Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Convergence of an extragradient-like iterative algorithm for monotone mappings and nonexpansive mappings

Yuan Qing[1] and Meijuan Shang[2,3*]

*Correspondence: meijuanshang@yahoo.com.cn
[2]Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing, 100044, China
[3]Department of Mathematics, Shijiazhuang University, Shijiazhuang, 050035, China
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate the problem of finding some common element in the set of common fixed points of an infinite family of nonexpansive mappings and in the set of solutions of variational inequalities based on an extragradient-like iterative algorithm. Strong convergence of the purposed iterative algorithm is obtained.

MSC: 47H05; 47H09; 47J25; 90C33

Keywords: inverse-strongly monotone mapping; fixed point; nonexpansive mapping; projection; strong convergence

1 Introduction

Iterative algorithms have been playing an important role in the approximation solvability, especially of nonlinear variational inequalities as well as of nonlinear equations in several fields such as mechanics, traffic, economics, information, medicine, and many others. The well-known convex feasibility problem which captures applications in various disciplines such as image restoration and radiation therapy treatment planning is to find a point in the intersection of common fixed point sets of a family of nonlinear mappings; see, for example, [1–11]. The Mann iterative algorithm is an efficient method to study the class of nonexpansive mappings. Indeed, Picard cannot converge even that the fixed point set of nonexpansive mappings is nonempty.

It is known that Mann iterative algorithm only has weak convergence for nonexpansive mappings in infinite-dimensional Hilbert spaces; see [12] for more details and the references therein. In many disciplines, including economics [13], image recovery [14], quantum physics [15–20], and control theory [21], problems arise in infinite dimension spaces. To improve the weak convergence of the Mann iterative algorithm, many authors considered using contractions to approximate nonexpansive mappings; for more details, see [22] and [23] and the references therein.

In this paper, we focus on the problem of finding some common element in the set of common fixed points of an infinite family of nonexpansive mappings and in the set of solutions of variational inequalities based on an extragradient-like iterative algorithm. Some deduced sub-results and applications are obtained.

2 Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty, closed, and convex subset of H. Let P_K be the metric projection from H onto K.

Recall that a mapping $B: K \to H$ is said to be inverse-strongly monotone iff there exists a positive real number μ such that

$$\langle Bx - By, x - y\rangle \geq \mu\|Bx - By\|^2, \quad \forall x, y \in K.$$

For such a case, B is also said to be μ-inverse-strongly monotone.

Recall that a mapping $T: K \to K$ is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K.$$

In this paper, we use $F(T)$ to denote the fixed point set of the mapping T.

Recall that a mapping $f: K \to K$ is said to be a contraction iff there exists a coefficient $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in K.$$

For such a case, f is also said to be an α-contraction.

Recall that a linear bounded operator $A: K \to K$ is strongly positive iff there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in K.$$

Recall that a set-valued mapping $S: H \to 2^H$ is said to be monotone iff $f \in Sx$ and $g \in Sy$ imply

$$\langle x - y, f - g\rangle \geq 0, \quad \forall x, y \in H.$$

A monotone mapping $S: H \to 2^H$ is maximal iff the graph of $G(S)$ of S is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping S is maximal iff for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(S)$ implies $f \in Sx$. Let $Q: C \to H$ be a monotone mapping and $N_K v$ be the normal cone to K at $v \in K$, *i.e.*, $N_K v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in K\}$, and define

$$Sv = \begin{cases} Qv + N_K v, & v \in K, \\ \emptyset, & v \notin K. \end{cases}$$

Then S is maximal monotone and $0 \in Sv$ iff $v \in VI(K, A)$; see [24] for more details.

Recall that the classical variational inequality is to find a $u \in K$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.1}$$

where $B: K \to H$ is a monotone mapping. It is known that $u \in K$ is a solution to (2.1) iff u is a fixed point of the mapping $P_K(I - \lambda B)$, where $\lambda > 0$ is a constant and I stands for the identity mapping. In this paper, we use $VI(K,B)$ to denote the solution set of the variational inequality (2.1).

Iterative algorithms for nonexpansive mappings have recently been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping T on a real Hilbert space H,

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax,x\rangle - \langle x,u\rangle, \tag{2.2}$$

where A is a linear bounded self-adjoint operator on H and u is a given point in H. In [25], it is proved that the sequence $\{x_n\}$ defined by the iterative algorithm

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0,$$

strongly converges to the unique solution of the minimization problem (2.2) provided that the sequence $\{\alpha_n\}$ satisfies certain restriction.

Recently, Marino and Xu [26] reconsidered the problem by viscosity approximation method. They investigated the following iterative algorithm:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0,$$

where A is a linear bounded self-adjoint operator on H, $T : H \to H$ is a nonexpansive mapping, and $f : H \to H$ is a contraction. They proved that the sequence $\{x_n\}$ generated in the above iterative process converges strongly to the unique solution of the following variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in K,$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax,x\rangle - h(x),$$

where h is a potential function for γf, that is, $h'(x) = \gamma f(x)$ for $x \in H$.

