$$k\big(D\big(Ax, Ax^*\big)\big)^2 \leq \|u\|\,\big\|x - x^*\big\| \leq D\big(Ax, Ax^*\big)\,\big\|x - x^*\big\|.$$

So,

$$kD\big(Ax, Ax^*\big) \leq \big\|x - x^*\big\| \quad \forall x \in D(T), x^* \in F(T). \tag{2.6}$$

Therefore, for all $x \in D(T)$, $y \in Tx$, $x^* \in F(T)$ such that $x \neq x^*$, using inequalities (2.5) and (2.6) and the fact that $j_q(x - x^*) = \|x - x^*\|^{q-2}j(x - x^*)$, we obtain

$$\begin{aligned}
\big\langle x - y, j_q\big(x - x^*\big)\big\rangle &= \big\|x - x^*\big\|^{q-2}\big\langle x - y, j\big(x - x^*\big)\big\rangle \\
&\geq k^{q-1}\big(D\big(Ax, Ax^*\big)\big)^q \\
&\geq k^{q-1}\|x - y\|^q.
\end{aligned}$$

$D(Ax, Ax^*) \geq \|x - x^*\|$ since $Tx^* = \{x^*\}$. This completes the proof. □

Lemma 2.5 *Let K be a nonempty closed subset of a real Banach space E and let $T : K \to P(K)$ be a k-strictly pseudo-contractive mapping. Assume that for every $x \in K$, Tx is weakly closed. Then $(I - T)$ is strongly demiclosed at zero.*

Proof Let $\{x_n\} \subseteq K$ be such that $x_n \to x$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Since K is closed, we have that $x \in K$. Since, for every n, Tx_n is proximinal, let $y_n \in Tx_n$ such that $\|x_n - y_n\| = d(x_n, Tx_n)$. Using Lemma 2.2, for each n, there exists $z_n \in Tx$ such that

$$\|y_n - z_n\| \leq D(Tx_n, Tx).$$

We then have

$$\begin{aligned}
\|x - z_n\| &\leq \|x - x_n\| + \|x_n - y_n\| + \|y_n - z_n\| \\
&\leq \|x - x_n\| + \|x_n - y_n\| + D(Tx_n, Tx) \\
&\leq \|x - x_n\| + \|x_n - y_n\| + L_k\|x_n - x\|.
\end{aligned}$$

Observing that $d(x, Tx) \leq \|x - z_n\|$, it then follows that

$$d(x, Tx) \leq \|x - x_n\| + \|x_n - y_n\| + L_k\|x_n - x\|.$$

Taking limit as $n \to \infty$, we have that $d(x, Tx) = 0$. Therefore $x \in Tx$. The proof is completed. □

Lemma 2.6 [23] *Let $q > 1$ and E be a smooth real Banach space. Then the following are equivalent*:

(i) *E is q-uniformly smooth.*

(ii) *There exists a constant $d_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q.$$

(iii) *There exists a constant $c_q > 0$ such that for all $x, y \in E$ and $\lambda \in [0, 1]$,*

$$\left\|(1 - \lambda)x + \lambda y\right\|^q \geq (1 - \lambda)\|x\|^q + \lambda\|y\|^q - w_q(\lambda)c_q\|x - y\|^q,$$

where $w_q(\lambda) := \lambda^q(1 - \lambda) + \lambda(1 - \lambda)^q$.

From now on, d_q denotes the constant that appeared in Lemma 2.6. Let $\mu := \min\{1, (\frac{qk^{q-1}}{d_q})^{\frac{1}{q-1}}\}$.

3 Main results

We prove the following theorem.

