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Halpern iteration for strongly quasinonexpansive mappings on a geodesic space with curvature bounded above by one

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Abstract

In this paper, we deal with the Halpern iterative scheme for a strongly quasinonexpansive mapping in the setting of a complete geodesic space with curvature bounded above by one. Our result can be applied to the image recovery problem. We also consider the approximation of a fixed point of a nonexpansive mapping and obtain convergence theorems, one of which is a supplement of the result by Piątek with an additional sufficient condition.

MSC: 47H09

Keywords: CAT(1) space; Halpern iteration; strongly quasinonexpansive; Δ-demiclosed; fixed point

1 Introduction

Halpern's iterative method [1] is one of the most effective methods to find a fixed point of a nonexpansive mapping, which guarantees strong convergence of the approximating sequence. A remarkable result for nonlinear mappings was obtained by Wittmann [2] in the setting of Hilbert spaces. Since then, it has been investigated by a large number of researchers, and they have obtained different types of strong convergence theorems for nonexpansive mappings and their variations.

On the other hand, the notion of a strongly nonexpansive mapping was first proposed by Bruck and Reich [3] as a generalization of firmly nonexpansive mappings. This mapping was later generalized to a strongly quasinonexpansive mapping by Bruck [4]. We propose a new definition of this mapping in the framework of CAT(1) space by imposing a natural bound on the diameter of the space.

The first result of convergence of the Halpern iteration on a complete CAT(0) space was obtained by Saejung [5] for nonexpansive mappings, and a similar result in the setting of a complete CAT(1) space was proposed by Piątek [6]. The combination of the Halpern iteration and strongly quasinonexpansive mappings was made recently. See [7, 8] and others.

In this paper, we deal with the Halpern iterative scheme for a strongly quasinonexpansive mapping in the setting of a complete CAT(1) space. Then we show that the main result can be applied to the problem of image recovery. We also consider the approximation of a fixed point of a nonexpansive mapping. We point out that the proof of the result by Piątek is not sufficient and we supplement it with an additional sufficient condition for the convergence of the iterative sequence.

2 Preliminaries

Let X be a metric space. An element z of X is said to be an asymptotic center of a sequence $\{x_n\}$ in X if

$$\limsup_{n\to\infty} d(x_n, z) = \inf_{x\in X} \limsup_{n\to\infty} d(x_n, x).$$

Moreover, $\{x_n\}$ is said to be Δ-convergent and z is said to be its Δ-limit if z is the unique asymptotic center of any subsequences of $\{x_n\}$.

A geodesic with endpoints $x, y \in X$ is defined as an isometric mapping from the closed segment $[0, l]$ of real numbers to X whose image connects x and y. If a geodesic exists for every $x, y \in X$, then X is called a geodesic space.

For a triangle $\triangle(x, y, z)$ in a geodesic space X satisfying $d(y, z) + d(z, x) + d(x, y) < 2\pi$, we can find the comparison triangle $\triangle(\overline{x}, \overline{y}, \overline{z})$ in $\mathbb{S}^2$, that is, each corresponding edge has the same length as that of the original triangle. If every two points p, q on the edges of any $\triangle(x, y, z)$ and their corresponding points $\overline{p}$, $\overline{q}$ satisfy that

$$d(p, q) \leq d_{\mathbb{S}^2}(\overline{p}, \overline{q}),$$

we call X a CAT(1) space, where $d_{\mathbb{S}^2}$ is the spherical metric on $\mathbb{S}^2$.

In this paper, we deal with only CAT(1) spaces; however, we remark that all the results can be easily generalized to CAT(κ) spaces with positive κ by changing the scale of the space.

For two points x, y in a CAT(1) space X with $d(x, y) < \pi$ and $t \in [0, 1]$, we denote by $tx \oplus (1 - t)y$ the point z on a geodesic segment between x and y such that $d(y, z) = td(x, y)$ and $d(x, z) = (1 - t)d(x, y)$. A subset C of X is said to be π-convex if $tx \oplus (1 - t)y$ belongs to C for every $x, y \in C$ with $d(x, y) < \pi$ and $t \in [0, 1]$.

We refer to [9] for more details on geodesic spaces including CAT(1) spaces.

For three points x, y, z in a CAT(1) space X with $d(y, z) + d(z, x) + d(x, y) < 2\pi$ and $t \in [0, 1]$, we know that the following inequality holds [10]:

$$\cos d(x, v) \sin d(y, z) \geq \cos d(x, y) \sin\bigl(td(y, z)\bigr) + \cos d(x, z) \sin\bigl((1 - t)d(y, z)\bigr),$$

where $v = ty \oplus (1 - t)z$. This simple inequality plays a very important role in this paper.

Let X be a complete CAT(1) space, C a nonempty closed π-convex subset of X and suppose that $d(x, C) = \inf_{y\in C} d(x, y) < \pi/2$ for every $x \in X$. Then we can define the metric projection P_C from X onto C; that is, for every $x \in X$, $P_C x \in C$ is the unique point satisfying

$$d(x, P_C x) = \inf_{y\in C} d(x, y).$$

Let X be a CAT(1) space. Let $T : X \to X$ and suppose that the set $F(T) = \{x \in X : x = Tx\}$ of fixed points is not empty. Then T is said to be quasinonexpansive if $d(Tx, p) \leq d(x, p)$ for every $x \in X$ and $p \in F(T)$. T is said to be strongly quasinonexpansive if it is quasinonexpansive, and for every $p \in F(T)$ and every sequence $\{x_n\}$ in X satisfying that $\sup_{n\in\mathbb{N}} d(x_n, p) <$

$\pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(Tx_n,p)) = 1$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. T is said to be Δ-demiclosed if for any Δ-convergent sequence $\{x_n\}$ in X, its Δ-limit belongs to $F(T)$ whenever $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The following lemmas are important for our main result.

