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A strong convergence theorem of common elements in Hilbert spaces

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Abstract

The purpose of this article is to investigate the convergence of an iterative process for equilibrium problems, fixed point problems and variational inequalities. Strong convergence of the purposed iterative process is obtained in the framework of Hilbert spaces.

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Keywords: equilibrium problem; variational inequality; quasi-nonexpansive; inverse-strongly monotone mapping

1 Introduction

Nonlinear analysis plays an important role in optimization problems, economics and transportation. The theory of variational inequalities has emerged as a rapidly growing area of research because of its applications; see [1–17] fore more details and the references therein. To study variational inequalities based on iterative methods has been attracting many authors' attention. For the iterative methods, the most popular method is the Mann iterative method which was introduced by Mann in 1953; see [18] and the references therein. The Mann iterative process has been proved to be weak convergence for nonexpansive mappings in infinite dimension spaces; see [19] and the reference therein. Recently, many authors studied the modification of Mann iterative methods. The most popular one is to use projections. We call the method a hybrid projection method; see [20] and the reference therein. In this paper, we study equilibrium problems, fixed point problems and variational inequalities based on the hybrid projection method. Strong convergence theorems for common solutions of the problems are established in infinite dimension Hilbert spaces.

2 Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $S: C \to C$ be a mapping. In this paper, we use $F(S)$ to denote the fixed point set of S.

Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be quasi-nonexpansive if $F(S) \neq \emptyset$ and

$$\|Sx - y\| \leq \|x - y\|, \quad \forall x \in C, y \in F(S).$$

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-strongly monotone. A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-inverse-strongly monotone. A is said to be Lipschitz if there exits a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be L-Lipschitz. A set-valued mapping $T : H \to 2^H$ is said to be monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle > 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and $A : C \to H$ is an inverse-strongly monotone mapping. In this paper, we consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F, A)$, *i.e.*,

$$EP(F, A) = \bigl\{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\bigr\}.$$

To study the generalized equilibrium problems (2.1), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F\bigl(tz + (1 - t)x, y\bigr) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Next, we give two special cases of the problem (2.1).

(I) If $A \equiv 0$, then the generalized equilibrium problem (2.1) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{2.2}$$

(II) If $F \equiv 0$, then the problem (2.1) is reduced to the following classical variational inequality:

$$\text{Find } x \in C \text{ such that } \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

It is known that $x \in C$ is a solution to (2.3) if and only if x is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping.

Recently, many authors studied the problems (2.1), (2.2) and (2.3) based on hybrid projection methods; see, for example, [21–36] and the references therein. Motivated by these results, we investigated the common element problems of the generalized equilibrium problem (2.1) and quasi-nonexpansive mappings based on the shrinking projection algorithm. A strong convergence theorem of common elements is established in the framework of Hilbert spaces.

In order to prove our main results, we also need the following definitions and lemmas.

The following lemma can be found in [4] and [9].

Lemma 2.1 *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

(a) *T_r is single-valued*;

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$*,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

(c) $F(T_r) = EP(F)$;

(d) *$EP(F)$ is closed and convex.*

Lemma 2.2 [37] *Let B be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, i.e.*,

$$N_C v = \left\{ w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C \right\}$$

and define a mapping M on C by

$$Mv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then M is maximal monotone and $0 \in Mv$ if and only if $\langle Bv, u - v\rangle \geq 0$ for all $u \in C$.

3 Main results

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and $A_m : C \to H$ be a κ_m-inverse-strongly monotone mapping for every $1 \leq m \leq N$, where N denotes some positive integer. Let $S : C \to C$ be a continuous quasi-nonexpansive mapping which is assumed to be demiclosed at zero and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Assume that $\mathcal{F} := \bigcap_{m=1}^N EP(F_m, A_m) \cap VI(C, B) \cap F(S) \neq \emptyset$. Let $\{\lambda_n\}$ be a positive sequence in $[0, 2\beta]$ and $\{r_{n,m}\}$ be a positive sequence in $[0, 2\kappa_m]$ for every $1 \leq m \leq N$. Let $\{\alpha_n\}, \{\beta_{n,1}\}, \ldots$ and $\{\beta_{n,N}\}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \operatorname{Proj}_C(\sum_{m=1}^N \beta_{n,m} u_{n,m} - \lambda_n B \sum_{m=1}^N \beta_{n,m} u_{n,m}), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases}$$

where $\{u_{n,m}\}$ is such that

$$F_m(u_{n,m}, u_m) + \langle A_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}} \langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \geq 0, \quad \forall u_m \in C$$

for each $1 \leq m \leq N$. Assume that the above sequence also satisfies the following restrictions:

