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Coupled common fixed point theorems for a pair of commuting mappings in partially ordered G -metric spaces

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Abstract

In this paper, we introduce a new contractive condition for a pair of commuting mappings in partially ordered G -metric spaces. Some new coupled coincidence point and coupled common fixed point theorems are obtained. An example is given to support the usability of our results. The results presented in this paper generalize and improve the corresponding results of Nashine and Shatanawi from partially ordered metric spaces to partially ordered G -metric spaces.

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Keywords: G -metric space; mixed g -monotone property; coupled coincidence point; coupled common fixed point; partial order

1 Introduction and preliminaries

In 2004, Ran and Reurings [1] showed the existence of fixed points of nonlinear contraction mappings in metric spaces endowed with a partial ordering and presented applications of their results to matrix equations. Subsequently, Nieto and Rodríguez-López [2, 3] extended the corresponding results in [1]. They proved the existence theorems of a unique fixed point for mappings in partially ordered sets and gave some applications to the existence of a unique solution for a first-order ordinary differential equation with periodic boundary conditions.

In 2005, the well-known Tarski theorem [4] was used in [5] to study the existence of solutions for fuzzy equations and to prove existence theorems for fuzzy differential equations.

It is well known that fixed point theory in partially ordered metric spaces as one of the most important tools of nonlinear analysis has been widely applied to matrix equations (for more details, see [1] and the references therein), ordinary differential equations (for more details, see [2, 3, 6] and [7] and the references therein), fuzzy differential equations (for more details, see [5] and the references therein), integral equations (for more details, see [8] and [9] and the references therein) and intermediate value theorems (for more details, see [10] and the references therein). For more results on partially ordered metric spaces, one refers to the results in [11–21].

In 2006, Bhaskar and Lakshmikantham [22] introduced the notion of a coupled fixed point and proved some interesting coupled fixed point theorems for the mappings satisfying a mixed monotone property; while Lakshmikantham and Ćirić [23] introduced the

concept of a mixed g -monotone mapping and proved coupled coincidence and coupled common fixed point theorems that extended the theorems due to Bhaskar and Lakshmikantham [22]. Subsequently, many authors obtained many coupled coincidence and coupled fixed point theorems in ordered metric spaces; see [24–29] and the references therein.

In 2006, Mustafa and Sims [30] introduced the notion of generalized metric spaces or simply G -metric spaces as a generalization of the concept of a metric space. Based on the concept of a G -metric space, many authors obtained many fixed point and common fixed point theorems for the mappings satisfying different contractive conditions; see [31–42] for more details. Fixed point problems also have been considered in partially ordered G -metric spaces; see [43–55] and the references therein for more details.

Inspired and motivated by the facts said above, in this paper we establish coupled coincidence point and coupled common fixed point theorems for a pair of commuting mappings in partially ordered G -metric spaces. An example is given to support the usability of our results. The results presented in this paper generalize and improve the corresponding results of Nashine and Shatanawi [29] from partially ordered metric spaces to partially ordered G -metric spaces.

First, we present some known definitions and propositions.

Definition 1.1 [30] Let X be a nonempty set, and let $G : X \times X \times X \rightarrow R^+$ be a function satisfying the following axioms:

- (G1) $G(x, y, z) = 0$ if $x = y = z$;
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables);
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G -metric on X and the pair (X, G) is called a G -metric space.

It is known that the function $G(x, y, z)$ on a G -metric space X is jointly continuous in all three of its variables, and $G(x, y, z) = 0$ if and only if $x = y = z$; see [30] and the references therein for more details.

Definition 1.2 [30] Let (X, G) be a G -metric space, and let $\{x_n\}$ be a sequence of points in X . A point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m, n \rightarrow \infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G -convergent to x .

Thus, if $x_n \rightarrow x$ in a G -metric space (X, G) , then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Proposition 1.1 [30] Let (X, G) be a G -metric space. Then the following are equivalent:

- (1) $\{x_n\}$ is G -convergent to x ;
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (4) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow \infty$.

Definition 1.3 [30] Let (X, G) be a G -metric space. A sequence $\{x_n\}$ is called a G -Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; that is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Definition 1.4 [30] A G -metric space (X, G) is said to be G -complete if every G -Cauchy sequence in (X, G) is G -convergent in X .

Proposition 1.2 [30] Let (X, G) be a G -metric space. Then the following are equivalent:

- (1) The sequence $\{x_n\}$ is G -Cauchy;
- (2) For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq k$.

Definition 1.5 [30] Let (X, G) and (X', G') be two G -metric spaces. Then $f : X \rightarrow X'$ is G -continuous at $x \in X$ if and only if it is G -sequentially continuous at x ; that is, whenever $\{x_n\}$ is G -convergent to x , $\{f(x_n)\}$ is G -convergent to $f(x)$.

Proposition 1.3 [30] Let (X, G) be a G -metric space. Then, for any $x, y \in X$, we have $G(x, y, y) \leq 2G(y, x, x)$.

Definition 1.6 [43] Let (X, G) be a G -metric space. A mapping $F : X \times X \rightarrow X$ is said to be continuous if for any two G -convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $\{F(x_n, y_n)\}$ is G -convergent to $F(x, y)$.