Recently, the problem of finding a common element in the fixed point set of a nonexpansive mapping and in the solution set of a variational inequality has been considered by many authors; see, for example, [27–40] and the references therein. In 2003, Takahashi and Toyoda [35] considered the following iterative algorithm:

$$x_1 \in K, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_K(x_n - \lambda_n Bx_n), \quad n \geq 1, \tag{2.3}$$

where $T : K \to K$ is a nonexpansive mapping, $B : K \to H$ is a μ-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\mu)$. They showed that the sequence $\{x_n\}$ generated in (2.3) weakly converges to some point $z \in F(T) \cap VI(K,B)$.

Iiduka and Takahashi [36] reconsidered the common element problem via the following iterative algorithm:

$$x_1 = x \in K, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)TP_K(x_n - \lambda_n Bx_n), \quad n \geq 1, \tag{2.4}$$

where $T : K \to K$ is a nonexpansive mapping, $B : K \to H$ is a μ-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\mu)$. They proved that the sequence $\{x_n\}$ strongly converges to some point $z \in F(T) \cap VI(K,B)$.

In this paper, we will consider an infinite family of nonexpansive mappings. More precisely, we consider the mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{2.5}$$

where $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \le \gamma_n \le 1$, $T_1, T_2, \ldots$ is an infinite family of mappings of K into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Regarding W_n, we have the following lemmas which are important to prove our main results.

Lemma 2.1 [41] *Let K be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of K into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then, for every $x \in K$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define the mapping W as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in K. \tag{2.6}$$

Such a mapping W is called W-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.

Throughout this paper, we will assume that $0 < \gamma_n \le b < 1$ for each $n \ge 1$.

Lemma 2.2 [41] *Let K be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of K into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for each $n \ge 1$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

In this paper, motivated by the above results, we investigate the problem of approximating a common element in the solution set of variational inequalities and in the common fixed point set of a family of nonexpansive mappings based on an extragradient-like iterative algorithm. Strong convergence theorems of common elements are established in the framework of Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 2.3 *In a real Hilbert space H, the following inequality holds*:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.4 [26] *Assume A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with the coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.5 [26] *Let H be a Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let T be a nonexpansive mapping with a fixed point $x_t \in H$ of the contraction $x \mapsto t\gamma f(x) + (I - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T, which solves the variational inequality*

$$\big\langle (A - \gamma f)\bar{x}, z - \bar{x}\big\rangle \leq 0, \quad \forall z \in F(T).$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\bar{x} = \bar{x}$.

Lemma 2.6 [42] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.7 [39] *Let K be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\{\gamma_n\}$ be a real sequence such that $0 < \gamma_n \leq b < 1$ for each $n \geq 1$. If C is any bounded subset of K, then* $\lim_{n\to\infty} \sup_{x\in C} \|Wx - W_n x\| = 0$.

3 Main results

Theorem 3.1 *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Let $B_i : K \to H$ be μ_i-inverse-strongly monotone mappings for each $i = 1, 2$, and $f : K \to K$ be an α-contraction. Let $A : K \to K$ be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Let $\{x_n\}$ be a sequence generated in the following extragradient-like iterative algorithm*:

$$\begin{cases} x_1 \in K, \\ y_n = P_K(x_n - \eta_n B_2 x_n), \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + (I - \alpha_n A) W_n P_K(I - \lambda_n B_1) y_n), \quad n \geq 1, \end{cases} \tag{3.1}$$

where P_K is the metric projection from H onto K, W_n is a mapping defined by (2.5), $\{\alpha_n\}$ is a real number sequence in $(0,1)$, and $\{\lambda_n\}$, $\{\eta_n\}$ are two positive real number sequences. Assume that $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(K, B_1) \cap VI(K, B_2) \neq \emptyset$, $0 < \gamma < \bar{\gamma}/\alpha$ and the following restrictions are satisfied:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, *and* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| \le \infty$;
(b) $\sum_{n=1}^{\infty}|\eta_{n+1}-\eta_n| < \infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$;
(c) $\{\eta_n\}, \{\lambda_n\} \subset [u, v]$, *where* $0 < u < v < 2\min\{\mu_1, \mu_2\}$.

Then the sequence $\{x_n\}$ *strongly converges to* $x^* \in F$, *where* $x^* = P_F(\gamma f + (I - A))x^*$.