(a) $\alpha_n \leq a < 1$;

(b) $\sum_{m=1}^N \beta_{n,m} = 1$ *and* $0 \leq b \leq \beta_{n,m} < 1$ *for each* $1 \leq m \leq N$;

(c) $0 < c \leq \lambda_n \leq d < 2\beta$ *and* $0 < e \leq r_{n,m} \leq f < 2\kappa_m$ *for each* $1 \leq m \leq N$,

where a, b, c, d, e and f are real numbers. Then the sequence $\{x_n\}$ strongly converges to $\operatorname{Proj}_{\mathcal{F}} x_1$.

Proof In view of Lemma 2.1, we see that

$$u_{n,m} = T_{r_{n,m}}(x_n - r_{n,m} A_m x_n), \quad \forall 1 \leq m \leq N.$$

Letting $p \in \mathcal{F}$, we obtain that

$$p = Sp = \operatorname{Proj}_C(I - \lambda_n B)p = T_{r_{n,m}}(p - r_{n,m} A_m p), \quad \forall m \in \{1, 2, \ldots, N\}.$$

In view of the restriction (c), we obtain that

$$\begin{aligned}
&\big\|(I - r_{n,m}A_m)x - (I - r_{n,m}A_m)y\big\|^2 \\
&\quad = \|x - y\|^2 - 2r_{n,m}\langle x - y, A_m x - A_m y\rangle + r_{n,m}^2\|A_m x - A_m y\|^2 \\
&\quad \le \|x - y\|^2 - r_{n,m}(2\kappa_m - r_{n,m})\|A_m x - A_m y\|^2 \\
&\quad \le \|x - y\|^2, \quad \forall x, y \in C.
\end{aligned}$$

This shows that $I - r_{n,m}A_m$ is nonexpansive for every $m \in \{1, 2, \ldots, N\}$. In view of the restriction (c), we also see that $I - \lambda_n B$ is nonexpansive.

Next, we show that C_n is closed and convex. In view of the assumption in the main body of the theorem, we see that $C_1 = C$ is closed and convex. Suppose that C_i is closed and convex for some $i \ge 1$. We show that C_{i+1} is closed and convex for the same i. Indeed, for any $\nu \in C_i$, we see that

$$\|y_i - \nu\| \le \|x_i - \nu\|$$

is equivalent to

$$\|y_i\|^2 - \|x_i\|^2 - 2\langle \nu, y_i - x_i\rangle \ge 0.$$

Thus C_{i+1} is closed and convex. This shows that C_n is closed and convex.

Next, we show that $\mathcal{F} \subset C_n$ for each $n \ge 1$. From the assumption, we see that $\mathcal{F} \subset C = C_1$. Assume that $\mathcal{F} \subset C_i$ for some $i \ge 1$. For any $\nu \in \mathcal{F} \subset C_i$, we see that

$$\begin{aligned}
\|y_i - \nu\| &= \big\|\alpha_i x_i + (1 - \alpha_i)Sz_i - \nu\big\| \\
&\le \alpha_i\|x_i - \nu\| + (1 - \alpha_i)\|z_i - \nu\| \\
&\le \alpha_i\|x_i - \nu\| + (1 - \alpha_i)\sum_{m=1}^{N}\beta_{i,m}\|u_{i,m} - \nu\| \\
&\le \alpha_i\|x_i - \nu\| + (1 - \alpha_i)\sum_{m=1}^{N}\beta_{i,m}\big\|T_{r_{i,m}}(I - r_{i,m}A_m)x_i - \nu\big\| \\
&\le \alpha_i\|x_i - \nu\| + (1 - \alpha_i)\sum_{m=1}^{N}\beta_{i,m}\big\|(I - r_{i,m}A_m)x_i - \nu\big\| \\
&\le \|x_i - \nu\|.
\end{aligned}$$