Definition 1.7 [22] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.8 [23] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.9 [23] Let X be a nonempty set. Then we say that the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are commutative if $gF(x, y) = F(gx, gy)$.

Definition 1.10 [22] Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y ; that is, for any $x, y \in X$, we have

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \Rightarrow \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \Rightarrow \quad F(x, y_1) \geq F(x, y_2).$$

Definition 1.11 [23] Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. The mapping F is said to have the mixed g -monotone property if F is monotone g -nondecreasing in its first argument and is monotone g -nonincreasing in its second argument; that is, for any $x, y \in X$, we have

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \quad \Rightarrow \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \quad \Rightarrow \quad F(x, y_1) \geq F(x, y_2).$$

2 Main results

Theorem 2.1 *Let (X, \leq) be a partially ordered set and (X, G) be a G -metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g -monotone property on X . Suppose there exist non-negative real numbers α, β, γ and L with $\beta + \gamma < 1$ such that*

$$\begin{aligned} &G(F(x, y), F(u, v), F(w, z)) \\ &\leq \alpha \min \{ G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx) \} \\ &\quad + \beta \min \{ G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu) \} \\ &\quad + \gamma \min \{ G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw) \} \\ &\quad + L \min \{ G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx) \} \end{aligned} \tag{2.1}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $g(x) \leq g(u) \leq g(w)$ and $g(y) \geq g(v) \geq g(z)$. Further suppose $F(X \times X) \subset g(X)$ and $g(X)$ is a complete subspace of X . Also suppose that X satisfies the following properties:

- (i) if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \leq x$ for all $n \in \mathbb{N}$;
- (ii) if a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist two elements $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then F and g have a coupled coincidence point. That is, there exists $(x, y) \in X \times X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Proof In view of the condition of the theorem, we see that there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Since $F(X \times X) \subset g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again since $F(X \times X) \subset g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n), \quad \forall n \geq 0. \tag{2.2}$$

Now we prove that

$$g(x_n) \leq g(x_{n+1}) \quad \text{and} \quad g(y_n) \geq g(y_{n+1}), \quad \forall n \geq 0. \tag{2.3}$$

We will use the mathematical induction. Let $n = 0$. Since $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0) \leq g(x_1)$ and $g(y_0) \geq g(y_1)$; that is, (2.3) hold for $n = 0$. We assume that (2.3) hold for some $n > 0$. As F has the mixed g -monotone property, $g(x_n) \leq g(x_{n+1})$ and $g(y_n) \geq g(y_{n+1})$, from (2.2) we get that

$$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n) \geq F(y_{n+1}, x_n). \tag{2.4}$$

In the same way, we obtain that

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n) \quad \text{and} \quad g(y_{n+2}) = F(y_{n+1}, x_{n+1}) \leq F(y_{n+1}, x_n). \tag{2.5}$$

By combining (2.4) and (2.5), we obtain $g(x_{n+1}) \leq g(x_{n+2})$ and $g(y_{n+1}) \geq g(y_{n+2})$. Thus, by the mathematical induction, we conclude that (2.3) hold for all $n \geq 0$.

We check easily that

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \dots \leq g(x_n) \leq g(x_{n+1}) \leq \dots$$

and

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq \dots \geq g(y_n) \geq g(y_{n+1}) \geq \dots$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} &G(gx_{n+1}, gx_n, gx_n) \\ &= G(F(x_n, y_n), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq \alpha \min \{ G(F(x_n, y_n), gx_n, gx_n), G(F(x_{n-1}, y_{n-1}), gx_n, gx_n), G(F(x_{n-1}, y_{n-1}), gx_n, gx_n) \} \\ &\quad + \beta \min \{ G(F(x_n, y_n), gx_{n-1}, gx_{n-1}), G(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_{n-1}), \\ &\quad G(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_{n-1}) \} \\ &\quad + \gamma \min \{ G(F(x_n, y_n), gx_{n-1}, gx_{n-1}), G(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_{n-1}), \\ &\quad G(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_{n-1}) \} \\ &\quad + L \min \{ G(F(x_n, y_n), gx_{n-1}, gx_{n-1}), G(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_{n-1}), \\ &\quad G(F(x_{n-1}, y_{n-1}), gx_n, gx_n) \} \\ &= \alpha \min \{ G(gx_{n+1}, gx_n, gx_n), G(gx_n, gx_n, gx_n), G(gx_n, gx_n, gx_n) \} \\ &\quad + \beta \min \{ G(gx_{n+1}, gx_{n-1}, gx_{n-1}), G(gx_n, gx_{n-1}, gx_{n-1}), G(gx_n, gx_{n-1}, gx_{n-1}) \} \\ &\quad + \gamma \min \{ G(gx_{n+1}, gx_{n-1}, gx_{n-1}), G(gx_n, gx_{n-1}, gx_{n-1}), G(gx_n, gx_{n-1}, gx_{n-1}) \} \\ &\quad + L \min \{ G(gx_{n+1}, gx_{n-1}, gx_{n-1}), G(gx_n, gx_{n-1}, gx_{n-1}), G(gx_n, gx_n, gx_n) \} \\ &= (\beta + \gamma) \min \{ G(gx_{n+1}, gx_{n-1}, gx_{n-1}), G(gx_n, gx_{n-1}, gx_{n-1}) \}. \end{aligned} \tag{2.6}$$