Proof First, we show that $I - \lambda_n B_1$ and $I - \eta_n B_2$ are nonexpansive. Indeed, we see from the restriction (c) that

$$\begin{aligned}\big\|(I-\lambda_nB_1)x-(I-\lambda_nB_1)y\big\|^2 &= \big\|x-y-\lambda_n(B_1x-B_1y)\big\|^2\\ &= \|x-y\|^2 - 2\lambda_n\langle x-y, B_1x-B_1y\rangle + \lambda_n^2\|B_1x-B_1y\|^2\\ &\le \|x-y\|^2 + \lambda_n(\lambda_n-2\mu_1)\|B_1x-B_1y\|^2\\ &\le \|x-y\|^2, \quad \forall x,y\in C.\end{aligned}$$

This shows that $I - \lambda_n B_1$ is nonexpansive, so is $I - \eta_n B_2$. Noticing the condition (a), we may assume, with no loss of generality, that $\alpha_n \le \|A\|^{-1}$ for each $n \ge 1$. It follows from Lemma 2.4 that $\|I - \alpha_n A\| \le 1 - \alpha_n\bar{\gamma}$.

Next, we show that the sequence $\{x_n\}$ is bounded. Letting $p \in F$, we see that

$$\|y_n - p\| = \big\|P_K(I-\eta_nB_2)x_n - P_K(I-\eta_nB_2)p\big\| \le \|x_n - p\|.$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &\le \big\|\alpha_n\big(\gamma f(x_n)-Ap\big) + (I-\alpha_nA)\big(W_nP_C(I-\lambda_nB)y_n - p\big)\big\|\\ &\le \alpha_n\big\|\gamma f(x_n)-Ap\big\| + (1-\alpha_n\bar{\gamma})\big\|W_nP_C(I-\lambda_nB)y_n-p\big\|\\ &\le \alpha_n\gamma\big\|f(x_n)-f(p)\big\| + \alpha_n\big\|\gamma f(p)-Ap\big\| + (1-\alpha_n\bar{\gamma})\|y_n-p\|\\ &= \big(1-\alpha_n(\bar{\gamma}-\gamma\alpha)\big)\|x_n-p\| + \alpha_n\big\|\gamma f(p)-Ap\big\|.\end{aligned}$$

By simple induction, we have

$$\|x_n - p\| \le \max\left\{\|x_1-p\|, \frac{\|Ap-\gamma f(p)\|}{\bar{\gamma}-\gamma\alpha}\right\},$$

which yields that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Notice that

$$\begin{aligned}&\|y_{n+1}-y_n\|\\ &\quad= \big\|P_K(I-\eta_{n+1}B_2)x_{n+1} - P_K(I-\eta_nB_2)x_n\big\|\\ &\quad\le \big\|(I-\eta_{n+1}B_2)x_{n+1} - (I-\eta_{n+1}B_2)x_n + (I-\eta_{n+1}B_2)x_n - (I-\eta_nB_2)x_n\big\|\\ &\quad\le \|x_{n+1}-x_n\| + |\eta_{n+1}-\eta_n|\|B_2x_n\|. \end{aligned} \tag{3.2}$$

Putting $\rho_n = P_K(I - \lambda_n B_1)y_n$, we have

$$\begin{aligned}&\|\rho_{n+1}-\rho_n\|\\ &\quad= \big\|P_K(I-\lambda_{n+1}B_1)y_{n+1} - P_K(I-\lambda_nB_1)y_n\big\|\end{aligned}$$

$$\begin{aligned}&\le \big\|(I-\lambda_{n+1}B_1)y_{n+1}-(I-\lambda_{n+1}B_1)y_n+(I-\lambda_{n+1}B_1)y_n-(I-\lambda_nB_1)y_n\big\|\\&\le \|y_{n+1}-y_n\|+|\lambda_{n+1}-\lambda_n|\|B_1y_n\|.\end{aligned}\tag{3.3}$$

Substituting (3.2) into (3.3), we arrive at

$$\|\rho_{n+1}-\rho_n\|\le\|x_{n+1}-x_n\|+M_1\big(|\eta_{n+1}-\eta_n|+|\lambda_{n+1}-\lambda_n|\big),\tag{3.4}$$

where M_1 is an appropriate constant such that

$$M_1\ge\max\Big\{\sup_{n\ge1}\|B_1y_n\|,\sup_{n\ge1}\|B_2x_n\|\Big\}.$$

Notice that

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\|\\&\quad\le\big\|(I-\alpha_{n+1}A)(W_{n+1}\rho_{n+1}-W_n\rho_n)-(\alpha_{n+1}-\alpha_n)AW_n\rho_n\\&\qquad+\gamma\big(\alpha_{n+1}\big(f(x_{n+1})-f(x_n)\big)+f(x_n)(\alpha_{n+1}-\alpha_n)\big)\big\|\\&\quad\le(1-\alpha_{n+1}\bar\gamma)\big(\|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|\big)+|\alpha_{n+1}-\alpha_n|\|AW_n\rho_n\|\\&\qquad+\gamma\big(\alpha_{n+1}\alpha\|x_{n+1}-x_n\|+\big\|f(x_n)\big\||\alpha_{n+1}-\alpha_n|\big).\end{aligned}\tag{3.5}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have from (2.6) that

