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# Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces

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## Abstract

In this paper, we introduce the notion of a cyclic  $(\psi, A, B)$ -contraction for the pair  $(f, T)$  of self-mappings on the set  $X$ . We utilize our definition to introduce some common fixed point theorems for the two mappings  $f$  and  $T$  under a set of conditions. Also, we introduce an example to support the validity of our results. As application of our results, we derive some common fixed point theorems of integral type.

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**Keywords:** metric spaces; common fixed point; altering distance function; almost contraction; ordered metric spaces; weakly increasing mappings

## 1 Introduction

In recent years many authors established interesting results in fixed point theory in (ordered) metric spaces. One of the popular topics in the fixed point theory is the cyclic contraction. Kirk *et al.* [1] established the first result in this interesting area. Meantime, other authors obtained important results in this area (see [1–12]).

We begin with the definition of a cyclic map.

**Definition 1.1** Let  $A$  and  $B$  be non-empty subsets of a metric space  $(X, d)$  and  $T: A \cup B \rightarrow A \cup B$ . Then  $T$  is called a *cyclic map* if  $T(A) \subseteq B$  and  $T(B) \subseteq A$ .

In 2003, Kirk *et al.* [1] gave the following interesting theorem in fixed point theory for a cyclic map.

**Theorem 1.1** ([1]) *Let  $A$  and  $B$  be nonempty closed subsets of a complete metric space  $(X, d)$ . Suppose that  $T: A \cup B \rightarrow A \cup B$  is a cyclic map such that*

$$d(Tx, Ty) \leq kd(x, y) \quad \forall x \in A, \forall y \in B.$$

*If  $k \in [0, 1)$ , then  $T$  has a unique fixed point in  $A \cap B$ .*

Recently, several authors proved many results in fixed point theory for cyclic mappings, satisfying various (nonlinear) contractive conditions (see [1–12]). Some of contractive conditions are based on functions called control functions which alter the distance between two points in a metric space. Such functions were introduced by Khan *et al.* [13].

**Definition 1.2** (altering distance function, [13]) The function  $\phi: [0, +\infty) \rightarrow [0, +\infty)$  is called an *altering distance function* if the following properties are satisfied:

- (1)  $\phi$  is continuous and nondecreasing;
- (2)  $\phi(t) = 0$  if and only if  $t = 0$ .

For some fixed point theorems based on an altering distance function, we refer the reader to [14–20].

Let  $X$  be a nonempty set. Then  $(X, d, \preceq)$  is called an *ordered metric space* if and only if  $(X, d)$  is a metric space and  $(X, \preceq)$  is a partially ordered set. Two elements  $x, y \in X$  are called *comparable* if  $x \preceq y$  or  $y \preceq x$ .

Altun *et al.* [21, 22] introduced the notion of weakly increasing mappings and proved some existing theorems. For some works in the theory of weakly increasing mappings, we refer the reader to [23, 24].

**Definition 1.3** ([21]) Let  $(X, \preceq)$  be a partially ordered set. Two mappings  $F, G: X \rightarrow X$  are said to be *weakly increasing* if  $Fx \preceq GFx$  and  $Gx \preceq FGx$  for all  $x \in X$ .

The purpose of this paper is to obtain common fixed point results for mappings satisfying nonlinear contractive conditions of a cyclic form based on the notion of an altering distance function.

## 2 Main result

We start with the following definition.

**Definition 2.1** Let  $(X, d, \preceq)$  be an ordered metric space and  $A, B$  be nonempty closed subsets of  $X$ . Let  $f, T: X \rightarrow X$  be two mappings. The pair  $(f, T)$  is called a *cyclic  $(\psi, A, B)$ -contraction* if

- (1)  $\psi$  is an altering distance function;
- (2)  $A \cup B$  has a cyclic representation w.r.t. the pair  $(f, T)$ ; that is,  $fA \subseteq B$ ,  $TB \subseteq A$  and  $X = A \cup B$ ;
- (3) There exists  $0 < \delta < 1$  such that for any comparable elements  $x, y \in X$  with  $x \in A$  and  $y \in B$ , we have

$$\psi(d(fx, Ty)) \leq \delta \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right). \tag{2.1}$$

**Definition 2.2** Let  $(X, \preceq)$  be a partially ordered set and  $A, B$  be closed subsets of  $X$  with  $X = A \cup B$ . Let  $f, T: X \rightarrow X$  be two mappings. The pair  $(f, T)$  is said to be  *$(A, B)$ -weakly increasing* if  $fx \preceq Tfx$  for all  $x \in A$  and  $Tx \preceq fTx$  for all  $x \in B$ .