Theorem 3.1 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. For arbitrary $x_1 \in K$ and $\lambda \in (0, \mu)$, let $\{x_n\}$ be a sequence defined iteratively by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{3.1}$$

where $y_n \in Tx_n$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof Let $x^* \in F(T)$. Then, using the recursion formula (3.1), Lemmas 2.6 and 2.4, we have

$$\begin{aligned} \left\|x_{n+1} - x^*\right\|^q &= \left\|x_n - x^* - \lambda(x_n - y_n)\right\|^q \\ &\leq \left\|x_n - x^*\right\|^q - \lambda q\left\langle x_n - y_n, j_q\left(x_n - x^*\right)\right\rangle + \lambda^q d_q\|x_n - y_n\|^q \\ &\leq \left\|x_n - x^*\right\|^q - q\lambda k^{q-1}\|x_n - y_n\|^q + \lambda^q d_q\|x_n - y_n\|^q \\ &= \left\|x_n - x^*\right\|^q - \lambda\left(qk^{q-1} - d_q\lambda^{q-1}\right)\|x_n - y_n\|^q. \end{aligned} \tag{3.2}$$

It follows that

$$\sum_{n=1}^{\infty}\|x_n - y_n\|^q < \infty.$$

Hence, $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Since $y_n \in Tx_n$, we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

A mapping $T : K \to CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. We note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

We now prove the following corollaries of Theorem 3.1.

Corollary 3.2 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous and hemicompact. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{3.3}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is hemicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ for some $p \in K$. Since T is continuous, we have $d(x_{n_j}, Tx_{n_j}) \to d(p, Tp)$. Therefore, $d(p, Tp) = 0$ and so $p \in F(T)$. Setting $x^* = p$ in the proof of Theorem 3.1, it follows from inequality (3.2) that $\lim_{n\to\infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p. This completes the proof. □

Corollary 3.3 *Let $q > 1$ be a real number and K be a nonempty, compact and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{3.4}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Observing that if K is compact, every map $T : K \to CB(K)$ is hemicompact, the proof follows from Corollary 3.2. □

Remark 2 In Corollary 3.2, the continuity assumption on T can be dispensed if we assume that for every $x \in K$, the set Tx is proximinal and weakly closed. In fact, we have the following result.

Corollary 3.4 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{3.5}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Following the same arguments as in the proof of Corollary 3.2, we have $x_{n_j} \to p$ and $\lim_{n\to\infty} d(x_{n_j}, Tx_{n_j}) = 0$. Furthermore, from Lemma 2.5, $(I - T)$ is strongly demiclosed at zero. It then follows that $p \in Tp$. Setting $x^* = p$ and following the same computations as in the proof of Theorem 3.1, we have from inequality (3.2) that $\lim \|x_n - p\|$ exists. Since $\{x_{n_j}\}$ converges strongly to p, it follows that $\{x_n\}$ converges strongly to $p \in F(T)$. The proof is completed. $\square$

A mapping $T : K \to CB(K)$ is said to satisfy Condition (I) if there exists a strictly increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d\big(x, T(x)\big) \geq f\big(d\big(x, F(T)\big)\big) \quad \forall x \in D.$$

Convergence theorems have been proved in real Hilbert spaces for multi-valued nonexpansive mappings T under the assumption that T satisfies Condition (I) (see, *e.g.*, [16, 24]). The following corollary extends such theorems to multi-valued strictly pseudo-contractive maps and to q-uniformly smooth real Banach spaces.

Corollary 3.5 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to P(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T satisfies Condition* (I). *Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{3.6}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Using the fact that T satisfies Condition (I), it follows that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Thus there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T)$ such that

$$\|x_{n_j} - p_j\| < \frac{1}{2^j} \quad \forall j \in \mathbb{N}.$$

By setting $x^* = p_j$ and following the same arguments as in the proof of Theorem 3.1, we obtain from inequality (3.2) that

$$\|x_{n_{j+1}} - p_j\| \leq \|x_{n_j} - p_j\| < \frac{1}{2^j}.$$

We now show that $\{p_j\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned}\|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}.\end{aligned}$$