Hence, we obtain

$$G(gx_{n+1}, gx_n, gx_n) \leq (\beta + \gamma)G(gx_n, gx_{n-1}, gx_{n-1}). \tag{2.7}$$

Similarly, using (2.1) and (2.2), we can prove that

$$G(gy_{n+1}, gy_n, gy_n) \leq (\beta + \gamma)G(gy_n, gy_{n-1}, gy_{n-1}). \tag{2.8}$$

Suppose that $\rho_n = G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n)$ and $\delta = \beta + \gamma$, then $0 \leq \delta < 1$. From (2.7) and (2.8), we have

$$\begin{aligned} \rho_n &= G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n) \\ &\leq \delta (G(gx_n, gx_{n-1}, gx_{n-1}) + G(gy_n, gy_{n-1}, gy_{n-1})) \end{aligned}$$

$$\begin{aligned}
 &= \delta \rho_{n-1} \leq \delta^2 \rho_{n-2} \leq \dots \\
 &\leq \delta^n \rho_0,
 \end{aligned}
 \tag{2.9}$$

which implies that

$$\lim_{n \rightarrow \infty} \rho_n = \lim_{n \rightarrow \infty} [G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n)] = 0,$$

and

$$\lim_{n \rightarrow \infty} G(gx_{n+1}, gx_n, gx_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} G(gy_{n+1}, gy_n, gy_n) = 0.$$

Next, we will prove that $\{gx_n\}$ and $\{gy_n\}$ are G -Cauchy sequences in $g(X)$. In fact, for each $n, m \in \mathbb{N}$, $m > n$, from (G5) we have

$$\begin{aligned}
 G(gx_m, gx_n, gx_n) &\leq G(gx_m, gx_{m-1}, gx_{m-1}) + G(gx_{m-1}, gx_{m-2}, gx_{m-2}) + \dots \\
 &\quad + G(gx_{n+1}, gx_n, gx_n)
 \end{aligned}
 \tag{2.10}$$

and

$$\begin{aligned}
 G(gy_m, gy_n, gy_n) &\leq G(gy_m, gy_{m-1}, gy_{m-1}) + G(gy_{m-1}, gy_{m-2}, gy_{m-2}) + \dots \\
 &\quad + G(gy_{n+1}, gy_n, gy_n).
 \end{aligned}
 \tag{2.11}$$

Therefore, combining inequalities (2.10), (2.11) and (2.9), we have

$$\begin{aligned}
 &G(gx_m, gx_n, gx_n) + G(gy_m, gy_n, gy_n) \\
 &\leq \rho_{m-1} + \rho_{m-2} + \dots + \rho_n \\
 &= (\delta^{m-1} + \delta^{m-2} + \dots + \delta^n) \rho_0 \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.
 \end{aligned}
 \tag{2.12}$$

This implies that $\{gx_n\}$ and $\{gy_n\}$ are G -Cauchy sequences in $g(X)$. Since $g(X)$ is a complete subspace of X , there exist $x, y \in X$ such that $gx_n \rightarrow gx$ and $gy_n \rightarrow gy$ as $n \rightarrow \infty$. Since $\{gx_n\}$ is a non-decreasing sequence and $\{gy_n\}$ is a non-increasing sequence, by the assumption conditions (i) and (ii), we have $gx_n \leq gx$ and $gy_n \geq gy$ for all $n \geq 0$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
 &G(gx_{n+1}, F(x, y), F(x, y)) \\
 &= G(F(x_n, y_n), F(x, y), F(x, y)) \\
 &\leq \alpha \min \{ G(F(x_n, y_n), gx_n, gx_n), G(F(x, y), gx_n, gx_n), G(F(x, y), gx_n, gx_n) \} \\
 &\quad + \beta \min \{ G(F(x_n, y_n), gx, gx), G(F(x, y), gx, gx), G(F(x, y), gx, gx) \} \\
 &\quad + \gamma \min \{ G(F(x_n, y_n), gx, gx), G(F(x, y), gx, gx), G(F(x, y), gx, gx) \} \\
 &\quad + L \min \{ G(F(x_n, y_n), gx, gx), G(F(x, y), gx, gx), G(F(x, y), gx_n, gx_n) \} \\
 &= \alpha \min \{ G(gx_{n+1}, gx_n, gx_n), G(F(x, y), gx_n, gx_n), G(F(x, y), gx_n, gx_n) \}
 \end{aligned}$$

$$\begin{aligned}
 &+ (\beta + \gamma) \min\{G(gx_{n+1}, gx, gx), G(F(x, y), gx, gx)\} \\
 &+ L \min\{G(gx_{n+1}, gx, gx), G(F(x, y), gx, gx), G(F(x, y), gx_n, gx_n)\}. \tag{2.13}
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in the above inequality, we obtain $G(gx, F(x, y), F(x, y)) \leq 0$. Hence $G(gx, F(x, y), F(x, y)) = 0$, and so $gx = F(x, y)$.