This shows that $\nu \in C_{i+1}$. This proves that $\mathcal{F} \subset C_n$. Notice that $x_n = \operatorname{Proj}_{C_n} x_1$. For each $\nu \in \mathcal{F} \subset C_n$, we have

$$\|x_1 - x_n\| \le \|x_1 - \nu\|.$$

In particular, we have

$$\|x_1 - x_n\| \le \|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\|.$$

This implies that $\{x_n\}$ is bounded. Since $x_n = \mathrm{Proj}_{C_n} x_1$ and $x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we arrive at

$$0 \le \langle x_1 - x_n, x_n - x_{n+1}\rangle \le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.$$

It follows that

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\|.$$

This implies that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. On the other hand, we have

$$\begin{aligned}
&\|x_n - x_{n+1}\|^2 \\
&\quad = \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&\quad = \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&\quad \le \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.1}$$

Notice that $x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1 \in C_{n+1}$. It follows that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|.$$

This in turn implies that

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \le 2\|x_n - x_{n+1}\|.$$

In view of (3.1), we obtain that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.2}$$

On the other hand, we have

$$\|x_n - y_n\| = (1 - \alpha_n)\|x_n - Sz_n\|.$$

It follows from (3.2) that

$$\lim_{n\to\infty} \|x_n - Sz_n\| = 0. \tag{3.3}$$

For any $p \in \mathcal{F}$, we have from Lemma 2.1 that

$$\begin{aligned}
\|u_{n,m} - p\|^2 &= \left\| T_{r_{n,m}}(I - r_{n,m}A_m)x_n - T_{r_{n,m}}(I - r_{n,m}A_m)p \right\|^2 \\
&\le \left\| (x_n - p) - r_{n,m}(A_m x_n - A_m p) \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - 2r_{n,m}\langle x_n - p, A_m x_n - A_m p\rangle + r_{n,m}^2 \|A_m x_n - A_m p\|^2 \\
&\le \|x_n - p\|^2 - r_{n,m}(2\kappa_m - r_{n,m})\|A_m x_n - A_m p\|^2, \\
&\quad \forall m \in \{1, 2, \dots, N\}. \qquad (3.4)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|y_n - p\|^2 &= \big\|\alpha_n x_n + (1-\alpha_n) S z_n - p\big\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|S z_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\sum_{m=1}^{N} \beta_{n,m}\|u_{n,m} - p\|^2. \qquad (3.5)
\end{aligned}
$$

Substituting (3.4) into (3.5), we arrive at

$$
\|y_n - p\|^2 \le \|x_n - p\|^2 - (1-\alpha_n)\sum_{m=1}^{N} \beta_{n,m} r_{n,m}(2\kappa_m - r_{n,m})\|A_m x_n - A_m p\|^2. \qquad (3.6)
$$

This in turn implies that

$$
\begin{aligned}
&(1-\alpha_n)\beta_{n,m} r_{n,m}(2\kappa_m - r_{n,m})\|A_m x_n - A_m p\|^2 \\
&\quad \le \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\quad \le \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|, \quad \forall m \in \{1, 2, \dots, N\}.
\end{aligned}
$$

In view of the restrictions (a)-(c), we obtain from (3.2) that

$$
\lim_{n\to\infty} \|A_m x_n - A_m p\| = 0, \quad \forall m \in \{1, 2, \dots, N\}. \qquad (3.7)
$$