From now on, by  $\psi$  we mean altering distance functions unless otherwise stated. In the rest of this paper,  $\mathbb{N}$  stands for the set of nonnegative integer numbers.

**Theorem 2.1** Let  $(X, d, \preceq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$ . Let  $f, T: X \rightarrow X$  be two mappings such that the pair  $(f, T)$  is  $(A, B)$ -weakly increasing. Assume the following:

- (1) The pair  $(f, T)$  is a cyclic  $(\psi, A, B)$ -contraction;
- (2)  $f$  or  $T$  is continuous.

Then  $f$  and  $T$  have a common fixed point.

*Proof* Choose  $x_0 \in A$ . Let  $x_1 = fx_0$ . Since  $fA \subseteq B$ , we have  $x_1 \in B$ . Also, let  $x_2 = Tx_1$ . Since  $TB \subseteq A$ , we have  $x_2 \in A$ . Continuing this process, we can construct a sequence  $\{x_n\}$  in  $X$  such  $x_{2n+1} = fx_{2n}$ ,  $x_{2n+2} = Tx_{2n+1}$ ,  $x_{2n} \in A$  and  $x_{2n+1} \in B$ .

Since  $f$  and  $T$  are  $(A, B)$ -weakly increasing, we have

$$x_1 = fx_0 \leq Tfx_0 = Tx_1 = x_2 \leq fTx_1 = fx_2 = x_3 \leq \dots$$

We divide our proof into the following steps.

Step 1: We will show that  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ .

Subcase 1: Suppose that  $x_{2n} = x_{2n+1}$  for some  $n \in \mathbb{N}$ . Since  $x_{2n}$  and  $x_{2n+1}$  are comparable elements in  $X$  with  $x_{2n} \in A$  and  $x_{2n+1} \in B$ , we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(fx_{2n}, Tx_{2n+1})) \\ &\leq \delta \psi \left( \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, Tx_{2n+1}), \right. \right. \\ &\quad \left. \left. \frac{1}{2}(d(x_{2n}, Tx_{2n+1}) + d(fx_{2n}, x_{2n+1})) \right\} \right) \\ &= \delta \psi \left( \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \right. \right. \\ &\quad \left. \left. \frac{1}{2}(d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})) \right\} \right) \\ &\leq \delta \psi(d(x_{2n+1}, x_{2n+2})). \end{aligned}$$

Since  $\delta < 1$ , we have  $\psi(d(x_{2n+1}, x_{2n+2})) = 0$  and hence  $x_{2n+2} = x_{2n+1}$ . Similarly, we may show that  $x_{2n+3} = x_{2n+2}$ . Hence  $\{x_n\}$  is a constant sequence in  $X$ , so it is a Cauchy sequence in  $(X, d)$ .

Subcase 2:  $x_{2n} \neq x_{2n+1}$  for all  $n \in \mathbb{N}$ . Given  $n \in \mathbb{N}$ . If  $n$  is even, then  $n = 2t$  for some  $t \in \mathbb{N}$ . Since  $x_{2t} \in A$ ,  $x_{2t+1} \in B$  and  $x_{2t}$ ,  $x_{2t+1}$  are comparable, we have

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(x_{2t+1}, x_{2t+2})) \\ &= \psi(d(fx_{2t}, Tx_{2t+1})) \\ &\leq \delta \psi \left( \max \left\{ d(x_{2t}, x_{2t+1}), d(x_{2t}, fx_{2t}), d(x_{2t+1}, Tx_{2t+1}), \right. \right. \\ &\quad \left. \left. \frac{1}{2}(d(x_{2t}, Tx_{2t+1}) + d(fx_{2t}, x_{2t+1})) \right\} \right) \\ &= \delta \psi \left( \max \left\{ d(x_{2t}, x_{2t+1}), d(x_{2t+1}, x_{2t+2}), \frac{1}{2}(d(x_{2t}, x_{2t+2}) + d(x_{2t+1}, x_{2t+1})) \right\} \right) \\ &\leq \delta \psi(\max\{d(x_{2t}, x_{2t+1}), d(x_{2t+1}, x_{2t+2})\}). \end{aligned}$$