$$\begin{aligned}\|W_{n+1}\rho_n-W_n\rho_n\|&=\|\gamma_1T_1U_{n+1,2}\rho_n-\gamma_1T_1U_{n,2}\rho_n\|\\&\le\gamma_1\|U_{n+1,2}\rho_n-U_{n,2}\rho_n\|\\&=\gamma_1\|\gamma_2T_2U_{u+1,3}\rho_n-\gamma_2T_2U_{n,3}\rho_n\|\\&\le\gamma_1\gamma_2\|U_{u+1,3}\rho_n-U_{n,3}\rho_n\|\\&\le\cdots\\&\le\gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\\&\le M_2\prod_{i=1}^n\gamma_i,\end{aligned}\tag{3.6}$$

where $M_2\ge0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\le M_2$ for each $n\ge1$. Substituting (3.4) and (3.6) into (3.5), we arrive at

$$\begin{aligned}\|x_{n+2}-x_{n+1}\|\le{}&\big(1-\alpha_{n+1}(\bar\gamma-\alpha\gamma)\big)\|x_{n+1}-x_n\|\\&+M_3\Bigg(\prod_{i=1}^n\gamma_i+|\alpha_{n+1}-\alpha_n|+|\lambda_{n+1}-\lambda_n|+|\eta_{n+1}-\eta_n|\Bigg),\end{aligned}\tag{3.7}$$

where M_3 is an appropriate constant such that

$$M_3=\max\Big\{M_1,M_2,\gamma\sup_{n\ge1}\big\{\big\|f(x_n)\big\|+\|AW_n\rho_n\|\big\}\Big\}.$$

From the restrictions (a) and (b), we obtain from Lemma 2.6 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Notice that

$$\begin{aligned}\|x_{n+1} - W_n\rho_n\| &= \left\|P_K\big(\alpha_n\gamma f(x_n) + (I - \alpha_n A)W_n P_K(I - \lambda_n B_1)y_n\big) - P_K(W_n\rho_n)\right\| \\ &\le \alpha_n\left\|\gamma f(x_n) - AW_n\rho_n\right\|.\end{aligned}$$

It follows from the restriction (a) that

$$\lim_{n\to\infty} \|W_n\rho_n - x_{n+1}\| = 0. \tag{3.9}$$

Notice that

$$\begin{aligned}\|y_n - p\|^2 &\le \left\|(x_n - p) - \eta_n(B_2x_n - B_2p)\right\|^2 \\ &\le \|x_n - p\|^2 - 2\eta_n\mu_2\|B_2x_n - B_2p\|^2 + \eta_n^2\|B_2x_n - B_2p\|^2 \\ &= \|x_n - p\|^2 + \big(\eta_n^2 - 2\eta_n\mu_2\big)\|B_2x_n - B_2p\|^2.\end{aligned} \tag{3.10}$$

In a similar way, we find that

$$\|\rho_n - p\|^2 \le \|x_n - p\|^2 + \big(\lambda_n^2 - 2\lambda\mu_1\big)\|B_1y_n - B_1p\|^2. \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \left\|\alpha_n\big(\gamma f(x_n) - Ap\big) + (I - \alpha_n A)(W_n\rho_n - p)\right\|^2 \\ &\le \big(\alpha_n\left\|\gamma f(x_n) - Ap\right\| + (1 - \alpha_n\bar{\gamma})\|\rho_n - p\|\big)^2 \\ &\le \alpha_n\left\|\gamma f(x_n) - Ap\right\|^2 + \|\rho_n - p\|^2 + 2\alpha_n\left\|\gamma f(x_n) - Ap\right\|\|\rho_n - p\|.\end{aligned} \tag{3.12}$$

Substituting (3.11) into (3.12) gives

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\left\|\gamma f(x_n) - Ap\right\|^2 + \|x_n - p\|^2 + \big(\lambda_n^2 - 2\lambda_n\mu_1\big)\|B_1y_n - B_1p\|^2 \\ &\quad + 2\alpha_n\left\|\gamma f(x_n) - Ap\right\|\|\rho_n - p\|.\end{aligned}$$

It follows from the restriction (c) that

$$\begin{aligned}&u(2\mu_1 - \nu)\|B_1y_n - B_1p\|^2 \\ &\quad\le \alpha_n\left\|\gamma f(x_n) - Ap\right\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\left\|\gamma f(x_n) - Ap\right\|\|\rho_n - p\| \\ &\quad\le \alpha_n\left\|\gamma f(x_n) - Bp\right\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| \\ &\qquad + 2\alpha_n\left\|\gamma f(x_n) - Ap\right\|\|\rho_n - p\|.\end{aligned}$$

In view of the restriction (a), we obtain from (3.8) that

$$\lim_{n\to\infty} \|B_1y_n - B_1p\| = 0. \tag{3.13}$$

From (3.12), we also have

$$\begin{aligned}\|x_{n+1}-p\|^2 \le{}& \alpha_n\|\gamma f(x_n)-Ap\|^2+\|y_n-p\|^2\\&+2\alpha_n\|\gamma f(x_n)-Ap\|\|\rho_n-p\|.\end{aligned}\tag{3.14}$$