On the other hand, we have from Lemma 2.1 that

$$
\begin{aligned}
\|u_{n,m} - p\|^2 &= \big\|T_{r_{n,m}}(I - r_{n,m}A_m)x_n - T_{r_{n,m}}(I - r_{n,m}A_m)p\big\|^2 \\
&\le \big\langle (I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p, u_{n,m} - p\big\rangle \\
&= \frac{1}{2}\Big(\big\|(I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p\big\|^2 + \|u_{n,m} - p\|^2 \\
&\quad - \big\|(I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p - (u_{n,m} - p)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_{n,m} - p\|^2 - \big\|x_n - u_{n,m} - r_{n,m}(A_m x_n - A_m p)\big\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_{n,m} - p\|^2 - \big(\|x_n - u_{n,m}\|^2 \\
&\quad - 2r_{n,m}\langle x_n - u_{n,m}, A_m x_n - A_m p\rangle + r_{n,m}^2\|A_m x_n - A_m p\|^2\big)\Big).
\end{aligned}
$$

This implies that

$$\|u_{n,m}-p\|^2 \le \|x_n-p\|^2 - \|x_n-u_{n,m}\|^2 + 2r_{n,m}\|x_n-u_{n,m}\|\|A_mx_n-A_mp\|. \tag{3.8}$$

Notice that

$$\begin{aligned}\|y_n-p\|^2 &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|Sz_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\sum_{m=1}^N\beta_{n,m}\|u_{n,m}-p\|^2. \end{aligned}\tag{3.9}$$

Substituting (3.8) into (3.9), we see that

$$\begin{aligned}\|y_n-p\|^2 \le{}& \|x_n-p\|^2+(1-\alpha_n)\sum_{m=1}^N\beta_{n,m}2r_{n,m}\|x_n-u_{n,m}\|\|A_mx_n-A_mp\|\\ &-(1-\alpha_n)\sum_{m=1}^N\beta_{n,m}\|x_n-u_{n,m}\|^2\\ \le{}& \|x_n-p\|^2+\sum_{m=1}^N 2r_{n,m}\|x_n-u_{n,m}\|\|A_mx_n-A_mp\|\\ &-(1-\alpha_n)\sum_{m=1}^N\beta_{n,m}\|x_n-u_{n,m}\|^2,\quad \forall 1\le m\le N. \end{aligned}\tag{3.10}$$

It follows that

$$\begin{aligned}&(1-\alpha_n)\beta_{n,m}\|x_n-u_{n,m}\|^2\\ &\quad\le \|x_n-p\|^2-\|y_n-p\|^2+\sum_{m=1}^N 2r_{n,m}\|x_n-u_{n,m}\|\|A_mx_n-A_mp\|\\ &\quad\le \bigl(\|x_n-p\|+\|y_n-p\|\bigr)\|x_n-y_n\|+\sum_{m=1}^N 2r_{n,m}\|x_n-u_{n,m}\|\|A_mx_n-A_mp\|,\\ &\qquad \forall 1\le m\le N. \end{aligned}\tag{3.11}$$

In view of the restrictions (a) and (b), we obtain from (3.2) and (3.7) that

$$\lim_{n\to\infty}\|x_n-u_{n,m}\|=0,\quad \forall 1\le m\le N. \tag{3.12}$$

Since $\{x_n\}$ is bounded, we may assume that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to some point x. It follows from (3.12) that $u_{n_i,m}$ converges weakly to x for every $m\in\{1,2,\dots,N\}$.

Next, we show that $x\in EP(F_m,A_m)$ for every $m\in\{1,2,\dots,N\}$. Since $u_{n,m}=T_{r_{n,m}}(x_n-r_{n,m}A_mx_n)$ for any $u\in C$, we have

$$F_m(u_{n,m},u_m)+\langle A_mx_n,u_m-u_{n,m}\rangle+\frac{1}{r_{n,m}}\langle u_m-u_{n,m},u_{n,m}-x_n\rangle\ge 0.$$

From the condition (A2), we see that

$$\langle A_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \geq F_m(u_m, u_{n,m}). \tag{3.13}$$

Replacing n by n_i, we arrive at

$$\langle A_m x_{n_i}, u_m - u_{n_i,m}\rangle + \left\langle u_m - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\right\rangle \geq F_m(u_m, u_{n_i,m}). \tag{3.14}$$