If

$$\max\{d(x_{2t}, x_{2t+1}), d(x_{2t+1}, x_{2t+2})\} = d(x_{2t+1}, x_{2t+2}),$$

then

$$\psi(d(x_{2t+1}, x_{2t+2})) \leq \delta \psi(d(x_{2t+1}, x_{2t+2})) < \psi(d(x_{2t+1}, x_{2t+2})),$$

which is a contradiction. Thus

$$\max\{d(x_{2t}, x_{2t+1}), d(x_{2t+1}, x_{2t+2})\} = d(x_{2t}, x_{2t+1}), \tag{2.2}$$

therefore

$$\psi(d(x_{2t+1}, x_{2t+2})) \leq \delta \psi(d(x_{2t}, x_{2t+1})). \tag{2.3}$$

If  $n$  is odd, then  $n = 2t + 1$  for some  $t \in \mathbb{N}$ . Since  $x_{2t+2}$  and  $x_{2t+1}$  are comparable with  $x_{2t+2} \in A$  and  $x_{2t+1} \in B$ , we have

$$\begin{aligned} & \psi(d(x_{n+2}, x_{n+1})) \\ &= \psi(d(x_{2t+3}, x_{2t+2})) \\ &= \psi(d(fx_{2t+2}, Tx_{2t+1})) \\ &\leq \delta \psi\left(\max\left\{d(x_{2t+2}, x_{2t+1}), d(x_{2t+2}, fx_{2t+2}), d(x_{2t+1}, Tx_{2t+1}), \right. \right. \\ &\quad \left. \left. \frac{1}{2}(d(x_{2t+2}, Tx_{2t+1}) + d(fx_{2t+2}, x_{2t+1}))\right\}\right) \\ &\leq \delta \psi\left(\max\left\{d(x_{2t+2}, x_{2t+1}), d(x_{2t+2}, x_{2t+3}), \frac{1}{2}(d(x_{2t+2}, x_{2t+2}) + d(x_{2t+3}, x_{2t+1}))\right\}\right) \\ &\leq \delta \psi(\max\{d(x_{2t+2}, x_{2t+1}), d(x_{2t+2}, x_{2t+3})\}). \end{aligned}$$

If

$$\max\{d(x_{2t+2}, x_{2t+1}), d(x_{2t+2}, x_{2t+3})\} = d(x_{2t+2}, x_{2t+3}),$$

then

$$\psi(d(x_{2t+3}, x_{2t+2})) \leq \delta \psi(d(x_{2t+2}, x_{2t+3})) < \psi(d(x_{2t+2}, x_{2t+3})),$$

which is a contradiction. Therefore

$$\max\{d(x_{2t+2}, x_{2t+1}), d(x_{2t+2}, x_{2t+3})\} = d(x_{2t+2}, x_{2t+1}), \tag{2.4}$$

and hence

$$\psi(d(x_{2t+3}, x_{2t+2})) \leq \delta \psi(d(x_{2t+2}, x_{2t+1})). \tag{2.5}$$

From (2.3) and (2.5), we have

$$\psi(d(x_{n+1}, x_{n+2})) \leq \delta \psi(d(x_n, x_{n+1})). \tag{2.6}$$

Since  $\psi$  is an altering distance function, we have  $\{d(x_{n+1}, x_{n+2}) : n \in \mathbb{N} \cup \{0\}\}$  is a bounded nonincreasing sequence. Thus there exists  $r \geq 0$  such that

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = r.$$

On letting  $n \rightarrow +\infty$  in (2.6), we have

$$\psi(r) \leq \delta \psi(r).$$

Since  $\delta < 1$ , we have  $\psi(r) = 0$  and hence  $r = 0$ . Thus

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Next, we show that  $\{x_n\}$  is a Cauchy sequence in the metric space  $(X, d)$ . It is sufficient to show that  $\{x_{2n}\}$  is a Cauchy sequence in  $(X, d)$ . Suppose to the contrary; that is,  $\{x_{2n}\}$  is not a Cauchy sequence in  $(X, d)$ . Then there exists  $\epsilon > 0$  for which we can find two subsequences  $\{x_{2m(i)}\}$  and  $\{x_{2n(i)}\}$  of  $\{x_{2n}\}$  such that  $n(i)$  is the smallest index for which

$$n(i) > m(i) > i, \quad d(x_{2m(i)}, x_{2n(i)}) \geq \epsilon. \tag{2.8}$$

This means that

$$d(x_{2m(i)}, x_{2n(i)-2}) < \epsilon. \tag{2.9}$$

From (2.8), (2.9) and the triangular inequality, we get that

$$\begin{aligned} \epsilon &\leq d(x_{2m(i)}, x_{2n(i)}) \leq d(x_{2m(i)}, x_{2n(i)-2}) + d(x_{2n(i)-2}, x_{2n(i)-1}) + d(x_{2n(i)-1}, x_{2n(i)}) \\ &< \epsilon + d(x_{2n(i)-2}, x_{2n(i)-1}) + d(x_{2n(i)-1}, x_{2n(i)}). \end{aligned}$$