This shows that $\{p_j\}$ is a Cauchy sequence in K and thus converges strongly to some $p \in K$. Using the fact that T is L-Lipschitzian and $p_j \to p$, we have

$$\begin{aligned} d(p_j, Tp) &\leq D(Tp_j, Tp) \\ &\leq L\|p_j - p\|, \end{aligned}$$

so that $d(p, Tp) = 0$ and thus $p \in Tp$. Therefore, $p \in F(T)$ and $\{x_{n_j}\}$ converges strongly to p. Setting $x^* = p$ in the proof of Theorem 3.1, it follows from inequality (3.2) that $\lim_{n\to\infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p. This completes the proof. □

Corollary 3.6 *Let $q > 1$ be a real number and K be a nonempty, compact and convex subset of a q uniformly smooth real Banach space E. Let $T : K \to P(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that for every $x \in K$, the set Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{3.7}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since $\{x_n\} \subseteq K$ and K is compact, $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ that converges strongly to some $p \in K$. Furthermore, from Lemma 2.5, $(I - T)$ is strongly demiclosed at zero. It then follows that $p \in Tp$. Setting $x^* = p$ and following the same arguments as in the proof of Theorem 3.1, we have from inequality (3.2) that $\lim \|x_n - p\|$ exists. Since $\{x_{n_j}\}$ converges strongly to q, it follows that $\{x_n\}$ converges strongly to $p \in F(T)$. This completes the proof. □

Corollary 3.7 *Let $q > 1$ be a real number and K be a nonempty compact convex subset of a q uniformly smooth real Banach space E. Let $T : K \to P(K)$ be a multi-valued nonexpansive mapping. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$,*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{3.8}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3 The recursion formula (3.1) of Theorem 3.1 is of the Krasnoselkii type (see, *e.g.*, [25]) and is known to be superior to the recursion formula of the Mann algorithm (see, *e.g.*, Mann [26]) in the following sense: (i) The recursion formula (3.1) requires less computation time than the formula of the Mann algorithm because the parameter λ in formula (3.1) is fixed in $(0, 1 - k)$, whereas in the algorithm of Mann, λ is replaced by a sequence $\{c_n\}$ in $(0, 1)$ satisfying the following conditions: $\sum_{n=1}^{\infty} c_n = \infty$, $\lim c_n = 0$. The c_n must be computed at each step of the iteration process. (ii) The Krasnoselskii-type algorithm usually yields rate of convergence as fast as that of a geometric progression, whereas the Mann algorithm usually has order of convergence of the form $o(1/n)$.

Remark 4 In [24], the authors replace the condition $Tp = \{p\}$ $\forall p \in F(T)$ with the following restriction on the sequence $y_n : y_n \in P_T x_n$, *i.e.*, $y_n \in Tx_n$ and $\|y_n - x_n\| = d(x_n, Tx_n)$. We observe that if, for example, the set Tx_n is a closed and convex subset of a real Hilbert space, then y_n is *unique* and is characterized by

$$\langle x_n - y_n, y_n - u_n\rangle \geq 0 \quad \forall u_n \in Tx_n.$$

Since this y_n has to be computed at each step of the iteration process, this makes the recursion formula difficult to use in any possible application.

Remark 5 The addition of *bounded* error terms to the recursion formula (3.1) leads to no generalization.

Remark 6 Our theorems in this paper are important generalizations of several important recent results in the following sense: (i) Our theorems extend results proved for multi-valued *nonexpansive* mappings in *real Hilbert spaces* (see, *e.g.*, [15–17, 20, 21]) to a much larger class of multi-valued *strictly pseudo-contractive mappings* and in a much larger class of *q-uniformly smooth real Banach spaces*. (ii) Our theorems are proved with the superior Krasnoselskii-type algorithm.

We give examples of multi-valued maps where, for each $x \in K$, the set Tx is proximinal and weakly closed.