Lemma 2.1 (Xu [11]) *Let $\{s_n\}$, $\{t_n\}$ and $\{u_n\}$ be sequences of real numbers such that $s_n \geq 0$ and $u_n \geq 0$ for every $n \in \mathbb{N}$, $\limsup_{n\to\infty} t_n \leq 0$, and $\sum_{n=0}^{\infty} u_n < \infty$. Let $\{\gamma_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \gamma_n = \infty$. If $s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n t_n + u_n$ for every $n \in \mathbb{N}$, then $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.2 (Saejung-Yotkaew [12]) *Let $\{s_n\}$ and $\{t_n\}$ be sequences of real numbers such that $s_n \geq 0$ for every $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence in $]0,1[$ such that $\sum_{n=0}^{\infty} \beta_n = \infty$. Suppose that $s_{n+1} \leq (1-\beta_n)s_n + \beta_n t_n$ for every $n \in \mathbb{N}$. If $\limsup_{k\to\infty} t_{n_k} \leq 0$ for every subsequence $\{n_k\}$ of $\mathbb{N}$ satisfying $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0$, then $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.3 (He-Fang-Lopez-Li [13]) *Let X be a complete CAT(1) space and $p \in X$. If a sequence $\{x_n\}$ in X satisfies that $\limsup_{n\to\infty} d(x_n,p) < \pi/2$ and that $\{x_n\}$ is Δ-convergent to $x \in X$, then $d(x,p) \leq \liminf_{n\to\infty} d(x_n,p)$.*

3 Main result

As the main theorem of this paper, we prove strong convergence of the iterative sequence to a fixed point of a strongly quasinonexpansive mapping. We adopt the Halpern iterative scheme to generate the sequence. We begin with the following basic lemma, which is one of the main tools for our results.

Lemma 3.1 *Let X be a CAT(1) space such that $d(v,v') < \pi$ for every $v, v' \in X$. Let $\alpha \in [0,1]$ and $u, y, z \in X$. Then*

$$\begin{aligned} &1 - \cos d\bigl(\alpha u \oplus (1-\alpha)y, z\bigr) \\ &\quad \leq (1-\beta)\bigl(1 - \cos d(y,z)\bigr) + \beta\left(1 - \frac{\cos d(u,z)}{\sin d(u,y)\tan(\frac{\alpha}{2} d(u,y)) + \cos d(u,y)}\right), \end{aligned}$$

where

$$\beta = \begin{cases} 1 - \frac{\sin((1-\alpha)d(u,y))}{\sin d(u,y)} & (u \neq y), \\ \alpha & (u = y). \end{cases}$$

Proof It is obvious if $u = y$. Otherwise, from the inequality

$$\cos d\bigl(\alpha u \oplus (1-\alpha)y, z\bigr) \geq \frac{\sin(\alpha d(u,y))}{\sin d(u,y)} \cos d(u,z) + (1-\beta)\cos d(y,z),$$

we have that

$$\begin{aligned} &1 - \cos d\bigl(\alpha u \oplus (1-\alpha)y, z\bigr) \\ &\quad \leq (1-\beta)\bigl(1 - \cos d(y,z)\bigr) + \beta\left(1 - \frac{\sin(\alpha d(u,y))}{\beta \sin d(u,y)} \cos d(u,z)\right). \end{aligned}$$

We also have that

$$\begin{aligned}\frac{\sin(\alpha d(u,y))}{\beta \sin d(u,y)} &= \frac{\sin(\alpha d(u,y))}{\sin d(u,y) - \sin((1-\alpha)d(u,y))} \\ &= \frac{\sin(\alpha d(u,y))}{\sin d(u,y)(1-\cos(\alpha d(u,y))) + \cos d(u,y)\sin(\alpha d(u,y))} \\ &= \frac{1}{\sin d(u,y)\tan(\frac{\alpha}{2}d(u,y)) + \cos d(u,y)}\end{aligned}$$

and hence we obtain the desired result. □

Remark On the same assumption, we have

$$\cos d\bigl(\alpha u \oplus (1-\alpha)y, z\bigr) \geq \alpha \cos d(u,z) + (1-\alpha)\cos d(y,z).$$

Indeed, it holds from the first inequality of the proof above together with

$$\frac{\sin(\alpha d(u,y))}{\sin d(u,y)} \geq \alpha \quad \text{and} \quad \frac{\sin((1-\alpha)d(u,y))}{\sin d(u,y)} \geq 1-\alpha.$$

Now, we show the main theorem.

Theorem 3.2 *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $T : X \to X$ be a strongly quasinonexpansive and Δ-demiclosed mapping, and suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in* $]0,1[$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *For given points $u, x_0 \in X$, let $\{x_n\}$ be the sequence in X generated by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v,v') < \pi/2$;
(b) $d(u, P_{F(T)}u) < \pi/4$ *and* $d(u, P_{F(T)}u) + d(x_0, P_{F(T)}u) < \pi/2$;
(c) $\sum_{n=0}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_{F(T)}u$.