Similarly, one can show that $gy = F(y, x)$. Thus we proved that F and g have a coupled coincidence point. This concludes the proof. \square

Theorem 2.2 *Let (X, \preceq) be a partially ordered set and (X, G) be a complete G -metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g -monotone property on X . Suppose there exist non-negative real numbers α, β, γ and L with $\beta + \gamma < 1$ such that*

$$\begin{aligned}
 &G(F(x, y), F(u, v), F(w, z)) \\
 &\leq \alpha \min\{G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx)\} \\
 &\quad + \beta \min\{G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu)\} \\
 &\quad + \gamma \min\{G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw)\} \\
 &\quad + L \min\{G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx)\} \tag{2.14}
 \end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $g(x) \preceq g(u) \preceq g(w)$ and $g(y) \succeq g(v) \succeq g(z)$. Further suppose $F(X \times X) \subset g(X)$ and g is continuous non-decreasing and commutes with F , and also suppose either

- (a) F is continuous or
- (b) X has the following property:
 - (i) if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;
 - (ii) if a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.

If there exist two elements $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point. That is, there exists $(x, y) \in X \times X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Proof Following the proof of Theorem 2.1, we will get two G -Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that $\{gx_n\}$ is a non-decreasing sequence and $\{gy_n\}$ is a non-increasing sequence in X . Since (X, G) is a complete G -metric space, there is $(x, y) \in X \times X$ such that $gx_n \rightarrow x$ and $gy_n \rightarrow y$.

Since g is continuous, we have $ggx_n \rightarrow gx$ and $ggy_n \rightarrow gy$ as $n \rightarrow \infty$.

First, suppose that (a) holds, that is, F is continuous. Then

$$F(gx_n, gy_n) \rightarrow F(x, y) \quad \text{and} \quad F(gy_n, gx_n) \rightarrow F(y, x).$$

On the other hand, since g commutes with F , we have

$$F(gx_n, gy_n) = gF(x_n, y_n) = ggx_{n+1} \rightarrow gx \quad \text{as } n \rightarrow \infty$$

and

$$F(gy_n, gx_n) = gF(y_n, x_n) = ggy_{n+1} \rightarrow gy \quad \text{as } n \rightarrow \infty.$$

Therefore, by the uniqueness of limit, we get $gx = F(x, y)$ and $gy = F(y, x)$.

Second, suppose that (b) holds. Since $\{gx_n\}$ is a non-decreasing sequence such that $gx_n \rightarrow x$, $\{gy_n\}$ is a non-increasing sequence such that $gy_n \rightarrow y$ and g is a non-decreasing function, by (i) and (ii) we get that $g(gx_n) \leq g(x)$ and $g(gy_n) \geq g(y)$ hold for all $n \in \mathbb{N}$.

By (2.14), we have

$$\begin{aligned} &G(g(gx_{n+1}), F(x, y), F(x, y)) \\ &= G(F(gx_n, gy_n), F(x, y), F(x, y)) \\ &\leq \alpha \min\{G(F(gx_n, gy_n), g(gx_n), g(gx_n)), G(F(x, y), g(gx_n), g(gx_n)), \\ &\quad G(F(x, y), g(gx_n), g(gx_n))\} \\ &\quad + \beta \min\{G(F(gx_n, gy_n), gx, gx), G(F(x, y), gx, gx), G(F(x, y), gx, gx)\} \\ &\quad + \gamma \min\{G(F(gx_n, gy_n), gx, gx), G(F(x, y), gx, gx), G(F(x, y), gx, gx)\} \\ &\quad + L \min\{G(F(gx_n, gy_n), gx, gx), G(F(x, y), gx, gx), G(F(x, y), g(x_n), g(gx_n))\} \\ &= \alpha \min\{G(g(gx_{n+1}), g(gx_n), g(gx_n)), G(F(x, y), g(gx_n), g(gx_n)), \\ &\quad G(F(x, y), g(gx_n), g(gx_n))\} \\ &\quad + (\beta + \gamma) \min\{G(g(gx_{n+1}), gx, gx), G(F(x, y), gx, gx)\} \\ &\quad + L \min\{G(g(gx_{n+1}), gx, gx), G(F(x, y), gx, gx), G(F(x, y), g(x_n), g(gx_n))\}. \end{aligned} \tag{2.15}$$

Letting $n \rightarrow \infty$, we get $G(gx, F(x, y), F(x, y)) \leq 0$, which implies that $G(gx, F(x, y), F(x, y)) = 0$, so that $gx = F(x, y)$.

Similarly, we can show that $gy = F(y, x)$. Thus we proved that F and g have a coupled coincidence point. □

Remark 2.1 Theorems 2.1 and 2.2 generalize and extend the corresponding results in [29, Theorems 2.1 and 2.2] from partially ordered metric spaces to partially ordered G -metric spaces.

In Theorem 2.2, if we take $g = I$ (I is an identity mapping), then we have the following corollary.