Example 1 Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Define $T : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$Tx = \big[f(x-), f(x+)\big] \quad \forall x \in \mathbb{R},$$

where $f(x-) := \lim_{y\to x^-} f(y)$ and $f(x+) := \lim_{y\to x^+} f(y)$. For every $x \in \mathbb{R}$, Tx is either a singleton or a closed and bounded interval. Therefore, Tx is always weakly closed and convex. Hence, for every $x \in \mathbb{R}$, the set Tx is proximinal and weakly closed.

Example 2 Let H be a real Hilbert space and $f : H \to \mathbb{R}$ be a convex continuous function. Let $T : H \to 2^H$ be the multi-valued map defined by

$$Tx = \partial f(x) \quad \forall x \in H,$$

where $\partial f(x)$ is the *subdifferential* of f at x and is defined by

$$\partial f(x) = \big\{z \in H : \langle z, y - x\rangle \leq f(y) - f(x)\ \forall y \in H\big\}.$$

It is well known that for every $x \in H$, $\partial f(x)$ is nonempty, weakly closed and convex. Therefore, since H is a real Hilbert space, it then follows that for every $x \in H$, the set Tx is proximinal and weakly closed. The subdifferential has deep connection with convex optimization problems.

The condition $Tp = \{p\}$ for all $p \in F(T)$, which is imposed in all our theorems of this paper, can actually be replaced by another condition (see, e.g., Shahzad and Zegeye [24]). This is done in Theorem 3.9.

Let K be a nonempty, closed and convex subset of a real Hilbert space, $T : K \to P(K)$ be a multi-valued map and $P_T : K \to CB(K)$ be defined by

$$P_T(x) := \{y \in Tx : \|x - y\| = d(x, Tx)\}.$$

We will need the following result.

Lemma 3.8 (Song and Cho [27]) *Let K be a nonempty subset of a real Banach space and $T : K \to P(K)$ be a multi-valued map. Then the following are equivalent*:

(i) $x^* \in F(T)$;

(ii) $P_T(x^*) = \{x^*\}$;

(iii) $x^* \in F(P_T)$. *Moreover*, $F(T) = F(P_T)$.

Remark 7 We observe from Lemma 3.8 that if $T : K \to P(K)$ is *any multi-valued map* with $F(T) \neq \emptyset$, then the corresponding multi-valued map P_T satisfies $P_T(p) = \{p\}$ for all $p \in F(P_T)$, the condition imposed in all our theorems and corollaries. Consequently, the examples of multi-valued maps $T : K \to CB(K)$ satisfying the condition $Tp = \{p\}$ for all $p \in F(T)$ abound.

Theorem 3.9 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued mapping such that $F(T) \neq \emptyset$. Assume that P_T is k-strictly pseudo-contractive. For arbitrary $x_1 \in K$ and $\lambda \in (0, \mu)$, let $\{x_n\}$ be a sequence defined iteratively by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n,$$

where $y_n \in P_T(x_n)$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

We conclude this paper with an example of a multi-valued map T for which P_T is k-strictly pseudo-contractive, the condition assumed in Theorem 3.9. Trivially, every non-expansive map is strictly pseudo-contractive.

Example 3 Let $E = \mathbb{R}$ with the usual metric and $T : \mathbb{R} \to CB(\mathbb{R})$ be the multi-valued map defined by

$$Tx = \begin{cases} [0, \frac{x}{2}], & x \in (0, \infty), \\ [\frac{x}{2}, 0], & x \in (-\infty, 0]. \end{cases}$$

Then P_T is strictly pseudo-contractive. In fact, $P_T(x) = \{\frac{x}{2}\}$ for all $x \in \mathbb{R}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