Proof Let $p = P_{F(T)}u$ and let

$$\begin{aligned} s_n &= 1 - \cos d(x_n, p), \\ t_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Tx_n)\tan(\frac{\alpha_n}{2}d(u, Tx_n)) + \cos d(u, Tx_n)}, \\ \beta_n &= \begin{cases} 1 - \frac{\sin((1-\alpha_n)d(u,Tx_n))}{\sin d(u,Tx_n)} & (u \neq Tx_n), \\ \alpha_n & (u = Tx_n) \end{cases}\end{aligned}$$

for $n \in \mathbb{N}$. Then, since T is quasinonexpansive, it follows from Lemma 3.1 that

$$s_{n+1} \leq (1-\beta_n)\bigl(1 - \cos d(Tx_n, p)\bigr) + \beta_n t_n \leq (1-\beta_n)s_n + \beta_n t_n$$

for every $n \in \mathbb{N}$. We also have that

$$\begin{aligned}\cos d(x_{n+1}, p) &= \cos d\bigl(\alpha_n u \oplus (1-\alpha_n) T x_n, p\bigr)\\ &\ge \alpha_n \cos d(u,p) + (1-\alpha_n)\cos d(Tx_n, p)\\ &\ge \alpha_n \cos d(u,p) + (1-\alpha_n)\cos d(x_n, p)\\ &\ge \min\bigl\{\cos d(u,p), \cos d(x_n,p)\bigr\}\end{aligned}$$

for all $n \in \mathbb{N}$. Thus we obtain that

$$\cos d(x_n, p) \ge \min\bigl\{\cos d(u,p), \cos d(x_0,p)\bigr\} = \cos \max\bigl\{d(u,p), d(x_0,p)\bigr\} > 0$$

for all $n \in \mathbb{N}$ and hence $\sup_{n\in\mathbb{N}} d(x_n, p) \le \max\{d(u,p), d(x_0,p)\} < \pi/2$.

Now, we see that each of the conditions (a), (b) and (c) implies that $\sum_{n=0}^{\infty} \beta_n = \infty$. For the cases of (a) and (b), let $M = \sup_{n\in\mathbb{N}} d(u, Tx_n)$. Then we show that $M < \pi/2$. For (a), it is trivial. For (b), since $\sup_{n\in\mathbb{N}} d(x_n, p) \le \max\{d(u,p), d(x_0,p)\}$, we have that

$$\begin{aligned}M &\le \sup_{n\in\mathbb{N}}\bigl(d(u,p) + d(Tx_n, p)\bigr)\\ &\le \sup_{n\in\mathbb{N}}\bigl(d(u,p) + d(x_n, p)\bigr)\\ &\le \max\bigl\{2d(u,p), d(u,p) + d(x_0,p)\bigr\}\\ &< \frac{\pi}{2}.\end{aligned}$$

So, in each case of (a) and (b), we have

$$\begin{aligned}\beta_n &\ge 1 - \frac{\sin((1-\alpha_n)M)}{\sin M}\\ &= \frac{2}{\sin M}\sin\left(\frac{\alpha_n}{2}M\right)\cos\left(\left(1-\frac{\alpha_n}{2}\right)M\right)\\ &\ge \alpha_n \cos M.\end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, it follows that $\sum_{n=0}^{\infty} \beta_n = \infty$. For the case of (c), we have that

$$\beta_n \ge 1 - \sin\frac{(1-\alpha_n)\pi}{2} = 1 - \cos\frac{\alpha_n \pi}{2} \ge \frac{\alpha_n^2 \pi^2}{8}$$

for every $n \in \mathbb{N}$. Therefore, from the condition (c) we have that $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any subsequence $\{s_{n_j}\}$ of $\{s_n\}$ satisfying that $\liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \ge 0$, we have that

$$\begin{aligned}0 &\le \liminf_{j\to\infty}(s_{n_j+1} - s_{n_j})\\ &= \liminf_{j\to\infty}\bigl(\cos d(x_{n_j}, p) - \cos d(x_{n_j+1}, p)\bigr)\\ &\le \liminf_{j\to\infty}\bigl(\cos d(x_{n_j}, p) - \bigl(\alpha_{n_j}\cos d(u,p) + (1-\alpha_{n_j})\cos d(Tx_{n_j}, p)\bigr)\bigr)\\ &= \liminf_{j\to\infty}\bigl(\cos d(x_{n_j}, p) - \cos d(Tx_{n_j}, p)\bigr)\end{aligned}$$

$$\leq \limsup_{j\to\infty}\bigl(\cos d(x_{n_j},p) - \cos d(Tx_{n_j},p)\bigr)$$
$$\leq 0.$$