Combining (3.10) with (3.14), we arrive at

$$\begin{aligned}\|x_{n+1}-p\|^2 \le{}& \alpha_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2+(\eta_n^2-2\eta_n\mu_2)\|B_2x_n-B_2p\|^2\\&+2\alpha_n\|\gamma f(x_n)-Ap\|\|\rho_n-p\|,\end{aligned}$$

which implies from the restriction (c) that

$$\begin{aligned}&u(2\mu_2-\nu)\|B_2x_n-B_2p\|^2\\&\quad\le \alpha_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\|\rho_n-p\|\\&\quad\le \alpha_n\|\gamma f(x_n)-Ap\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\|\rho_n-p\|.\end{aligned}$$

In view of the restriction (a), we obtain from (3.8) that

$$\lim_{n\to\infty}\|B_2x_n-B_2p\|=0.\tag{3.15}$$

On the other hand, we see from the firm expansivity of P_K that

$$\begin{aligned}\|y_n-p\|^2 ={}& \|P_K(I-\eta_nB_2)x_n-P_K(I-\eta_nB_2)p\|^2\\ \le{}& \langle (I-\eta_nB_2)x_n-(I-\eta_nB_2)p,y_n-p\rangle\\ ={}& \frac{1}{2}\big(\|(I-\eta_nB_2)x_n-(I-\eta_nB_2)p\|^2+\|y_n-p\|^2\\&-\|(I-\eta_nB_2)x_n-(I-\eta_nB_2)p-(y_n-p)\|^2\big)\\ \le{}& \frac{1}{2}\big(\|x_n-p\|^2+\|y_n-p\|^2-\|(x_n-y_n)-\eta_n(B_2x_n-B_2p)\|^2\big)\\ ={}& \frac{1}{2}\big(\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n\|^2-\eta_n^2\|B_2x_n-B_2p\|^2\\&+2\eta_n\langle x_n-y_n,B_2x_n-B_2p\rangle\big),\end{aligned}$$

which yields

$$\|y_n-p\|^2\le\|x_n-p\|^2-\|x_n-y_n\|^2+2\eta_n\|x_n-y_n\|\|B_2x_n-B_2p\|.\tag{3.16}$$

In the same way, we can obtain that

$$\|\rho_n-p\|^2\le\|x_n-p\|^2-\|\rho_n-y_n\|^2+2\lambda_n\|\rho_n-y_n\|\|B_1y_n-B_1p\|.\tag{3.17}$$

Substituting (3.16) into (3.14) yields

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n \big\|\gamma f(x_n)-Ap\big\|^2+\|x_n-p\|^2-\|x_n-y_n\|^2\\&\quad+2\eta_n\|x_n-y_n\|\|B_2x_n-B_2p\|+2\alpha_n\big\|\gamma f(x_n)-Ap\big\|\|\rho_n-p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n-y_n\|^2 &\le \alpha_n \big\|\gamma f(x_n)-Ap\big\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2\eta_n\|x_n-y_n\|\|B_2x_n-B_2p\|+2\alpha_n\big\|\gamma f(x_n)-Ap\big\|\|\rho_n-p\|\\&\le \alpha_n \big\|\gamma f(x_n)-Ap\big\|^2+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_{n+1}-x_n\|\\&\quad+2\eta_n\|x_n-y_n\|\|B_2x_n-B_2p\|+2\alpha_n\big\|\gamma f(x_n)-Ap\big\|\|\rho_n-p\|.\end{aligned}$$

In view of (3.7) and (3.15), we see from the restriction (a) that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.18}$$

Similarly, we can obtain that

$$\lim_{n\to\infty}\|\rho_n-y_n\|=0. \tag{3.19}$$

Observe that

$$\|\rho_n-W_n\rho_n\|\le\|y_n-\rho_n\|+\|x_n-y_n\|+\|x_n-x_{n+1}\|+\|x_{n+1}-W_n\rho_n\|.$$

It follows from (3.7), (3.9), (3.18) and (3.19) that

$$\lim_{n\to\infty}\|W_n\rho_n-\rho_n\|=0. \tag{3.20}$$

From Lemma 2.7, we have $\|W\rho_n-W_n\rho_n\|\to 0$ as $n\to\infty$. This in turn implies that

$$\lim_{n\to\infty}\|W\rho_n-\rho_n\|=0. \tag{3.21}$$

Next, we show that

$$\limsup_{n\to\infty}\big\langle\gamma f\big(x^*\big)-Ax^*,x_n-x^*\big\rangle\le 0. \tag{3.22}$$