Corollary 2.1 *Let (X, \leq) be a partially ordered set and (X, G) be a complete G -metric space. Let $F : X \times X \rightarrow X$ be a mapping such that F has the mixed monotone property on X . Suppose there exist non-negative real numbers α, β, γ and L with $\beta + \gamma < 1$ such that*

$$\begin{aligned} &G(F(x, y), F(u, v), F(w, z)) \\ &\leq \alpha \min\{G(F(x, y), x, x), G(F(u, v), x, x), G(F(w, z), x, x)\} \\ &\quad + \beta \min\{G(F(x, y), u, u), G(F(u, v), u, u), G(F(w, z), u, u)\} \end{aligned}$$

$$\begin{aligned}
 & + \gamma \min\{G(F(x, y), w, w), G(F(u, v), w, w), G(F(w, z), w, w) \\
 & + L \min\{G(F(x, y), u, u), G(F(u, v), w, w), G(F(w, z), x, x)\} \tag{2.16}
 \end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $x \leq u \leq w$ and $y \geq v \geq z$, and also suppose either

- (a) F is continuous or
- (b) X has the following property:
 - (i) if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \leq x$ for all $n \in \mathbb{N}$;
 - (ii) if a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist two elements $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then F and g have a coupled fixed point. That is, there exists $(x, y) \in X \times X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Also, if we take $\alpha = \beta = \gamma = 0$ in Theorem 2.2, then we get the following.

Corollary 2.2 Let (X, \leq) be a partially ordered set and (X, G) be a complete G -metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g -monotone property on X . Suppose there exists a non-negative real number L such that

$$\begin{aligned}
 & G(F(x, y), F(u, v), F(w, z)) \\
 & \leq L \min\{G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx)\} \tag{2.17}
 \end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $g(x) \leq g(u) \leq g(w)$ and $g(y) \geq g(v) \geq g(z)$. Further suppose $F(X \times X) \subset g(X)$ and g is continuous non-decreasing and commutes with F , and also suppose either

- (a) F is continuous or
- (b) X has the following property:
 - (i) if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \leq x$ for all $n \in \mathbb{N}$;
 - (ii) if a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist two elements $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then F and g have a coupled coincidence point. That is, there exists $(x, y) \in X \times X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Corollary 2.3 Let (X, \leq) be a partially ordered set and (X, G) be a complete G -metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g -monotone property on X . Suppose there exist non-negative real numbers α, β, γ and L with $\beta + \gamma < 1$ such that

$$\begin{aligned}
 & G(F(x, y), F(u, v), F(w, z)) \\
 & \leq (\alpha + \beta + \gamma) \min\{G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx), \\
 & G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu), G(F(x, y), gw, gw)\},
 \end{aligned}$$

$$\begin{aligned}
 &G(F(u, v), gw, gw), G(F(w, z), gw, gw) \} \\
 &+ L \min \{ G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx) \} \tag{2.18}
 \end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $g(x) \leq g(u) \leq g(w)$ and $g(y) \geq g(v) \geq g(z)$. Further suppose $F(X \times X) \subset g(X)$ and g is continuous non-decreasing and commutes with F , and also suppose either

- (a) F is continuous or
- (b) X has the following property:
 - (i) if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \leq x$ for all $n \in \mathbb{N}$;
 - (ii) if a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist two elements $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then F and g have a coupled coincidence point. That is, there exists $(x, y) \in X \times X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Proof By noting that if α, β and γ are non-negative real numbers, from (2.18) we have

$$\begin{aligned}
 &G(F(x, y), F(u, v), F(w, z)) \\
 &\leq (\alpha + \beta + \gamma) \min \{ G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx), \\
 &G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu), G(F(x, y), gw, gw), \\
 &G(F(u, v), gw, gw), G(F(w, z), gw, gw) \} \\
 &+ L \min \{ G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx) \} \\
 &\leq \alpha \min \{ G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx) \} \\
 &+ \beta \min \{ G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu) \} \\
 &+ \gamma \min \{ G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw) \} \\
 &+ L \min \{ G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx) \}. \tag{2.19}
 \end{aligned}$$

From Theorem 2.2, we see that F and g have a coupled coincidence point $(x, y) \in X \times X$; that is, $F(x, y) = gx$ and $F(y, x) = gy$. □

Remark 2.2 Corollaries 2.1, 2.2 and 2.3 generalize and extend the corresponding results in [29, Corollaries 2.1, 2.2 and 2.3] from partially ordered metric spaces to partially ordered G -metric spaces.

Now we will prove the existence and uniqueness theorem of a coupled common fixed point. Note that if (X, \leq) is a partially ordered set, then we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, \quad (u, v) \leq (x, y) \iff x \geq u, \quad y \leq v.$$

Theorem 2.3 In addition to the hypotheses of Theorem 2.1, suppose that $L = 0$ and for every $(x, y), (y^*, x^*) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F and g have a unique coupled

common fixed point; that is, there exists a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Proof From Theorem 2.1, the set of coupled coincidence points of F and g is nonempty. Suppose (x, y) and (x^*, y^*) are coupled coincidence points of F and g , that is, $gx = F(x, y)$, $gy = F(y, x)$, $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$, then

$$g(x) = g(x^*) \quad \text{and} \quad g(y) = g(y^*). \tag{2.20}$$

By assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_{n+1}) = F(u_n, v_n) \quad \text{and} \quad g(v_{n+1}) = F(v_n, u_n) \quad \forall n \in \mathbb{N}.$$

Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$ and $y_0^* = y^*$. In the same way, we define the sequences $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(x_n^*)\}$ and $\{g(y_n^*)\}$. It is easy to show that

$$g(x_n) \rightarrow F(x, y), \quad g(y_n) \rightarrow F(y, x), \quad g(x_n^*) \rightarrow F(x^*, y^*), \quad g(y_n^*) \rightarrow F(y^*, x^*).$$