Thus we have that $\lim_{j\to\infty}(\cos d(x_{n_j},p) - \cos d(Tx_{n_j},p)) = 0$. Using the inequality $\sup_{n\in\mathbb{N}} d(Tx_n,p) < \pi/2$, we also have $\lim_{j\to\infty}(\cos d(x_{n_j},p)/\cos d(Tx_{n_j},p)) = 1$. Since T is strongly quasinonexpansive, it follows that $\lim_{j\to\infty} d(x_{n_j}, Tx_{n_j}) = 0$. Let $\{v_k\}$ be a Δ-convergent subsequence of $\{x_{n_j}\}$ such that $\lim_{k\to\infty} d(u,v_k) = \liminf_{j\to\infty} d(u,x_{n_j})$. Then, since T is Δ-demiclosed and $\lim_{k\to\infty} d(v_k, Tv_k) = 0$, the Δ-limit z of $\{v_k\}$ belongs to $F(T)$. Using Lemma 2.3 and the definitions of the Δ-limit and the metric projection, we have that

$$\liminf_{j\to\infty} d(u, Tx_{n_j}) = \liminf_{j\to\infty} d(u,x_{n_j}) = \lim_k d(u,v_k) \geq d(u,z) \geq d(u,p).$$

Therefore, we obtain that

$$\begin{aligned}\limsup_{j\to\infty} t_{n_j} &= \limsup_{j\to\infty}\left(1 - \frac{\cos d(u,p)}{\sin d(u,Tx_{n_j})\tan(\frac{\alpha_{n_j}}{2}d(u,Tx_{n_j})) + \cos d(u,Tx_{n_j})}\right)\\ &= \limsup_{j\to\infty}\left(1 - \frac{\cos d(u,p)}{0 + \cos d(u,Tx_{n_j})}\right)\\ &\leq 0.\end{aligned}$$

By Lemma 2.2, we have that $\lim_{n\to\infty} s_n = 0$, that is, $\{x_n\}$ converges to $p = P_{F(T)}u$, and we finish the proof. □

4 Application to the image recovery problem

In the setting of Hilbert spaces, the image recovery problem can be formulated as to find the nearest point in the intersection of a family of closed convex subsets from a given point by using the metric projection of each subset. In this section, we consider this problem in the setting of complete CAT(1) spaces. As the simplest case, we deal with only two closed convex subsets C_1 and C_2 such that $C_1 \cap C_2 \neq \emptyset$ and generate an iterative sequence converging to the nearest point in $C_1 \cap C_2$ from a given point.

First, we observe some properties of the metric projection defined on a CAT(1) space. Let X be a complete CAT(1) space, C a nonempty closed π-convex subset of X and suppose that $d(x,C) = \inf_{y\in C} d(x,y) < \pi/2$ for every $x \in X$. Then we can prove that the metric projection $P : X \to C$ is a strongly quasinonexpansive and Δ-demiclosed mapping such that $F(P_C) = C$. Indeed, it is known that P_C is quasinonexpansive; see [14]. Let $\{x_n\}$ be a sequence in X and $p \in C$ such that $\sup_{n\in\mathbb{N}} d(x_n,p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(P_Cx_n,p)) = 1$. Then, from the property of metric projection, we have that

$$\cos d(x_n,P_Cx_n)\cos d(P_Cx_n,p) \geq \cos d(x_n,p)$$

for every $n \in \mathbb{N}$. Therefore, we have

$$1 \geq \cos d(x_n,P_Cx_n) \geq \frac{\cos d(x_n,p)}{\cos d(P_Cx_n,p)}$$

and thus $\lim_{n\to\infty} \cos d(x_n, P_C x_n) = 1$, that is, $\lim_{n\to\infty} d(x_n, P_C x_n) = 0$. Hence, P_C is strongly quasinonexpansive.

On the other hand, let $\{x_n\}$ be such that $\lim_{n\to\infty} d(x_n, P_C x_n) = 0$ and assume that $\{x_n\}$ is Δ-convergent to $x_\infty \in X$. Then $\{P_C x_n\}$ is also Δ-convergent to x_∞. Since $\{P_C x_n\}$ is a sequence in a closed π-convex subset C, we have that its Δ-limit x_∞ belongs to C, that is, $x_\infty \in F(P_C)$ [14]. It shows that P_C is Δ-demiclosed.

For two strongly quasinonexpansive and Δ-demiclosed mappings having common fixed points, we can create a new strongly quasinonexpansive and Δ-demiclosed mapping whose fixed points are common fixed points of given two mappings. For example, as we have seen above, metric projections to closed and convex sets are strongly quasinonexpansive and Δ-demiclosed. Thus, for given two metric projections to closed convex sets whose intersection is nonempty, the following method is applicable. It is useful to solve the image recovery problem.

Lemma 4.1 *Let X be a* CAT(1) *space and y_0, y_1 and y elements of X such that $d(y_0, y) + d(y_1, y) + d(y_0, y_1) < 2\pi$. Then*

$$\cos d\left(\frac{1}{2}y_0 \oplus \frac{1}{2}y_1, y\right) \cos \frac{d(y_0, y_1)}{2} \geq \min\bigl\{\cos d(y_0, y), \cos d(y_1, y)\bigr\}.$$

Proof It is obvious if $y_0 = y_1$. Otherwise, we have that

$$\begin{aligned} \cos d\left(\frac{1}{2}y_0 \oplus \frac{1}{2}y_1, y\right) \sin d(y_0, y_1) &\geq \bigl(\cos d(y_0, y) + \cos d(y_1, y)\bigr) \sin \frac{d(y_0, y_1)}{2} \\ &\geq 2 \min\bigl\{\cos d(y_0, y), \cos d(y_1, y)\bigr\} \sin \frac{d(y_0, y_1)}{2}. \end{aligned}$$