To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle\gamma f\big(x^*\big)-Ax^*,x_n-x^*\big\rangle=\lim_{i\to\infty}\big\langle\gamma f\big(x^*\big)-Ax^*,x_{n_i}-x^*\big\rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume, without loss of generality, that $x_{n_i}\rightharpoonup p$. From (3.18) and (3.19), we also have $y_{n_i} \rightharpoonup p$ and $z_{n_i} \rightharpoonup p$, respectively. Notice that $p \in F$. Indeed, let us first show that $p \in VI(K, B_1)$. Put

$$Sw = \begin{cases} B_1 v + N_K v, & v \in K, \\ \emptyset, & v \notin K. \end{cases}$$

Then S is maximal monotone. Let $(v, w) \in G(S)$. Since $w - B_1 v \in N_K v$ and $\rho_n \in K$, we have

$$\langle v - \rho_n, w - B_1 v \rangle \geq 0.$$

On the other hand, we have from $\rho_n = P_K(I - \lambda_n B_1)y_n$ that

$$\langle v - \rho_n, \rho_n - (I - \lambda_n B_1)y_n \rangle \geq 0$$

and hence

$$\left\langle v - \rho_n, \frac{\rho_n - y_n}{\lambda_n} + B_1 y_n \right\rangle \geq 0.$$

It follows that

$$\begin{aligned} \langle v - \rho_{n_i}, w \rangle &\geq \langle v - \rho_{n_i}, B_1 v \rangle \\ &\geq \langle v - \rho_{n_i}, B_1 v \rangle - \left\langle v - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} + B_1 y_{n_i} \right\rangle \\ &\geq \left\langle v - \rho_{n_i}, B_1 v - \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} - B_1 y_{n_i} \right\rangle \\ &= \langle v - \rho_{n_i}, B_1 v - B_1 \rho_{n_i} \rangle + \langle v - \rho_{n_i}, B_1 \rho_{n_i} - B_1 y_{n_i} \rangle - \left\langle v - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - \rho_{n_i}, B_1 \rho_{n_i} - B_1 y_{n_i} \rangle - \left\langle v - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} \right\rangle, \end{aligned}$$

which implies that $\langle v - p, w \rangle \geq 0$. We have $p \in B_1^{-1}0$ and hence $p \in VI(K, B_1)$. In a similar way, we can show $p \in VI(K, B_2)$. Next, let us show $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Since Hilbert spaces satisfy Opial's condition, we see from (3.21) that

$$\begin{aligned} \liminf_{i \to \infty} \|\rho_{n_i} - p\| &< \liminf_{i \to \infty} \|\rho_{n_i} - Wp\| \\ &= \liminf_{i \to \infty} \|\rho_{n_i} - W\rho_{n_i} + W\rho_{n_i} - Wp\| \\ &\leq \liminf_{i \to \infty} \|W\rho_{n_i} - Wp\| \\ &\leq \liminf_{i \to \infty} \|\rho_{n_i} - p\|, \end{aligned}$$

which derives a contradiction. Thus, we have $p \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. From Lemma 2.5, we see that there exists a unique x^* such that $x^* = P_F(\gamma f + (I - A))x^*$. It follows that

$$\limsup_{n \to \infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle = \langle \gamma f(x^*) - Ax^*, p - x^* \rangle \leq 0.$$

That is, (3.22) holds. It follows from Lemma 2.3 that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|\alpha_n(\gamma f(x_n)-Ax^*)+(I-\alpha_nA)(W_n\rho_n-x^*)\|^2\\ &\le (1-\alpha_n\bar{\gamma})^2\|W_n\rho_n-x^*\|^2+2\alpha_n\langle\gamma f(x_n)-Ax^*,x_{n+1}-x^*\rangle\\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2+\alpha\gamma\alpha_n(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\ &\quad+2\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\ &= \frac{(1-2\alpha_n\bar{\gamma}+\alpha_n\alpha\gamma)}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2\\ &\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\ &\le \left[1-\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2\\ &\quad+\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\left(\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_4\right),\end{aligned}$$

where M_4 is an appropriate constant such that $M_4 \ge \sup_{n\ge1}\|x_n-x^*\|^2$. Put $b_n=\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\alpha\gamma}$ and $c_n=\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_4$. That is,

$$\|x_{n+1}-x^*\|^2 \le (1-b_n)\|x_n-x^*\|+b_nc_n. \tag{3.23}$$

In view of the restrictions (a) and (b), we see from (3.22) that

$$\lim_{n\to\infty} b_n=0,\qquad \sum_{n=1}^{\infty} b_n=\infty\quad\text{and}\quad \limsup_{n\to\infty} c_n\le 0.$$

Apply Lemma 2.6 to (3.23) to conclude that $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

If $B_2=0$, the zero mapping, then Theorem 3.1 is reduced to the following.