On letting  $i \rightarrow +\infty$  in the above inequalities and using (2.7), we have

$$\lim_{i \rightarrow +\infty} d(x_{2m(i)}, x_{2n(i)}) = \epsilon. \tag{2.10}$$

Again, from (2.8) and the triangular inequality, we get that

$$\begin{aligned} \epsilon &\leq d(x_{2m(i)}, x_{2n(i)}) \\ &\leq d(x_{2n(i)}, x_{2n(i)-1}) + d(x_{2n(i)-1}, x_{2m(i)}) \\ &\leq d(x_{2n(i)}, x_{2n(i)-1}) + d(x_{2n(i)-1}, x_{2m(i)+1}) + d(x_{2m(i)+1}, x_{2m(i)}) \\ &\leq d(x_{2n(i)}, x_{2n(i)-1}) + d(x_{2n(i)-1}, x_{2m(i)}) + 2d(x_{2m(i)+1}, x_{2m(i)}) \\ &\leq 2d(x_{2n(i)}, x_{2n(i)-1}) + d(x_{2n(i)}, x_{2m(i)}) + 2d(x_{2m(i)+1}, x_{2m(i)}). \end{aligned}$$

Letting  $i \rightarrow +\infty$  in the above inequalities and using (2.7) and (2.10), we get that

$$\begin{aligned} \lim_{i \rightarrow +\infty} d(x_{2m(i)}, x_{2n(i)}) &= \lim_{i \rightarrow +\infty} d(x_{2m(i)+1}, x_{2n(i)-1}) \\ &= \lim_{i \rightarrow +\infty} d(x_{2m(i)+1}, x_{2n(i)}) \\ &= \lim_{i \rightarrow +\infty} d(x_{2m(i)}, x_{2n(i)-1}) \\ &= \epsilon. \end{aligned}$$

Since  $x_{2m(i)}$  and  $x_{2n(i)-1}$  are comparable with  $x_{2m(i)} \in A$  and  $x_{2n(i)-1} \in B$ , we have

$$\begin{aligned} \psi(d(x_{2m(i)+1}, x_{2n(i)})) &= \psi(d(fx_{2m(i)}, Tx_{2n(i)-1})) \\ &\leq \delta \psi \left( \max \left\{ d(x_{2m(i)}, x_{2n(i)-1}), d(x_{2m(i)}, fx_{2m(i)}), d(x_{2n(i)-1}, Tx_{2n(i)-1}), \right. \right. \\ &\quad \left. \left. \frac{1}{2}(d(x_{2m(i)}, Tx_{2n(i)-1}) + d(fx_{2m(i)}, x_{2n(i)-1})) \right\} \right) \\ &= \delta \psi \left( \max \left\{ d(x_{2m(i)}, x_{2n(i)-1}), d(x_{2m(i)}, x_{2m(i)+1}), d(x_{2n(i)-1}, x_{2n(i)}), \right. \right. \\ &\quad \left. \left. \frac{1}{2}(d(x_{2m(i)}, x_{2n(i)}) + d(x_{2m(i)+1}, x_{2n(i)-1})) \right\} \right). \end{aligned}$$

Letting  $i \rightarrow +\infty$  and using the continuity of  $\psi$ , we get that

$$\psi(\epsilon) \leq \delta \psi(\epsilon).$$

Since  $\delta < 1$ , we have  $\psi(\epsilon) = 0$  and hence  $\epsilon = 0$ , a contradiction. Thus  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ .

Step 2: Existence of a common fixed point.