For t_m with $0 < t_m \leq 1$ and $u_m \in C$, let $u_{t_m} = t_m u_m + (1 - t_m)x$. Since $u_m \in C$ and $x \in C$, we have $u_{t_m} \in C$ for every $1 \leq m \leq N$. It follows from (3.14) that

$$\begin{aligned}
&\langle u_{t_m} - u_{n_i,m}, A_m u_{t_m}\rangle \\
&\quad\geq \langle u_{t_m} - u_{n_i,m}, A_m u_{t_m}\rangle - \langle A_m x_{n_i,m}, u_{t_m} - u_{n_i,m}\rangle - \left\langle u_{t_m} - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\right\rangle \\
&\qquad + F_m(u_{t_m}, u_{n_i,m}) \\
&\quad= \langle u_{t_m} - u_{n_i,m}, A_m u_{t_m} - A_m u_{n_i,m}\rangle + \langle u_{t_m} - u_{n_i,m}, A_m u_{n_i,m} - A_m x_{n_i}\rangle \\
&\qquad - \left\langle u_{t_m} - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\right\rangle + F_m(u_{t_m}, u_{n_i,m}).
\end{aligned} \tag{3.15}$$

From (3.12), we have $A_m u_{n_i,m} - A_m x_{n_i} \to 0$ as $i \to \infty$ for every $1 \leq m \leq N$. On the other hand, we obtain from the monotonicity of A_m that $\langle u_{t_m} - u_{n_i,m}, A_m u_{t_m} - A_m u_{n_i,m}\rangle \geq 0$. It follows from (A4) that

$$\langle u_{t_m} - x, A_m u_{t_m}\rangle \geq F_m(u_{t_m}, x), \quad \forall 1 \leq m \leq N. \tag{3.16}$$

From (A1) and (A4), we obtain from (3.16) that

$$\begin{aligned}
0 &= F_m(u_{t_m}, u_{t_m}) \leq t_m F_m(u_{t_m}, u_m) + (1 - t_m)F_m(u_{t_m}, x) \\
&\leq t_m F_m(u_{t_m}, u_m) + (1 - t_m)\langle u_{t_m} - x, A_m u_{t_m}\rangle \\
&= t_m F_m(u_{t_m}, u_m) + (1 - t_m)t_m\langle u_m - x, A_m u_{t_m}\rangle,
\end{aligned}$$

which yields that

$$F_m(u_{t_m}, u_m) + (1 - t_m)\langle u_m - x, A_m u_{t_m}\rangle \geq 0, \quad \forall 1 \leq m \leq N.$$

Letting $t_m \to 0$ in the above inequality for every $1 \leq m \leq N$, we arrive at

$$F_m(x, u_m) + \langle u_m - x, A_m \xi\rangle \geq 0, \quad \forall 1 \leq m \leq N.$$

This shows that $x \in EP(F_m, A_m)$ for every $1 \leq m \leq N$, that is, $x \in \bigcap_{m=1}^N EP(F_m, A_m)$. Putting $w_n = \sum_{m=1}^N \beta_{n,m} u_{n,m}$, we see that

$$\|w_n - p\| \leq \|x_n - p\|$$

and

$$\begin{aligned}
\|y_n - p\|^2 &= \big\|\alpha_n x_n + (1-\alpha_n)Sz_n - p\big\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Sz_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\big\|(I-\lambda_n B)w_n - p\big\|^2 \\
&\le \|x_n - p\|^2 - (1-\alpha_n)\lambda_n(2\beta - \lambda_n)\|Bw_n - Bp\|^2. \qquad (3.17)
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
(1-\alpha_n)\lambda_n(2\beta-\lambda_n)\|Bw_n - Bp\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\le \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|. \qquad (3.18)
\end{aligned}$$