Dividing above by $2\sin(d(y_0, y_1)/2)$, we get the conclusion. □

Corollary 4.2 *Let T_0 and T_1 be quasinonexpansive mappings from X to X, x_0 and x_1 elements of X, and p an element of $F(T_0) \cap F(T_1)$. Then*

$$\cos d\left(\frac{1}{2}T_0 x \oplus \frac{1}{2}T_1 x, p\right) \cos \frac{d(T_0 x, T_1 x)}{2} \geq \cos d(x, p).$$

Lemma 4.3 *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for arbitrary v and v' of X, and T_0 and T_1 quasinonexpansive mappings from X to X such that $F(T_0) \cap F(T_1) \neq \emptyset$. Then*

$$F\left(\frac{1}{2}T_0 \oplus \frac{1}{2}T_1\right) = F(T_0) \cap F(T_1).$$

Proof It is obvious that $F(\frac{1}{2}T_0 \oplus \frac{1}{2}T_1) \supset F(T_0) \cap F(T_1)$. We will show the opposite inclusion. We denote $T = \frac{1}{2}T_0 \oplus \frac{1}{2}T_1$. Let $z \in F(T)$. Then, for arbitrary $p \in F(T_0) \cap F(T_1)$, from Corollary 4.2, we have that

$$\cos d(z, p) \sin \frac{d(T_0 z, T_1 z)}{2} = \cos d(Tz, p) \sin \frac{d(T_0 z, T_1 z)}{2} \geq \cos d(z, p),$$

that is, $T_0 z = T_1 z$. Hence, $z = Tz = T_0 z = T_1 z$, which means $z \in F(T_0) \cap F(T_1)$. □

Lemma 4.4 *Let X be a* CAT(1) *space such that $d(v,v') < \pi/2$ for arbitrary v and v' of X. Let T_0 and T_1 be mappings from X to X such that $F(T_0) \cap F(T_1) \neq \emptyset$. If both T_0 and T_1 are strongly quasinonexpansive, then so is $\frac{1}{2}T_0 \oplus \frac{1}{2}T_1$.*

Proof We denote $T = \frac{1}{2}T_0 \oplus \frac{1}{2}T_1$. By Corollary 4.2, for $x \in X$ and $p \in F(T_0) \cap F(T_1)$, we have

$$\cos d(Tx,p) \geq \cos d(Tx,p) \cos \frac{d(T_0x, T_1x)}{2} \geq \cos d(x,p),$$

that is, $d(Tx,p) \leq d(x,p)$ and hence T is quasinonexpansive. Moreover, for a sequence $\{x_n\}$ in X and a point p in $F(T)$ such that $\sup_{n\in\mathbb{N}} d(x_n,p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(Tx_n,p)) = 1$, by Lemma 4.1, we have

$$\begin{aligned}\cos d(Tx_n,p) &\geq \cos d(Tx_n,p) \cos \frac{d(T_0x_n, T_1x_n)}{2} \\ &\geq \min\{\cos d(T_0x_n,p), \cos d(T_1x_n,p)\}.\end{aligned}$$

So, there exist two disjoint subsets $\{m_i\}$ and $\{n_i\}$ of $\mathbb{N}$ such that $\mathbb{N} = \{m_i\} \cup \{n_i\}$ and

$$\begin{aligned}&d(Tx_{m_i},p) \leq d(T_0x_{m_i},p) \quad \text{for all } m_i \quad \text{and} \\ &d(Tx_{n_i},p) \leq d(T_1x_{n_i},p) \quad \text{for all } n_i.\end{aligned}$$

We may assume that both $\{m_i\}$ and $\{n_i\}$ are infinite sets without loss of generality. From Lemma 4.3, p is in $F(T_0)$ and thus

$$1 \geq \frac{\cos d(x_{m_i},p)}{\cos d(T_0x_{m_i},p)} \geq \frac{\cos d(x_{m_i},p)}{\cos d(Tx_{m_i},p)} \to 1,$$

which means $\lim_{i\to\infty}(\cos d(x_{m_i},p)/\cos d(T_0x_{m_i},p)) = 1$. Since T_0 is strongly quasinonexpansive, we have that $\lim_{i\to\infty} d(T_0x_{m_i},x_{m_i}) = 0$. By Corollary 4.2, we have

$$\cos \frac{d(T_0x_{m_i}, T_1x_{m_i})}{2} \geq \frac{\cos d(x_{m_i},p)}{\cos d(Tx_{m_i},p)} \to 1,$$

that is, $\lim_{i\to\infty} d(T_0x_{m_i},T_1x_{m_i}) = 0$. Then it follows that $\lim_{i\to\infty} d(T_1x_{m_i},x_{m_i}) = 0$. Similarly, we have that $\lim_{i\to\infty} d(T_1x_{n_i},x_{n_i}) = 0$ and $\lim_{i\to\infty} d(T_0x_{n_i},x_{n_i}) = 0$. Consequently, we have that $\lim_{n\to\infty} d(T_0x_n,x_n) = 0$ and $\lim_{n\to\infty} d(T_1x_n,x_n) = 0$. Hence, we obtain that $\lim_{n\to\infty} d(Tx_n,x_n) = 0$, which is the desired result. □