Since  $(X, d)$  is complete and  $\{x_n\}$  is a Cauchy sequence in  $X$ , we have  $\{x_n\}$  converges to some  $u \in X$ , that is,  $\lim_{n \rightarrow \infty} d(x_n, u) = 0$ . Therefore

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} x_{2n-1} = \lim_{n \rightarrow +\infty} x_{2n} = u. \tag{2.11}$$

Since  $x_{2n}$  is a sequence in  $A$ ,  $A$  is closed and  $x_{2n} \rightarrow u$ , we have  $u \in A$ . Also, since  $x_{2n+1}$  is a sequence in  $B$ ,  $B$  is closed and  $x_{2n+1} \rightarrow u$ , we have  $u \in B$ . Now, we show that  $u$  is a fixed point of  $f$  and  $T$ . Without loss of generality, we may assume that  $f$  is continuous, since  $x_{2n} \rightarrow u$ , we get  $x_{2n+1} = fx_{2n} \rightarrow fu$ . By the uniqueness of limit, we have  $u = fu$ . Now, we show that  $u = Tu$ . Since  $u \leq u$  with  $u \in A$  and  $u \in B$ , we have

$$\begin{aligned} \psi(d(u, Tu)) &= \psi(d(fu, Tu)) \\ &\leq \delta \psi \left( \max \left\{ d(fu, Tu), d(u, fu), d(u, Tu), \frac{1}{2}(d(u, Tu) + d(fu, u)) \right\} \right) \\ &= \delta \psi(d(u, Tu)). \end{aligned}$$

Since  $\delta < 1$ , we get that  $d(u, Tu) = 0$  and hence  $u = Tu$ . □

Theorem 2.1 can be proved without assuming the continuity of  $f$  or the continuity of  $T$ . For this instance, we assume that  $X$  satisfies the following property:

(P) If  $(x_n)$  is a nondecreasing sequence in  $X$  with  $x_n \rightarrow x$ , then  $x_n \preceq x$ .

Now, we state and prove the following result.

**Theorem 2.2** *Let  $(X, d, \preceq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$ . Let  $f, T: X \rightarrow X$  be two mappings such that the pair  $(f, T)$  is  $(A, B)$ -weakly increasing. Assume the following:*

- (1) *The pair  $(f, T)$  is a cyclic  $(\psi, A, B)$ -contraction;*
- (2)  *$X$  satisfies property (P).*

*Then  $f$  and  $T$  have a common fixed point.*

*Proof* We follow the proof of Theorem 2.1 step by step to construct a nondecreasing sequence  $(x_n)$  in  $X$  with  $x_{2n} \in A$ ,  $x_{2n+1} \in B$  and  $x_{2n} \rightarrow u$  for some  $u \in X$ . Since  $x_{2n} \rightarrow u$ ,  $x_{2n+1} \rightarrow u$ ,  $A$  and  $B$  are closed subsets of  $X$ , we get  $u \in A \cap B$ . Using property (P), we get  $x_n \preceq u$  for all  $n \in \mathbb{N}$ . Since  $x_{2n} \in A$  and  $u \in B$ , we have

$$\begin{aligned} & \psi(d(x_{2n+1}, Tu)) \\ &= \psi(d(fx_{2n}, Tu)) \\ &\leq \delta \psi \left( \left\{ d(x_{2n}, u), d(x_{2n}, fx_{2n}), d(u, Tu), \frac{1}{2}(d(x_{2n}, Tu) + d(fx_{2n}, u)) \right\} \right) \\ &= \delta \psi \left( \left\{ d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, Tu), \frac{1}{2}(d(x_{2n}, Tu) + d(x_{2n+1}, u)) \right\} \right). \end{aligned}$$

Letting  $n \rightarrow +\infty$  in the above inequality, we get  $\psi(d(u, Tu)) \leq \delta \psi(d(u, Tu))$ . Since  $\delta < 1$ , we get  $d(u, Tu) = 0$ , hence  $u = Tu$ . Similarly, we may show that  $u = fu$ . Thus  $u$  is a common fixed point of  $f$  and  $T$ . □

Taking  $\psi = I_{[0, +\infty)}$  (the identity function) in Theorem 2.1, we have the following result.

**Corollary 2.1** *Let  $(X, d, \preceq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$ . Let  $f, T: X \rightarrow X$  be two mappings such that the pair  $(f, T)$  is  $(A, B)$ -weakly increasing and  $A \cup B$  has a cyclic representation with respect to the pair  $(f, T)$ . Suppose that there exists  $0 < \delta < 1$  such that for any two comparable elements  $x, y \in X$  with  $x \in A$  and  $y \in B$ , we have*

$$d(fx, Ty) \leq \delta \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\}.$$

*If  $f$  or  $T$  is continuous, then  $f$  and  $T$  have a common fixed point.*

The continuity of  $f$  or  $T$  in Corollary 2.1 can be dropped.