Since

$$(F(x, y), F(y, x)) = (g(x_1), g(y_1)) = (g(x), g(y))$$

and

$$(F(u, v), F(v, u)) = (g(u_1), g(v_1))$$

are comparable, we see that $g(x) \leq g(u_1)$ and $g(y) \geq g(v_1)$. It is easy to show that $(g(x), g(y))$ and $(g(u_n), g(v_n))$ are comparable, that is, $g(x) \leq g(u_n)$ and $g(y) \geq g(v_n)$ for all $n \geq 1$. Thus from (2.1), we have

$$\begin{aligned} &G(gx, gu_{n+1}, gu_{n+1}) \\ &= G(F(x, y), F(u_n, v_n), F(u_n, v_n)) \\ &\leq \alpha \min \{ G(F(x, y), gx, gx), F(u_n, v_n), gx, gx), G(F(u_n, v_n), gx, gx) \} \\ &\quad + \beta \min \{ G(F(x, y), gu_n, gu_n), G(F(u_n, v_n), gu_n, gu_n), F(u_n, v_n), gu_n, gu_n) \} \\ &\quad + \gamma \min \{ G(F(x, y), gu_n, gu_n), G(F(u_n, v_n), gu_n, gu_n), G(F(u_n, v_n), gu_n, gu_n) \}. \end{aligned}$$

Since $F(x, y) = g(x)$ and $g(u_{n+1}) = F(u_n, v_n)$, we have

$$\begin{aligned} &G(gx, gu_{n+1}, gu_{n+1}) \\ &\leq (\beta + \gamma) \min \{ G(gx, gu_n, gu_n), G(gu_{n+1}, gu_n, gu_n) \}. \end{aligned}$$

Therefore, we have

$$G(gx, gu_{n+1}, gu_{n+1}) \leq (\beta + \gamma)G(gx, gu_n, gu_n). \tag{2.21}$$

Similarly, using (2.1), $g(y) = y$ and $g(v_{n+1}) = F(v_n, u_n)$, we can prove that

$$G(gy, gv_{n+1}, gv_{n+1}) \leq (\beta + \gamma)G(gy, gv_n, gv_n). \tag{2.22}$$

From (2.21) and (2.22), we have

$$\begin{aligned} &G(gx, gu_{n+1}, gu_{n+1}) + G(gy, gv_{n+1}, gv_{n+1}) \\ &\leq (\beta + \gamma)[G(gx, gu_n, gu_n) + G(gy, gv_n, gv_n)] \\ &\leq (\beta + \gamma)^2[G(gx, gu_{n-1}, gu_{n-1}) + G(gy, gv_{n-1}, gv_{n-1})] \\ &\leq \dots \\ &\leq (\beta + \gamma)^{n+1}[G(gx, gu_0, gu_0) + G(gy, gv_0, gv_0)]. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} [G(gx, gu_n, gu_n) + G(gy, gv_n, gv_n)] = 0,$$

which implies that

$$\lim_{n \rightarrow \infty} G(gx, gu_n, gu_n) = \lim_{n \rightarrow \infty} G(gy, gv_n, gv_n) = 0. \tag{2.23}$$

Similarly, we show that

$$\lim_{n \rightarrow \infty} G(gx^*, gu_n, gu_n) = \lim_{n \rightarrow \infty} G(gy^*, gv_n, gv_n) = 0. \tag{2.24}$$

By the rectangle inequality (G5), Proposition 1.3, (2.23) and (2.24), we obtain

$$\begin{aligned} G(gx, gx^*, gx^*) &\leq G(gx, gu_n, gu_n) + G(gu_n, gx^*, gx^*) \\ &\leq (gx, gu_n, gu_n) + 2G(gx^*, gu_n, gu_n) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} G(gy, gy^*, gy^*) &\leq G(gy, gv_n, gv_n) + G(gv_n, gy^*, gy^*) \\ &\leq (gy, gv_n, gv_n) + 2G(gy^*, gv_n, gv_n) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

It implies that $g(x) = g(x^*)$ and $g(y) = g(y^*)$. Thus we have (2.20) holds. This implies that $(g(x), g(y)) = (g(x^*), g(y^*))$.

Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$, by the commutativity of F and g , we have

$$g(gx) = g(F(x, y)) = F(gx, gy) \quad \text{and} \quad g(gy) = g(F(y, x)) = F(gy, gx). \tag{2.25}$$

Denote $g(x) = z, g(y) = w$. Then from (2.25) we have

$$g(z) = F(z, w) \quad \text{and} \quad g(w) = F(w, z). \tag{2.26}$$

Thus (z, w) is a coupled coincidence point of F and g . Then from (2.25) with $x^* = z$ and $y^* = w$ it follows $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \quad \text{and} \quad g(w) = w. \tag{2.27}$$

From (2.26) and (2.27), we get

$$z = g(z) = F(z, w) \quad \text{and} \quad w = g(w) = F(w, z).$$

Therefore, (z, w) is a coupled common fixed point of F and g . To prove the uniqueness, assume that (p, q) is another coupled common fixed point, then by (2.25) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. This completes the proof of Theorem 2.3. \square

In Theorem 2.3, if we take $g = I$ (I is an identity mapping), then we have the following corollary.