Corollary 3.2 *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Let $B_1: K\to H$ be μ_1-inverse-strongly monotone mappings and $f: K\to K$ be an α-contraction. Let $A: K\to K$ be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma}>0$. Assume that $0<\gamma<\bar{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*:

$$\begin{cases} x_1\in K,\\ x_{n+1}=P_K(\alpha_n\gamma f(x_n)+(I-\alpha_nA)W_nP_K(I-\lambda_nB_1)x_n), \quad n\ge1,\end{cases}$$

where P_K is the metric projection from H onto K, W_n is a mapping defined by (2.5), $\{\alpha_n\}$ *is a real number sequence in* $(0,1)$, *and* $\{\lambda_n\}$ *is a positive real number sequence. Assume that* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(K,B_1) \neq \emptyset$ *and the following restrictions are satisfied*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$;

(b) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(c) $\{\lambda_n\} \subset [u,v]$, *where* $0 < u < v < 2\mu_1$.

Then the sequence $\{x_n\}$ *strongly converges to* $x^* \in F$, *where* $x^* = P_F(\gamma f + (I - A))x^*$.

Remark 3.3 Corollary 3.2 includes the corresponding results in Iiduka and Takahashi [36] as a special case.

As an application of our main results, we consider another class of important nonlinear operators: strict pseudocontractions.

Recall that a mapping $S : K \to K$ is said to be a κ-strict pseudocontraction if there exists a constant $\kappa \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \left\| (I - S)x - (I - S)y \right\|^2, \quad \forall x, y \in K.$$

It is easy to see that the class of κ-strict pseudocontractions strictly includes the class of nonexpansive mappings as a special case.

Putting $B = I - S$, where $S : K \to K$ is a κ-strict pseudocontraction, we know that B is $\frac{1-\kappa}{2}$-inverse-strongly monotone; see [43] and the references therein.

Corollary 3.4 *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $S_i : K \to K$ be κ_i-inverse-strongly monotone mappings for each $i = 1, 2$ and $f : K \to K$ be an α-contraction. Let $A : K \to K$ be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in K, \\ y_n = (1 - \eta_n)x_n + \eta_n S_2 x_n, \\ x_{n+1} = P_K(\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n((1 - \lambda_n)y_n + \lambda_n S_1 y_n)), \quad n \geq 1, \end{cases}$$

where P_K is the metric projection from H onto K, W_n is a mapping defined by (2.5), $\{\alpha_n\}$ *is a real number sequence in* $(0,1)$, *and* $\{\lambda_n\}$, $\{\eta_n\}$ *are two positive real number sequences. Assume that* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap F(S_1) \cap F(S_2) \neq \emptyset$ *and the following restrictions are satisfied*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$;

(b) $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(c) $\{\eta_n\}, \{\lambda_n\} \subset [u,v]$, *where* $0 < u < v < 2\min\{\mu_1, \mu_2\}$.

Then the sequence $\{x_n\}$ *strongly converges to* $x^* \in F$, *where* $x^* = P_F(\gamma f + (I - A))x^*$.

Proof Put $B_1 = I - S_1$ and $B_2 = I - S_2$. Then B_1 is $(1 - \kappa_1)/2$-inverse-strongly monotone and B_2 is $(1-\kappa_2)/2$-inverse-strongly monotone, respectively. We have $F(S_1) = VI(K,B_1)$, $F(S_2) = VI(K,B_2)$, $P_K(I - \lambda_n B_1)y_n = (1 - \lambda_n)y_n + \lambda_n T_1 y_n$ and $P_K(I - \eta_n B_2)x_n = (1 - \eta_n)x_n + \eta_n T_2 x_n$. The desired conclusion can be immediately obtained from Theorem 3.1. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors participated in the design of this work and performed equally. Both authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Hangzhou Normal University, Hangzhou, 310036, China. [2]Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing, 100044, China. [3]Department of Mathematics, Shijiazhuang University, Shijiazhuang, 050035, China.

Acknowledgements

This research was supported by the Natural Science Foundation of Hebei Province (A2010001943), the Science Foundation of Shijiazhuang Science and Technology Bureau (121130971) and the Science Foundation of Beijing Jiaotong University (2011YJS075).