In view of the restriction (a)-(c), we obtain from (3.2) that

$$\lim_{n\to\infty}\|Bw_n - Bp\| = 0. \qquad (3.19)$$

On the other hand, we have from the firm nonexpansivity of Proj_C that

$$\begin{aligned}
\|z_n - p\|^2 &= \big\|\mathrm{Proj}_C(I-\lambda_n B)w_n - \mathrm{Proj}_C(I-\lambda_n B)p\big\|^2 \\
&\le \big\langle (I-\lambda_n B)w_n - (I-\lambda_n B)p, z_n - p\big\rangle \\
&= \frac{1}{2}\Big(\big\|(I-\lambda_n B)w_n - (I-\lambda_n B)p\big\|^2 + \|z_n - p\|^2 \\
&\quad - \big\|(I-\lambda_n B)w_n - (I-\lambda_n B)p - (z_n - p)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\|w_n - p\|^2 + \|z_n - p\|^2 - \big\|w_n - z_n - \lambda_n(Bw_n - Bp)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|w_n - z_n\|^2 \\
&\quad + 2\lambda_n\langle w_n - z_n, Bw_n - Bp\rangle - \lambda_n^2\|Bw_n - Bp\|^2\Big).
\end{aligned}$$

This implies that

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|w_n - z_n\|^2 + 2\lambda_n\|w_n - z_n\|\|Bw_n - Bp\|,$$

from which it follows that

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Sz_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\le \|x_n - p\|^2 - (1-\alpha_n)\|w_n - z_n\|^2 \\
&\quad + 2\lambda_n\|w_n - z_n\|\|Bw_n - Bp\|.
\end{aligned}$$

Hence, we get that

$$\begin{aligned}(1-\alpha_n)\|w_n-z_n\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2+2\lambda_n\|w_n-z_n\|\|Bw_n-Bp\| \\ &\le \bigl(\|x_n-p\|+\|y_n-p\|\bigr)\|x_n-y_n\|+2\lambda_n\|w_n-z_n\|\|Bw_n-Bp\|.\end{aligned}$$

In view of the restriction (a), we obtain from (3.2) and (3.19) that

$$\lim_{n\to\infty}\|w_n-z_n\|=0. \tag{3.20}$$

Note that

$$\|z_n-x_n\|\le\|z_n-w_n\|+\|w_n-x_n\|\le\|z_n-w_n\|+\sum_{m=1}^{N}\beta_{n,m}\|u_{n,m}-x_n\|.$$

In view of (3.12) and (3.20), we get that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.21}$$

Next, we prove that $x\in VI(C,B)$. In fact, let M be the maximal monotone mapping defined by

$$My=\begin{cases}By+N_Cy, & y\in C,\\ \emptyset, & y\notin C.\end{cases}$$

For any given $(s,t)\in G(T)$, we have $t-Bs\in N_Cs$. Since $z_n\in C$, by the definition of N_C, we have

$$\langle s-z_n,t-Bs\rangle\ge 0. \tag{3.22}$$

In view of the algorithm, we obtain that

$$\bigl\langle s-z_n,z_n-(I-\lambda_nB)w_n\bigr\rangle\ge 0$$

and hence

$$\left\langle s-z_n,\frac{z_n-w_n}{\lambda_n}+Bw_n\right\rangle\ge 0. \tag{3.23}$$

Since B is monotone, we obtain from (3.23) that

$$\begin{aligned}\langle s-z_{n_i},t\rangle &\ge \langle s-z_{n_i},Bs\rangle \\ &\ge \langle s-z_{n_i},Bs\rangle-\left\langle s-z_{n_i},\frac{z_{n_i}-w_{n_i}}{\lambda_{n_i}}+Bw_{n_i}\right\rangle \\ &= \langle s-z_{n_i},Bs-Bz_{n_i}\rangle+\langle s-z_{n_i},Bz_{n_i}-Bw_{n_i}\rangle\end{aligned}$$

$$\begin{aligned}&-\left\langle s - z_{n_i}, \frac{z_{n_i} - w_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\geq \langle s - z_{n_i}, Bz_{n_i} - Bw_{n_i}\rangle - \left\langle s - z_{n_i}, \frac{z_{n_i} - w_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}$$

It follows from (3.21) that $z_{n_i} \rightharpoonup x$. On the other hand, we have that B is $\frac{1}{\beta}$-Lipschitz continuous. It follows from (3.20) that

$$\langle s - x, t\rangle \geq 0.$$

Notice that M is maximal monotone and hence $0 \in Mx$. This shows that $x \in VI(C, B)$. Notice that

$$\|x_n - Sx_n\| \leq \|x_n - Sz_n\| + \|Sz_n - Sx_n\|.$$

We find from (3.3) and (3.21) that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.24}$$