**Corollary 2.2** *Let  $(X, d, \preceq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$ . Let  $f, T: X \rightarrow X$  be two mappings such that the pair  $(f, T)$  is  $(A, B)$ -weakly increasing and such that  $A \cup B$  has a cyclic representation with respect to the pair*

$(f, T)$ . Suppose that there exists  $0 < \delta < 1$  such that for any two comparable elements  $x, y \in X$  with  $x \in A$  and  $y \in B$ , we have

$$d(fx, Ty) \leq \delta \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\}.$$

If  $X$  satisfies property (P), then  $f$  and  $T$  have a common fixed point.

By taking  $f = T$  in Theorem 2.1, we have the following result.

**Corollary 2.3** Let  $(X, d, \preceq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$  with  $X = A \cup B$ . Let  $f: X \rightarrow X$  be a mapping such that  $fx \preceq f(fx)$  for all  $x \in X$ . Suppose that there exists  $0 < \delta < 1$  such that for all  $x \in A$  and  $y \in B$ , we have

$$\psi(d(fx, fy)) \leq \delta \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{1}{2}(d(x, fy) + d(fx, y)) \right\} \right).$$

Assume the following:

- (1)  $f$  is a cyclic map;
- (2)  $f$  is continuous.

Then  $f$  has a fixed point.

The continuity of  $f$  in Corollary 2.3 can be dropped.

**Corollary 2.4** Let  $(X, d, \preceq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$  with  $X = A \cup B$ . Let  $f: X \rightarrow X$  be a mapping such that  $fx \preceq f(fx)$  for all  $x \in X$ . Suppose that there exists  $0 < \delta < 1$  such that for all  $x \in A$  and  $y \in B$ , we have

$$\psi(d(fx, fy)) \leq \delta \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{1}{2}(d(x, fy) + d(fx, y)) \right\} \right).$$

Assume the following:

- (1)  $f$  is a cyclic map;
- (2)  $X$  satisfies property (P).

Then  $f$  has a fixed point.

Taking  $A = B = X$  in Theorem 2.1, we have the following result.

**Corollary 2.5** Let  $(X, d, \preceq)$  be an ordered complete metric space. Let  $f, T: X \rightarrow X$  be two weakly increasing mappings. Suppose that there exists  $0 < \delta < 1$  such that for any two comparable elements  $x, y \in X$ , we have

$$\psi(d(fx, Ty)) \leq \delta \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right).$$

If  $f$  or  $T$  is continuous, then  $f$  and  $T$  have a common fixed point.

The continuity of  $f$  or  $T$  in Corollary 2.5 can be dropped.



**Corollary 2.6** *Let  $(X, d, \preceq)$  be an ordered complete metric space. Let  $f, T: X \rightarrow X$  be two weakly increasing mappings. suppose that there exists  $0 < \delta < 1$  such that for any two comparable elements  $x, y \in X$ , we have*

$$\psi(d(fx, Ty)) \leq \delta \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right).$$

*If  $X$  satisfies property (P), then  $f$  and  $T$  have a common fixed point.*

To support the validity of our results, we introduce the following nontrivial example.

**Example 2.1** On  $X = \{0, 1, 2, 3, \dots\}$ , consider

$$d: X \times X \rightarrow X, \quad d(x, y) = \begin{cases} 0 & \text{if } x = y; \\ \max\{x, y\} & \text{if } x \neq y. \end{cases}$$

We introduce a relation on  $X$  by  $x \preceq y$  if and only if  $y \leq x$ . Define  $f, T: X \rightarrow X$  by the formulae

$$fx = \begin{cases} 0 & \text{if } x = 0; \\ x - 1 & \text{if } x \geq 1, \end{cases}$$

and

$$Tx = \begin{cases} 0 & \text{if } x = 0, 1, 2; \\ x - 3 & \text{if } x \geq 3. \end{cases}$$

Also, define  $\psi: [0, +\infty) \rightarrow [0, +\infty)$  by  $\psi(t) = te^t$ . Let  $A = \{0, 2, 4, 6, \dots\}$  and  $B = \{0, 1, 3, 5, \dots\}$ . Then