Lemma 4.5 *Let X be a* CAT(1) *space such that $d(v,v') < \pi/2$ for arbitrary v and v' of X. Let T_0 and T_1 be mappings from X to X such that $F(T_0) \cap F(T_1) \neq \emptyset$. If both T_0 and T_1 are Δ-demiclosed, then so is $\frac{1}{2}T_0 \oplus \frac{1}{2}T_1$.*

Proof We denote $T = \frac{1}{2}T_0 \oplus \frac{1}{2}T_1$. Let $\{x_n\}$ be a sequence in X and x_∞ an element of X such that $d(Tx_n,x_n) \to 0$ and suppose that $\{x_n\}$ is Δ-convergent to x_∞. Then, by Corollary 4.2, we have

$$\cos \frac{d(T_0x_n, T_1x_n)}{2} \geq \frac{\cos d(x_n,p)}{\cos d(Tx_n,p)} \to 1,$$

that is, $\lim_{n\to\infty} d(T_0x_n, T_1x_n) = 0$. Thus we have

$$d(T_0x_n, Tx_n) \leq \frac{d(T_0x_n, T_1x_n)}{2} \to 0.$$

Since T_0 is Δ-demiclosed, we have that $T_0x_\infty = x_\infty$. In a similar fashion, we have that $T_1x_\infty = x_\infty$. Hence $Tx_\infty = x_\infty$, that is, T is Δ-demiclosed. □

Let X be a complete CAT(1) space such that $d(v,v') < \pi/2$ for every $v, v' \in X$, and let C_0 and C_1 be closed convex subsets of X having the nonempty intersection. Then, for the metric projections P_{C_0} and P_{C_1}, the mapping $\frac{1}{2}P_{C_0} \oplus \frac{1}{2}P_{C_1}$ is strongly quasinonexpansive and Δ-demiclosed. Moreover, the set of its fixed points is $C_0 \cap C_1$. Applying these facts to Theorem 3.2, we obtain the following result for the image recovery problem for two convex subsets.

Theorem 4.6 *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let C_0 and C_1 be closed convex subsets of X such that $C_0 \cap C_1 \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in* $]0,1[$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *For given points $u, x_0 \in X$, let $\{x_n\}$ be the sequence in X generated by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)\left(\frac{1}{2}P_{C_0}x_n \oplus \frac{1}{2}P_{C_1}x_n\right)$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:
(a) $\sup_{v,v'\in X} d(v,v') < \pi/2$;
(b) $d(u, P_{C_0\cap C_1}u) < \pi/4$ *and* $d(u, P_{C_0\cap C_1}u) + d(x_0, P_{C_0\cap C_1}u) < \pi/2$;
(c) $\sum_{n=0}^{\infty} \alpha_n^2 = \infty$.
Then $\{x_n\}$ converges to $P_{C_0\cap C_1}u$.

5 Approximation to a fixed point of nonexpansive mappings

At the end of this paper, we prove two convergence theorems of iterative schemes which approximate a fixed point of a nonexpansive mapping. Firstly, we apply the main result Theorem 3.2 to this problem. We begin with the following lemmas.

Lemma 5.1 *A nonexpansive mapping defined on a* CAT(1) *space is Δ-demiclosed.*

Proof Let $S : X \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a Δ-convergent sequence in X with the Δ-limit $x_\infty \in X$ and suppose that $\lim_{n\to\infty} d(x_n, Sx_n) = 0$. We will prove that $x_\infty = Sx_\infty$. If $x_\infty \neq Sx_\infty$, then, by the uniqueness of the asymptotic center, we have that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, x_\infty) &< \limsup_{n\to\infty} d(x_n, Sx_\infty) \\ &\leq \limsup_{n\to\infty} \bigl(d(x_n, Sx_n) + d(Sx_n, Sx_\infty)\bigr) \\ &\leq \limsup_{n\to\infty} \bigl(d(x_n, Sx_n) + d(x_n, x_\infty)\bigr) \\ &= \limsup_{n\to\infty} d(x_n, x_\infty),\end{aligned}$$

a contradiction. Hence, we have that S is Δ-demiclosed. □

Lemma 5.2 *Let X be a* CAT(1) *space such that $d(v', v'') + d(v'', v) + d(v, v') < 2\pi$ for every $v, v', v'' \in X$. Let $S : X \to X$ be a nonexpansive mapping with a nonempty set of fixed points $F(S)$. Then the mapping $T : X \to X$ defined by*

$$Tx = \frac{1}{2}x \oplus \frac{1}{2}Sx$$

for $x \in X$ is a strongly quasinonexpansive and Δ-demiclosed mapping such that $F(T) = F(S)$.