Next, we prove that $x \in F(S)$. Since S is demiclosed at zero, we see that $x \in F(S)$. This proves that $x \in \mathcal{F}$. Notice that $\operatorname{Proj}_{\mathcal{F}} x_1 \subset C_{n+1}$ and $x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1$, we have

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\|.$$

On the other hand, we have

$$\begin{aligned}\|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\| &\leq \|x_1 - x\| \\ &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\|.\end{aligned}$$

We, therefore, obtain that

$$\|x_1 - x\| = \lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\|.$$

This implies $x_{n_i} \to x = \operatorname{Proj}_{\mathcal{F}} x_1$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we obtain that $x_n \to \operatorname{Proj}_{\mathcal{F}} x_1$ as $n \to \infty$. This completes the proof. □

If S is an identity mapping, we obtain from Theorem 3.1 the following.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and $A_m : C \to H$ be a κ_m-inverse-strongly monotone mapping for every $1 \leq m \leq N$, where N denotes some positive integer. Let $B : C \to H$ be a β-inverse-strongly monotone mapping. Assume that $\mathcal{F} := \bigcap_{m=1}^N EP(F_m, A_m) \cap VI(C, B) \neq \emptyset$. Let $\{\lambda_n\}$ be a positive sequence in $[0, 2\beta]$ and $\{r_{n,m}\}$*

be a positive sequence in $[0, 2\kappa_m]$ *for every* $1 \le m \le N$. *Let* $\{\alpha_n\}, \{\beta_{n,1}\}, \dots$ *and* $\{\beta_{n,N}\}$ *be sequences in* $[0,1]$. *Let* $\{x_n\}$ *be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\operatorname{Proj}_C(\sum_{m=1}^N \beta_{n,m}u_{n,m} - \lambda_n B \sum_{m=1}^N \beta_{n,m}u_{n,m}), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases}$$

where $\{u_{n,m}\}$ *is such that*

$$F_m(u_{n,m}, u_m) + \langle A_m x_n, u_m - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge 0, \quad \forall u_m \in C$$

for each $1 \le m \le N$. *Assume that the above sequence also satisfies the following restrictions*:

(a) $\alpha_n \le a < 1$;
(b) $\sum_{m=1}^N \beta_{n,m} = 1$ *and* $0 \le b \le \beta_{n,m} < 1$ *for each* $1 \le m \le N$;
(c) $0 < c \le \lambda_n \le d < 2\beta$ *and* $0 < e \le r_{n,m} \le f < 2\kappa_m$ *for each* $1 \le m \le N$,

where a, b, c, d, e *and* f *are real numbers. Then the sequence* $\{x_n\}$ *strongly converges to* $\operatorname{Proj}_{\mathcal{F}} x_1$.

If $N = 1$, we obtain from Theorem 3.1 the following.

Corollary 3.3 *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4) *and* $A : C \to H$ *be a* κ-*inverse-strongly monotone mapping. Let* $S : C \to C$ *be a continuous quasi-nonexpansive mapping which is assumed to be demiclosed at zero and let* $B : C \to H$ *be a* β-*inverse-strongly monotone mapping. Assume that* $\mathcal{F} := EP(F,A) \cap VI(C,B) \cap F(S) \neq \emptyset$. *Let* $\{\lambda_n\}$ *be a positive sequence in* $[0, 2\beta]$ *and* $\{r_n\}$ *be a positive sequence in* $[0, 2\kappa]$. *Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$. *Let* $\{x_n\}$ *be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S \operatorname{Proj}_C(u_n - \lambda_n B u_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases}$$

where $\{u_n\}$ *is such that*

$$F(u_n, u) + \langle A x_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \quad \forall u_m \in C.$$

Assume that the above sequence also satisfies the following restrictions:

(a) $\alpha_n \le a < 1$;
(b) $0 < b \le \lambda_n \le c < 2\beta$ *and* $0 < d \le r_n \le e < 2\kappa$ *for each* $1 \le m \le N$,

where a, b, c, d and e are real numbers. Then the sequence $\{x_n\}$ *strongly converges to* $\mathrm{Proj}_{\mathcal{F}} x_1$.