Received: 31 December 2012 Accepted: 21 February 2013 Published: 25 March 2013

References

1. Bauschke, HH, Borwein, JM: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**, 367-426 (1996)
2. Qin, X, Cho, SY, Kang, SM: Common fixed points of a pair of non-expansive mappings with applications to convex feasibility problems. Glasg. Math. J. **52**, 241-252 (2010)
3. Kotzer, T, Cohen, N, Shamir, J: Image restoration by a novel method of parallel projection onto constraint sets. Optim. Lett. **20**, 1772-1774 (1995)
4. Hao, Y, Wang, X, Tong, A: Weak and strong convergence theorems for two finite families of asymptotically nonexpansive mappings in Banach spaces. Adv. Fixed Point Theory **2**, 417-432 (2012)
5. Cho, SY, Kang, SM: Some results on asymptotically hemi-pseudocontractive mappings in the intermediate sense. Fixed Point Theory Appl. **13**, 153-164 (2012)
6. Qin, X, Cho, SY, Kang, SM: On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings. Appl. Math. Comput. **215**, 3874-3883 (2010)
7. Manro, S, Kumar, S, Bhatia, SS: Common fixed point theorems in intuitionistic fuzzy metric spaces using occasionally weakly compatible maps. J. Math. Comput. Sci. **2**, 73-81 (2012)
8. Lv, S, Wu, C: Convergence of iterative algorithms for a generalized variational inequality and a nonexpansive mapping. Eng. Math. Lett. **1**, 44-57 (2012)
9. Qin, X, Su, Y: Strong convergence theorems for relatively nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 1958-1965 (2007)
10. Cho, SY, Kang, SM: Hybrid projection algorithms for treating common fixed points of a family of demicontinuous pseudocontractions. Appl. Math. Lett. **25**, 854-857 (2012)
11. Qin, X, Cho, YJ, Kang, SM, Zhou, H: Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings. Appl. Math. Lett. **22**, 1051-1055 (2009)
12. Genel, A, Lindenstruss, J: An example concerning fixed points. Isr. J. Math. **22**, 81-86 (1975)
13. Khan, MA, Yannelis, NC: Equilibrium Theory in Infinite Dimensional Spaces. Springer, New York (1991)
14. Combettes, PL: The Convex Feasibility Problem in Image Recovery. In: Hawkes, P (ed.) Advances in Imaging and Electron Physics, vol. 95, pp. 155-270. Academic Press, New York (1996)
15. Dautray, R, Lions, JL: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 1. Springer, New York (1988)
16. Dautray, R, Lions, JL: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 2. Springer, New York (1989)
17. Dautray, R, Lions, JL: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 3. Springer, New York (1990)
18. Dautray, R, Lions, JL: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 4. Springer, New York (1991)
19. Dautray, R, Lions, JL: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 5. Springer, New York (1992)
20. Dautray, R, Lions, JL: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 6. Springer, New York (1993)
21. Fattorini, HO: Infinite-Dimensional Optimization and Control Theory. Cambridge University Press, Cambridge (1999)
22. Reich, S: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287-292 (1980)
23. Qin, X, Su, Y: Approximation of a zero point of accretive operator in Banach spaces. J. Math. Anal. Appl. **329**, 415-424 (2007)
24. Rockafellar, RT: On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 75-88 (1970)
25. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**, 659-678 (2003)
26. Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **318**, 43-52 (2006)
27. Qin, X, Shang, M, Su, Y: Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems. Math. Comput. Model. **48**, 1033-1046 (2008)
28. Cho, SY, Kang, SM: Approximation of fixed points of pseudocontraction semigroups based on a viscosity iterative process. Appl. Math. Lett. **24**, 224-228 (2011)
29. Zegeye, H, Shahzad, N: Strong convergence theorem for a common point of solution of variational inequality and fixed point problem. Adv. Fixed Point Theory **4**, 374-397 (2012)

30. Luo, H, Wang, Y: Iterative approximation for the common solutions of a infinite variational inequality system for inverse-strongly accretive mappings. J. Math. Comput. Sci. **2**, 1660-1670 (2012)
31. Cho, SY, Kang, SM: Approximation of common solutions of variational inequalities via strict pseudocontractions. Acta Math. Sci. **32**, 1607-1618 (2012)
32. Ye, J, Huang, J: Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces. J. Math. Comput. Sci. **1**, 1-18 (2011)
33. Qin, X, Cho, SY, Kang, SM: Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems. J. Comput. Appl. Math. **234**, 750-760 (2010)
34. Yang, S, Li, W: Iterative solutions of a system of equilibrium problems in Hilbert spaces. Adv. Fixed Point Theory **1**, 15-26 (2011)
35. Takahashi, W, Toyoda, M: Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **118**, 417-428 (2003)
36. Iiduka, H, Takahashi, W: Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings. Nonlinear Anal. **61**, 341-350 (2005)
37. Hao, Y: On variational inclusion and common fixed point problems in Hilbert spaces with applications. Appl. Math. Comput. **217**, 3000-3010 (2010)
38. Qin, X, Cho, SY, Kang, SM: Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J. Comput. Appl. Math. **225**, 20-30 (2009)
39. Chang, SS, Lee, HWJ, Chan, CK: A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal. **70**, 3307-3319 (2009)
40. Qin, X, Cho, YJ, Kang, SM: Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications. Nonlinear Anal. **72**, 99-112 (2010)
41. Shimoji, K, Takahashi, W: Strong convergence to common fixed points of infinite nonexpansive mappings and applications. Taiwan. J. Math. **5**, 387-404 (2001)
42. Liu, L: Ishikawa-type and Mann-type iterative processes with errors for constructing solutions of nonlinear equations involving *m*-accretive operators in Banach spaces. Nonlinear Anal. **34**, 307-317 (1998)
43. Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. **20**, 197-228 (1967)

doi:10.1186/1687-1812-2013-67
Cite this article as: Qing and Shang: **Convergence of an extragradient-like iterative algorithm for monotone mappings and nonexpansive mappings.** *Fixed Point Theory and Applications* 2013 **2013**:67.