- (1)  $(X, d, \preceq)$  is a complete ordered metric space;
- (2)  $A \cup B$  has a cyclic representation with respect to the pair  $(f, T)$ ;
- (3) The pair  $(f, T)$  is weakly  $(A, B)$ -increasing;
- (4)  $X$  satisfies property (P);
- (5) For every two comparable elements  $x, y \in X$  with  $x \in A$  and  $y \in B$ , we have

$$\psi(d(fx, Ty)) \leq e^{-1} \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right).$$

*Proof* The proof of part (1) is clear. Since  $fA = \{0, 1, 3, 5, \dots\} \subseteq B$  and  $TB = \{0, 2, 4, \dots\} \subseteq A$ , we conclude that  $A \cup B$  has a cyclic representation with respect to the pair  $(f, T)$ . To prove part (3), given  $x \in A$ . If  $x \in \{0, 1, 2, 3, 4\}$ , then  $T(fx) = 0$ . Thus  $Tfx \leq fx$  and hence  $fx \preceq T(fx)$ . If  $x \geq 5$ , then  $fx = x - 1$  and  $T(fx) = T(x - 1) = x - 4$ . Thus  $T(fx) \leq fx$  and hence  $fx \preceq T(fx)$ . Therefore  $fx \preceq T(fx)$  for all  $x \in A$ . Similarly, we may show that  $Tx \preceq f(Tx)$  for all  $x \in B$ . So, the pair  $(f, T)$  is weakly  $(A, B)$ -increasing. To prove part (4), let  $\{x_n\}$  be a nondecreasing sequence such that  $x_n \rightarrow x \in X$ . Then  $d(x_n, x) \rightarrow d(x, x) = 0$ . So,  $x_n = x$  for all  $n$  except for finitely many. Since  $(x_n)$  is a nondecreasing with respect to  $\preceq$ , we have  $x_1 \geq x_2 \geq x_3 \dots$ . Since  $x_n = x$  for all but finitely many, then there exists  $k \in \mathbb{N}$  such that  $x_1 \geq \dots \geq x_{k-1} \geq$

$x_n = x$  for all  $n \geq k$ . So,  $x_n \geq x$  for all  $n \in \mathbb{N}$  and hence  $x_n \leq x$  for all  $n \in \mathbb{N}$ . Thus  $X$  satisfies property (P). To prove part (5), given two comparable elements  $x, y \in X$  with  $x \in A$  and  $y \in B$ . We divide the proof into the following cases:

• Case one:  $x = 0$  and  $y \in \{0, 1, 3\}$ . Here, we have  $fx = Ty = 0$  and hence  $\psi(d(fx, Ty)) = 0$ . Thus

$$\psi(d(fx, Ty)) \leq e^{-1} \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right).$$

• Case two:  $x \geq 2$  and  $y \geq 5$ . Here  $fx = x - 1$  and  $Ty = y - 3$ . Since  $x \in A$  and  $y \in B$ , then  $x = 2t$  and  $y = 2n + 1$  for some  $t, n \in \mathbb{N}$ .

If  $fx = Ty$ , then  $x - 1 = y - 3$  and hence  $2t - 1 = 2n - 2$ . Thus  $2t = 2n - 1$ , which is impossible. If  $x - 1 > y - 3$ , then  $d(fx, Ty) = x - 1$ . Thus

$$\begin{aligned} \psi(d(fx, Ty)) &= (x - 1)e^{x-1} \\ &\leq e^{-1}xe^x \\ &= e^{-1}\psi(x) \\ &= e^{-1}\psi(d(x, fx)) \\ &\leq e^{-1} \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right). \end{aligned}$$

If  $y - 3 > x - 1$ , then  $d(fx, Ty) = y - 3$ . Thus

$$\begin{aligned} \psi(d(fx, Ty)) &= (y - 3)e^{y-3} \\ &\leq e^{-3}ye^y \\ &\leq e^{-1}ye^y \\ &= e^{-1}\psi(y) \\ &= e^{-1}\psi(d(y, Ty)) \\ &\leq e^{-1} \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right). \end{aligned}$$

• Case three:  $x = 0$  and  $y \geq 5$ . Here  $fx = 0$  and  $Ty = y - 3$ . Thus

$$\begin{aligned} \psi(d(fx, Ty)) &= \psi(y - 3) \\ &= (y - 3)e^{y-3} \\ &\leq e^{-1}ye^y \\ &= e^{-1}\psi(y) \\ &= e^{-1}\psi(d(y, Ty)) \\ &\leq e^{-1} \psi \left( \max \left\{ d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y)) \right\} \right). \end{aligned}$$