If B is a zero operator, we obtain from Theorem 3.1 the following.

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and $A_m : C \to H$ be a κ_m-inverse-strongly monotone mapping for every $1 \le m \le N$, where N denotes some positive integer. Let $S : C \to C$ be a continuous quasi-nonexpansive mapping which is assumed to be demiclosed at zero. Assume that $\mathcal{F} := \bigcap_{m=1}^{N} EP(F_m, A_m) \cap F(S) \neq \emptyset$. Let $\{r_{n,m}\}$ be a positive sequence in $[0, 2\kappa_m]$ for every $1 \le m \le N$. Let $\{\alpha_n\}, \{\beta_{n,1}\}, \ldots$ and $\{\beta_{n,N}\}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S \sum_{m=1}^{N} \beta_{n,m} u_{n,m}, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1, \end{cases}$$

where $\{u_{n,m}\}$ is such that

$$F_m(u_{n,m}, u_m) + \langle A_m x_n, u_m - u_{n,m} \rangle + \frac{1}{r_{n,m}} \langle u_m - u_{n,m}, u_{n,m} - x_n \rangle \ge 0, \quad \forall u_m \in C$$

for each $1 \le m \le N$. Assume that the above sequence also satisfies the following restrictions:

(a) $\alpha_n \le a < 1$;

(b) $\sum_{m=1}^{N} \beta_{n,m} = 1$ *and* $0 \le b \le \beta_{n,m} < 1$ *for each* $1 \le m \le N$;

(c) $0 < c \le r_{n,m} \le d < 2\kappa_m$ *for each* $1 \le m \le N$,

where a, b, c and d are real numbers. Then the sequence $\{x_n\}$ *strongly converges to* $\mathrm{Proj}_{\mathcal{F}} x_1$.

Finally, we consider the following optimization problem: Find an x^* such that

$$\begin{cases} \varphi_1(x^*) = \min_{x \in C} \varphi_1(x), \\ \varphi_2(x^*) = \min_{x \in C} \varphi_2(x), \\ \vdots \\ \varphi_N(x^*) = \min_{x \in C} \varphi_N(x), \end{cases}$$

where $\varphi_m : C \to \mathbb{R}$ is a convex and lower semicontinuous function for each $1 \le m \le N$, where $N \ge 1$ is some positive integer.

Theorem 3.5 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let φ_m be a proper convex and lower semicontinuous function for every $1 \le m \le N$, where N denotes some positive integer. Assume that $\mathcal{F} := OP(\varphi) \cap VI(C, B) \cap F(S) \neq \emptyset$, $OP(\varphi)$ denotes the solution set of the above optimization problem. Let $\{\alpha_n\}, \{\beta_{n,1}\}, \ldots$ and $\{\beta_{n,N}\}$ be* *sequences in* $[0,1]$. *Let* $\{x_n\}$ *be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{m=1}^{N}\beta_{n,m}u_{n,m}, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases}$$

where $\{u_{n,m}\}$ *is such that*

$$\varphi_m(u_m) - \varphi_m(u_{n,m}) + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge 0, \quad \forall u_m \in C$$

for each $1 \le m \le N$. *Assume that the above sequence also satisfies the following restrictions*:

(a) $\alpha_n \le a < 1$;
(b) $\sum_{m=1}^{N}\beta_{n,m} = 1$ *and* $0 \le b \le \beta_{n,m} < 1$ *for each* $1 \le m \le N$;
(c) $0 < c \le r_{n,m} \le d < \infty$ *for each* $1 \le m \le N$,

where a, b, c *and* d *are real numbers*. *Then the sequence* $\{x_n\}$ *strongly converges to* $\operatorname{Proj}_{\mathcal{F}} x_1$.

Proof Putting $S = I$, $A_m = B = 0$ and $F_m(x,y) = \varphi(y) - \varphi(x)$, we find from Theorem 3.1 the desired conclusion immediately. □

Competing interests
The author declares that he has no competing interests.

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