Corollary 2.4 *In addition to the hypotheses of Corollary 2.1, suppose that $L = 0$, and for every $(x, y), (y^*, x^*) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F has a unique coupled fixed point; that is, there exists a unique $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.*

Theorem 2.4 *In addition to the hypotheses of Theorem 2.1, if gx_0 and gy_0 are comparable and $L = 0$, then F and g have a coupled coincidence point (x, y) such that $gx = F(x, y) = F(y, x) = gy$.*

Proof By Theorem 2.1 we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_n \rightarrow gx$ and $gy_n \rightarrow gy$, where (x, y) is a coincidence point of F and g . Suppose $gx_0 \preceq gy_0$, then it is an easy matter to show that $gx_n \preceq gy_n$ for all $n \in \mathbb{N}$. Thus, by (2.1) we have

$$\begin{aligned} &G(gx_n, gy_n, gy_n) \\ &= G(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) \\ &\leq \alpha \min \{ G(F(x_{n-1}, y_{n-1}), gx_{n-1}, gx_{n-1}), G(F(y_{n-1}, x_{n-1}), gx_{n-1}, gx_{n-1}), \\ &\quad G(F(y_{n-1}, x_{n-1}), gx_{n-1}, gx_{n-1}) \} \\ &\quad + \beta \min \{ G(F(x_{n-1}, y_{n-1}), gy_{n-1}, gy_{n-1}), G(F(y_{n-1}, x_{n-1}), gy_{n-1}, gy_{n-1}), \\ &\quad G(F(y_{n-1}, x_{n-1}), gy_{n-1}, gy_{n-1}) \} \\ &\quad + \gamma \min \{ G(F(x_{n-1}, y_{n-1}), gy_{n-1}, gy_{n-1}), G(F(y_{n-1}, x_{n-1}), gy_{n-1}, gy_{n-1}), \end{aligned}$$

$$\begin{aligned} & G(F(y_{n-1}, x_{n-1}), gy_{n-1}, gy_{n-1}) \} \\ = & \alpha \min \{ G(gx_n, gx_{n-1}, gx_{n-1}), G(gy_n, gx_{n-1}, gx_{n-1}) \} \\ & + \beta \min \{ G(gx_n, gy_{n-1}, gy_{n-1}), G(gy_n, gy_{n-1}, gy_{n-1}) \} \\ & + \gamma \min \{ G(gx_n, gy_{n-1}, gy_{n-1}), G(gy_n, gy_{n-1}, gy_{n-1}) \}. \end{aligned}$$

By taking the limit as $n \rightarrow \infty$, we get $G(gx, gy, gy) = 0$. Hence $F(x, y) = gx = gy = F(y, x)$.

A similar argument can be used if $gy_0 \leq gx_0$. This completes the proof of Theorem 2.4. \square

In Theorem 2.4, if we take $g = I$ (I is an identity mapping), then we have the following corollary.

Corollary 2.5 *In addition to the hypotheses of Theorem 2.1, if x_0 and y_0 are comparable and $L = 0$, then F has a coupled coincidence point of the form (x, x) .*

Remark 2.3 Theorems 2.3 and 2.4 generalize and extend the corresponding results in [29, Theorems 2.3 and 2.4] from partially ordered metric spaces to partially ordered G -metric spaces.

Remark 2.4 Corollaries 2.4 and 2.5 generalize and extend the corresponding results in [29, Corollaries 2.4 and 2.5] from partially ordered metric spaces to partially ordered G -metric spaces.

Now, we introduce an example to support Theorem 2.1.

Example 2.1 Let $X = [0, 1]$, then (X, \leq) is a partially ordered set with a natural ordering of real numbers. Let $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Set F and g as

$$\begin{aligned} g : X &\rightarrow X, \quad g(x) = x^2 \quad \text{and} \\ F : X \times X &\rightarrow X, \quad F(x, y) = \begin{cases} \frac{x^2 - y^2}{16}, & x \geq y, \\ 0, & x < y. \end{cases} \end{aligned}$$

Then

- (1) (X, G) is a complete G -metric space;
- (2) $g(X) = X$ is complete;
- (3) $F(X \times X) \subset g(X) = X$;
- (4) X satisfies (i) and (ii) of Theorem 2.1;
- (5) F has the mixed g -monotone property;
- (6) For any $L \in [0, \infty)$, F and g satisfy

$$\begin{aligned} & G(F(x, y), F(u, v), F(w, z)) \\ & \leq \frac{1}{8} \min \{ G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx) \} \\ & \quad + \frac{1}{8} \min \{ G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu) \} \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{8} \min \{ G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw) \} \\
 &+ L \min \{ G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx) \} \quad (2.28)
 \end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $gx \leq gu \leq gw$ and $gy \geq gv \geq gz$. Thus by Theorem 2.1, F and g have a coupled coincidence point. Moreover, $(0, 0)$ is a coupled fixed point of F .

Proof The proofs of (1)-(5) are clear. The proof of (6) is divided into the following cases.