Proof It is obvious that $F(T) = F(S)$ by definition and, since both the identity mapping I and S are quasinonexpansive, for $x \in X$ and $p \in F(T) = F(S)$, we have that

$$\cos d(Tx, p) \geq \frac{1}{2}\cos d(x, p) + \frac{1}{2}\cos d(Sx, p) \geq \cos d(x, p).$$

Thus T is quasinonexpansive. Let $\{x_n\}$ be a sequence in X such that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$, and suppose that $\lim_{n\to\infty}(\cos d(x_n, p)/\cos d(Tx_n, p)) = 1$. Then we have

$$\begin{aligned}\cos d(Tx_n, p)\sin d(x_n, Sx_n) &\geq \sin\frac{d(x_n, Sx_n)}{2}\big(\cos d(x_n, p) + \cos d(Sx_n, p)\big)\\ &\geq 2\sin\frac{d(x_n, Sx_n)}{2}\cos d(x_n, p)\end{aligned}$$

for every $n \in \mathbb{N}$. It follows that

$$\cos d(Tx_n, p)\cos\frac{d(x_n, Sx_n)}{2} \geq \cos d(x_n, p)$$

and since $\sup_{n\in\mathbb{N}} d(Tx_n, p) \leq \sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$, we have

$$\begin{aligned}1 &\geq \limsup_{n\to\infty}\cos d(x_n, Tx_n)\\ &\geq \liminf_{n\to\infty}\cos d(x_n, Tx_n)\\ &= \liminf_{n\to\infty}\cos\frac{d(x_n, Sx_n)}{2}\\ &\geq \lim_{n\to\infty}\frac{\cos d(x_n, p)}{\cos d(Tx_n, p)}\\ &= 1.\end{aligned}$$

It implies that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and hence T is strongly quasinonexpansive.

For the Δ-demiclosedness of T, use Lemmas 4.5 and 5.1 with the fact that the identity mapping is also Δ-demiclosed. □

Applying this lemma and the results in the previous section to Theorem 3.2, we obtain the following convergence theorem of an iterative scheme approximating a fixed point of a nonexpansive mapping.

Theorem 5.3 *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $S : X \to X$ be a nonexpansive mapping and suppose that $F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a real* *sequence in* $]0,1[$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *For given points* $u, x_0 \in X$, *let* $\{x_n\}$ *be the sequence in* X *generated by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)\left(\frac{1}{2}x_n \oplus \frac{1}{2}Sx_n\right)$$

for $n \in \mathbb{N}$. *Suppose that one of the following conditions holds*:

(a) $\sup_{v,v'\in X} d(v,v') < \pi/2$;

(b) $d(u, P_{F(S)}u) < \pi/4$ *and* $d(u, P_{F(S)}u) + d(x_0, P_{F(S)}u) < \pi/2$;

(c) $\sum_{n=0}^{\infty}\alpha_n^2 = \infty$.

Then $\{x_n\}$ *converges to* $P_{F(S)}u$.

The next convergence theorem of an iterative scheme on CAT(1) spaces was first proposed by Piątek [6]. The theorem deals with the Halpern-type iterative sequence. Although the result itself is correct, a part of the proof does not seem to be exact. Precisely, in the proof of the convergence theorem for the explicit iteration process, the author makes use of Xu's lemma, Lemma 2.1 in this paper. However, the conditions required for this lemma are not verified. We attempt to prove the following theorem as a supplement of the aforementioned result, and moreover, we find another coefficient condition which guarantees convergence of the iterative scheme.

Before showing the result, we need the following lemma which is analogous to [6, Lemma 3.3]. The assumption for the length of the edges of the triangle is improved.

Lemma 5.4 *Let* X *be a* CAT(1) *space. For* $M \in]0,\pi[$, *let* $u, v, w \in X$ *be such that* $d(u,v) \le M$ *and* $d(u,w) \le M$. *For a given* $\alpha \in]0,1[$, *let* $v' = \alpha u \oplus (1-\alpha)v$ *and* $w' = \alpha u \oplus (1-\alpha)w$. *If* $d(v,w) + d(w,u) + d(u,v) < 2\pi$ *and* $\sin((1-\alpha)M) \le \sin M$, *then*

$$d(v', w') \le \frac{\sin((1-\alpha)M)}{\sin M} d(v,w).$$

Proof Consider the comparison triangle $\triangle(\overline{u},\overline{v},\overline{w})$ of $\triangle(u,v,w)$ on $\mathbb{S}^2$ and let $\overline{v}'$ and $\overline{w}'$ be the comparison points of v' and w', respectively. Let

$$U = d_{\mathbb{S}^2}(\overline{v},\overline{w}), \qquad V = d_{\mathbb{S}^2}(\overline{w},\overline{u}), \qquad W = d_{\mathbb{S}^2}(\overline{u},\overline{v}),$$

and $U' = d_{\mathbb{S}^2}(\overline{v}',\overline{w}')$. Then, letting $\alpha' = 1-\alpha$, we have that

$$\begin{aligned}
&\sin V \sin W\left(\frac{\sin^2\alpha' M}{\sin^2 M}(1-\cos U) - \bigl(1-\cos U'\bigr)\right) \\
&= \sin V \sin W\left(\frac{\sin^2\alpha' M}{\sin^2 M}(1-\cos U) - 1\right) \\
&\quad + (\cos U - \cos V\cos W)\sin\alpha' V\sin\alpha' W + \sin V\sin W\cos\alpha' V\cos\alpha' W \\
&= (1-\cos U)\left(\sin V\sin W\frac{\sin^2\alpha' M}{\sin^2 M} - \sin\alpha' V\sin\alpha' W\right) \\
&\quad + \bigl((1-\cos V\cos W)\sin\alpha' V\sin\alpha' W - \sin V\sin W\bigl(1-\cos\alpha' V\cos\alpha' W\bigr)\bigr).
\end{aligned}$$