1. Brouwer, LEJ: Uber Abbildung von Mannigfaltigkeiten. Math. Ann. **71**(4), 598 (1912)
2. Kakutani, S: A generalization of Brouwer's fixed point theorem. Duke Math. J. **8**(3), 457-459 (1941)
3. Nash, JF: Non-cooperative games. Ann. Math. **54**, 286-295 (1951)
4. Nash, JF: Equilibrium points in *n*-person games. Proc. Natl. Acad. Sci. USA **36**(1), 48-49 (1950)
5. Geanakoplos, J: Nash and Walras equilibrium via Brouwer. Econ. Theory **21**, 585-603 (2003)
6. Nadler, SB Jr.: Multivalued contraction mappings. Pac. J. Math. **30**, 475-488 (1969)
7. Downing, D, Kirk, WA: Fixed point theorems for set-valued mappings in metric and Banach spaces. Math. Jpn. **22**(1), 99-112 (1977)
8. Filippov, AF: Differential equations with discontinuous right hand side. Sb. Math. **51**, 99-128 (1960). English transl. Trans. Am. Math. Soc. **42**, 199-232 (1964)
9. Chang, KC: The obstacle problem and partial differential equations with discontinuous nonlinearities. Commun. Pure Appl. Math. **33**, 117-146 (1980)
10. Erbe, L, Krawcewicz, W: Existence of solutions to boundary value problems for impulsive second order differential inclusions. Rocky Mt. J. Math. **22**, 1-20 (1992)
11. Frigon, M, Granas, A, Guennoun, Z: A note on the Cauchy problem for differential inclusions. Topol. Methods Nonlinear Anal. **1**, 315-321 (1993)
12. Deimling, K: Multivalued Differential Equations. De Gruyter, Berlin (1992)
13. Chidume, C: Geometric Properties of Banach Spaces and Nonlinear Iterations. Lecture Notes in Mathematics, vol. 1965. Springer, Berlin (2009)
14. Lindenstrauss, J, Tzafriri, L: Classical Banach Spaces II: Function Spaces. Ergebnisse Math. Grenzgebiete, vol. 97. Springer, Berlin (1979)
15. Sastry, KPR, Babu, GVR: Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point. Czechoslov. Math. J. **55**, 817-826 (2005)
16. Panyanak, B: Mann and Ishikawa iteration processes for multi-valued mappings in Banach spaces. Comput. Math. Appl. **54**, 872-877 (2007)
17. Song, Y, Wang, H: Erratum to "Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces" [Comput. Math. Appl. **54**, 872-877 (2007)]. Comput. Math. Appl. **55**, 2999-3002 (2008)
18. Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. **20**, 197-228 (1967)
19. Chidume, CE, Chidume, CO, Djitte, N, Minjibir, MS: Convergence theorems for fixed points of multi-valued strictly pseudo-contractive mappings in Hilbert spaces (2012, to appear)
20. Khan, SH, Yildirim, I, Rhoades, BE: A one-step iterative scheme for two multi-valued nonexpansive mappings in Banach spaces. Comput. Math. Appl. **61**, 3172-3178 (2011)
21. Abbas, M, Khan, SH, Khan, AR, Agarwal, RP: Common fixed points of two multi-valued nonexpansive mappings by one-step iterative scheme. Appl. Math. Lett. **24**, 97-102 (2011)
22. Garcia-Falset, J, Lorens-Fuster, E, Suzuki, T: Fixed point theory for a class of generalized nonexpansive mappings. J. Math. Anal. Appl. **375**, 185-195 (2011)
23. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(12), 1127-1138 (1991)
24. Shahzad, N, Zegeye, H: On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces. Nonlinear Anal. **71**, 838-844 (2009)
25. Krasnosel'skiĭ, MA: Two observations about the method of successive approximations. Usp. Mat. Nauk **10**, 123-127 (1955)
26. Mann, WR: Mean value methods in iterations. Proc. Am. Math. Soc. **4**, 506-510 (1953)
27. Song, Y, Cho, YJ: Some notes on Ishikawa iteration for multi-valued mappings. Bull. Korean Math. Soc. **48**(3), 575-584 (2011). doi:10.4134/BKMS.2011.48.3.575

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