Case 1. If $x \geq y$. In this case, we have $z \leq v \leq y \leq x \leq u \leq w$, and so

$$F(x, y) = \frac{x^2 - y^2}{16}, \quad F(u, v) = \frac{u^2 - v^2}{16}, \quad F(w, z) = \frac{w^2 - z^2}{16}.$$

Hence, we get

$$\begin{aligned}
 &G(F(x, y), F(u, v), F(w, z)) \\
 &= G\left(\frac{x^2 - y^2}{16}, \frac{u^2 - v^2}{16}, \frac{w^2 - z^2}{16}\right) \\
 &= \frac{w^2 - z^2 - (x^2 - y^2)}{8} \leq \frac{w^2}{8} \leq \frac{15w^2}{64} \leq \frac{15w^2 + w^2 - x^2 + y^2}{64} \\
 &= \frac{1}{8} \cdot \frac{16w^2 - x^2 + y^2}{8} = \frac{1}{8} G\left(\frac{x^2 - y^2}{16}, w^2, w^2\right) \\
 &= \frac{1}{8} \min \left\{ G\left(\frac{x^2 - y^2}{16}, w^2, w^2\right), G\left(\frac{u^2 - v^2}{16}, w^2, w^2\right), G\left(\frac{w^2 - z^2}{16}, w^2, w^2\right) \right\} \\
 &= \frac{1}{8} \min \{ G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw) \} \\
 &\leq \frac{1}{8} \min \{ G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx) \} \\
 &\quad + \frac{1}{8} \min \{ G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu) \} \\
 &\quad + \frac{1}{8} \min \{ G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw) \} \\
 &\quad + L \min \{ G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx) \}.
 \end{aligned}$$

Case 2. $x < y$. We divide the study in two sub-cases:

(a) If $u \geq v$, then $z \leq v \leq u \leq w$, so $w \geq z$. Therefore, we get

$$F(x, y) = 0, \quad F(u, v) = \frac{u^2 - v^2}{16}, \quad F(w, z) = \frac{w^2 - z^2}{16}.$$

Hence, we have

$$\begin{aligned}
 &G(F(x, y), F(u, v), F(w, z)) \\
 &= G\left(0, \frac{u^2 - v^2}{16}, \frac{w^2 - z^2}{16}\right) \\
 &= \frac{w^2 - z^2}{8} \leq \frac{w^2}{8} \leq \frac{15w^2}{64} \leq \frac{15w^2 + w^2 - u^2 + v^2}{64}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{8} \cdot \frac{16w^2 - u^2 + v^2}{8} = \frac{1}{8} G\left(\frac{u^2 - v^2}{16}, w^2, w^2\right) \\
 &= \frac{1}{8} \min\left\{G(0, w^2, w^2), G\left(\frac{u^2 - v^2}{16}, w^2, w^2\right), G\left(\frac{w^2 - z^2}{16}, w^2, w^2\right)\right\} \\
 &= \frac{1}{8} \min\{G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw)\} \\
 &\leq \frac{1}{8} \min\{G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx)\} \\
 &\quad + \frac{1}{8} \min\{G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu)\} \\
 &\quad + \frac{1}{8} \min\{G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw)\} \\
 &\quad + L \min\{G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx)\}.
 \end{aligned}$$

(b) If $u < v$, hence $F(x, y) = 0 = F(u, v)$; the case where $w < z$ is obvious because we get $F(w, z) = 0$. If $w \geq z$, we have $F(w, z) = \frac{w^2 - z^2}{16}$. Therefore

$$\begin{aligned}
 &G(F(x, y), F(u, v), F(w, z)) \\
 &= G\left(0, 0, \frac{w^2 - z^2}{16}\right) \\
 &= \frac{w^2 - z^2}{8} \leq \frac{w^2}{8} \leq \frac{15w^2}{64} \leq \frac{15w^2 + z^2}{64} \\
 &= \frac{1}{8} \cdot \frac{15w^2 + z^2}{8} = \frac{1}{8} G\left(\frac{w^2 - z^2}{16}, w^2, w^2\right) \\
 &= \frac{1}{8} \min\left\{G(0, w^2, w^2), G(0, w^2, w^2), G\left(\frac{w^2 - z^2}{16}, w^2, w^2\right)\right\} \\
 &= \frac{1}{8} \min\{G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw)\} \\
 &\leq \frac{1}{8} \min\{G(F(x, y), gx, gx), G(F(u, v), gx, gx), G(F(w, z), gx, gx)\} \\
 &\quad + \frac{1}{8} \min\{G(F(x, y), gu, gu), G(F(u, v), gu, gu), G(F(w, z), gu, gu)\} \\
 &\quad + \frac{1}{8} \min\{G(F(x, y), gw, gw), G(F(u, v), gw, gw), G(F(w, z), gw, gw)\} \\
 &\quad + L \min\{G(F(x, y), gu, gu), G(F(u, v), gw, gw), G(F(w, z), gx, gx)\}.
 \end{aligned}$$

In all the above cases, inequality (2.1) of Theorem 2.1 is satisfied for $\alpha = \beta = \gamma = \frac{1}{8}$ and any $L \geq 0$. All the required hypotheses of Theorem 2.1 are satisfied. Clearly, F and g have a coupled coincidence point. Moreover, $(0, 0)$ is a coupled fixed point of F . □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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