- Case four:  $x \geq 2$  and  $y \in \{0, 1, 3\}$ . Here  $fx = x - 1$  and  $Ty = 0$ .

$$\begin{aligned} \psi(d(fx, Ty)) &= \psi(d(x - 1, 0)) = \psi(x - 1) \\ &= (x - 1)e^{x-1} \\ &\leq e^{-1}xe^x \\ &= e^{-1}\psi(x) \\ &= e^{-1}\psi(d(x, fx)) \\ &\leq e^{-1}\psi\left(\max\left\{d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y))\right\}\right). \end{aligned}$$

Note that  $f$  and  $T$  satisfy all the hypotheses of Theorem 2.1. Hence  $f$  and  $T$  have a fixed point. Here 0 is the fixed point of  $f$  and  $T$ . □

### 3 Applications

Denote by  $\Lambda$  the set of functions  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  satisfying the following hypotheses:

- (h1)  $\mu$  is a Lebesgue-integrable mapping on each compact of  $[0, +\infty)$ ;
- (h2) For every  $\epsilon > 0$ , we have

$$\int_0^\epsilon \mu(t) dt > 0.$$

**Theorem 3.1** *Let  $(X, d, \leq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$ . Let  $f, T: X \rightarrow X$  be two mappings such that the pair  $(f, T)$  is  $(A, B)$ -weakly increasing and  $A \cup B$  has a cyclic representation w.r.t. the pair  $(f, T)$ . Suppose that there exist  $\delta \in [0, 1)$  and  $\mu \in \Lambda$  such that for any two comparable elements  $x, y \in X$  with  $x \in A$  and  $y \in B$ , we have*

$$\int_0^{d(fx, Ty)} \mu(s) ds \leq \delta \int_0^{\max\{d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y))\}} \mu(s) ds.$$

*If  $f$  or  $T$  is continuous, then  $f$  and  $T$  have a common fixed point.*

*Proof* Follows from Theorem 2.1 by defining  $\psi: [0, +\infty) \rightarrow [0, +\infty)$  via  $\psi(t) = \int_0^t \mu(s) ds$  and noting that  $\psi$  is an altering distance function. □

The continuity of  $f$  or  $T$  in Theorem 3.1 can be dropped.

**Theorem 3.2** *Let  $(X, d, \leq)$  be an ordered complete metric space and  $A, B$  be nonempty closed subsets of  $X$ . Let  $f, T: X \rightarrow X$  be two mappings such that  $(f, T)$  is  $(A, B)$ -weakly increasing and  $A \cup B$  has a cyclic representation w.r.t. the pair  $(f, T)$ . Suppose that there exist  $\delta \in [0, 1)$  and  $\mu \in \Lambda$  such that for any two comparable elements  $x, y \in X$  with  $x \in A$  and  $y \in B$ , we have*

$$\int_0^{d(fx, Ty)} \mu(s) ds \leq \delta \int_0^{\max\{d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y))\}} \mu(s) ds.$$

*If  $X$  satisfies property (P), then  $f$  and  $T$  have a common fixed point.*

By taking  $A = B = X$  in Theorems 3.1 and 3.2, we have the following results.

**Corollary 3.1** *Let  $(X, d, \preceq)$  be an ordered complete metric space. Let  $f, T: X \rightarrow X$  be two weakly increasing mappings. Suppose that there exist  $\delta \in [0, 1)$  and  $\mu \in \Lambda$  such that for any two comparable elements  $x, y \in X$ , we have*

$$\int_0^{d(fx, Ty)} \mu(s) ds \leq \delta \int_0^{\max\{d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y))\}} \mu(s) ds.$$

*If  $f$  or  $T$  is continuous, then  $f$  and  $T$  have a common fixed point.*

**Corollary 3.2** *Let  $(X, d, \preceq)$  be an ordered complete metric space. Let  $f, T: X \rightarrow X$  be two weakly increasing mappings. Suppose that there exist  $\delta \in [0, 1)$  and  $\mu \in \Lambda$  such that for any two comparable elements  $x, y \in X$ , we have*

$$\int_0^{d(fx, Ty)} \mu(s) ds \leq \delta \int_0^{\max\{d(x, y), d(x, fx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(fx, y))\}} \mu(s) ds.$$

*If  $X$  satisfies property (P), then  $f$  and  $T$  have a common fixed point.*

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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