Since the functions $f_V(t) = \sin tV / \sin tM$, $f_W(t) = \sin tW / \sin tM$, and $g(t) = (1 - \cos tV \times \cos tW)/(\sin tV \sin tW)$ are all increasing on $[0,1]$, it follows that

$$\sin V \sin W \frac{\sin^2 \alpha' M}{\sin^2 M} - \sin \alpha' V \sin \alpha' W \geq 0,$$

and

$$(1 - \cos V \cos W) \sin \alpha' V \sin \alpha' W - \sin V \sin W \bigl(1 - \cos \alpha' V \cos \alpha' W\bigr) \geq 0.$$

Therefore, we have that

$$\sin V \sin W \left(\frac{\sin^2 \alpha' M}{\sin^2 M}(1 - \cos U) - \bigl(1 - \cos U'\bigr) \right) \geq 0.$$

Using the assumption that $\sin \alpha' M \leq \sin M$, we obtain that

$$\begin{aligned} \sin \frac{U'}{2} &= \sqrt{\frac{1 - \cos U'}{2}} \leq \frac{\sin \alpha' M}{\sin M} \sqrt{\frac{1 - \cos U}{2}} \\ &= \frac{\sin \alpha' M}{\sin M} \sin \frac{U}{2} \leq \sin\left(\frac{\sin \alpha' M}{\sin M} \frac{U}{2} \right) \end{aligned}$$

and, by the CAT(1) inequality, it follows that

$$d\bigl(v', w'\bigr) \leq d_{\mathbb{S}^2}\bigl(\overline{v}', \overline{w}'\bigr) = U' \leq \frac{\sin((1-\alpha)M)}{\sin M} U = \frac{\sin((1-\alpha)M)}{\sin M} d(v, w),$$

which is the desired result. □

Theorem 5.5 *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $T : X \to X$ be a nonexpansive mapping and suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in* $[0,1]$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. *For given points $u, x_0 \in X$, let $\{x_n\}$ be the sequence in X generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v, v' \in X} d(v, v') < \pi/2$;
(b) $d(u, P_{F(T)} u) < \pi/4$ *and* $d(u, P_{F(T)} u) + d(x_0, P_{F(T)} u) < \pi/2$;
(c) $\sum_{n=0}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_{F(T)} u$.

We employ the method used in [6] for some parts of the proof.

Proof From the definition of $\{x_n\}$, using Lemma 5.4, we have that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d\bigl(x_n, \alpha_n u \oplus (1 - \alpha_n) T x_{n-1}\bigr) + d\bigl(\alpha_n u \oplus (1 - \alpha_n) T x_{n-1}, x_{n+1}\bigr) \\ &= |\alpha_n - \alpha_{n-1}| d(u, T x_{n-1}) + d\bigl(\alpha_n u \oplus (1 - \alpha_n) T x_{n-1}, x_{n+1}\bigr) \end{aligned}$$

$$\begin{aligned}&\le |\alpha_n - \alpha_{n-1}| d(u, Tx_{n-1}) + \frac{\sin((1-\alpha_n)M_n)}{\sin M_n} d(Tx_{n-1}, Tx_n) \\ &\le |\alpha_n - \alpha_{n-1}| d(u, Tx_{n-1}) + \frac{\sin((1-\alpha_n)M_n)}{\sin M_n} d(x_{n-1}, x_n),\end{aligned}$$

where $M_n = \max\{d(u, Tx_n), d(u, Tx_{n-1})\}$ for each $n \in \mathbb{N}$. Let

$$\gamma_n = \begin{cases} 1 - \frac{\sin((1-\alpha_n)M_n)}{\sin M_n} & (M_n \neq 0), \\ \alpha_n & (M_n = 0) \end{cases}$$

for all $n \in \mathbb{N}$. Then, as in the proof of Theorem 3.2, we have that each of the conditions (a), (b) and (c) implies that $\sum_{n=0}^{\infty} \gamma_n = \infty$. In particular, in cases of (a) and (b), for $M = \sup_{v,v' \in X} d(v, v')$ and $M = \max\{2d(u,p), d(u,p) + d(x_0,p)\}$ with $p = P_{F(T)}u$, respectively, it holds that

$$\gamma_n \ge \alpha_n \cos M_n \ge \alpha_n \cos M,$$

and in case of (c), it holds that

$$\gamma_n \ge \frac{\alpha_n^2 \pi^2}{8}.$$

Then, by using Lemma 2.1 with the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, we have that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. It follows that

$$0 \le d(x_n, Tx_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + \alpha_n d(u, Tx_n)$$

and thus $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let p, $\{s_n\}$, $\{t_n\}$, $\{\beta_n\}$ be as in the proof of Theorem 3.2 again. Then by Lemma 3.1, we have that

$$s_{n+1} \le (1 - \beta_n) s_n + \beta_n t_n$$

for every $n \in \mathbb{N}$. Since every nonexpansive mapping is Δ-demiclosed, we can use the same technique as the proof of Theorem 3.2 and then we obtain that

$$\begin{aligned}\limsup_{n\to\infty} t_n &= \limsup_{n\to\infty} \left(1 - \frac{\cos d(u,p)}{\sin d(u, Tx_n) \tan(\frac{\alpha_n}{2} d(u, Tx_n)) + \cos d(u, Tx_n)} \right) \\ &\le 0.\end{aligned}$$

Consequently, we have that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.1. Hence, $\{x_n\}$ converges to $p = P_{F(T)}u$, which is the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed in this work on an equal basis. All authors read and approved the final manuscript.

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