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Generating function for q-Eulerian polynomials and their decomposition and applications

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Abstract

The aim of this paper is to define a generating function for q-Eulerian polynomials and numbers attached to any character χ of the finite cyclic group G. We derive many functional equations, q-difference equations and partial deferential equations related to these generating functions. By using these equations, we find many properties of q-Eulerian polynomials and numbers. Using the generating element of the finite cyclic group G and the generating element of the subgroups of G, we show that the generating function with a conductor f can be written as a sum of the generating function with conductors which are less than f.

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1 Introduction

The theory of the family of Eulerian polynomials and numbers (Frobenius-Euler polynomials and numbers) has become a very important part of analytic number theory and other sciences, for example, engineering, computer, geometric design and mathematical physics. Euler numbers are related to the Brouwer fixed point theorem and vector fields. Therefore, q-Eulerian type numbers may be related to Brouwer fixed point theorem and vector fields [1].

Recently, many different special functions have been used to define and investigate q-Eulerian numbers and polynomials, see details [1–39]. Therefore, we construct and investigate various properties of q-Eulerian numbers and polynomials, which are related to the many known polynomials and numbers such as Frobenius-Euler polynomials and numbers, Apostol-Euler polynomials and numbers, Euler polynomials and numbers.

Recently in [2–4], the authors defined a relationship between algebra and analysis. In detail, they made a new approximation between the subgroup and monoid presentation and special generating functions (such as Stirling numbers, Array polynomials *etc.*). In this paper, since our priority aim is to define special numbers and polynomials, it is worth depicting these references as well. Then in this paper, applying any group character χ of the finite cyclic group G to a special generating function (which has been defined in [5]), we give a generalization of q-Eulerian polynomials and numbers (*q-Apostol-type Frobenius-Euler*

polynomials and numbers) and investigate their properties and some useful functional equations. Using a generating element of the subgroups of G and a generating element of G, we also decompose our generating function attached to the characters of G, and so we obtain a new decomposition of q-Eulerian numbers and polynomials attached to the characters of a subgroup of G and the Dirichlet character of G. This decomposition enables us to compute q-Apostol-type Frobenius-Euler polynomials and numbers more easily.

Throughout this paper, we assume that $q \in \mathbb{C}$, the set of complex numbers, with $|q| < 1$

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q} & \text{if } q \neq 1, \\ x & \text{if } q = 1. \end{cases}$$

We use the following standard notions.

$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$ and also, as usual, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

1.1 Characters of a group G

We recall the definition and some properties of the character of an arbitrary group $(G, \cdot)$ (see [6]).

A non-zero complex-valued function χ defined on G is called a character of G if for all $a, b \in G$,

$$\chi(a \cdot b) = \chi(a)\chi(b).$$

In particular, if G is a finite group with the identity element e, then $\chi(e) = 1$ and $\chi(a)$ is a root of unity. In [6, Theorem 6.8], it is proved that a finite abelian group G of order n has exactly n distinct characters.

Let G be the group of reduced residue classes module positive integer f. Corresponding to each character χ of G, the Dirichlet character χ_D is defined as follows:

$$\chi_D(n) = \begin{cases} \chi(n) & \text{if } \gcd(n, f) = 1, \\ 0 & \text{if } \gcd(n, f) > 1. \end{cases}$$

Hence it is easily observed that

$$\chi(nf + a) = \chi(a)$$

for all $n, a \in \mathbb{N}$. In this note, f is called the conductor of the character of a group G of reduced residue classes module a positive integer f.

1.2 q-Eulerian polynomials and numbers

In [5], Simsek defined and studied some properties of q-Apostol type Frobenius-Euler polynomials and numbers.

Definition 1.1 Let $a, b \in \mathbb{R}^+$ ($a \neq b$) and $u \in \mathbb{C} \setminus \{1\}$.

(i) The q-Apostol type Frobenius-Euler numbers

$$\mathcal{H}_n(u; a, b; \lambda; q) \quad (\lambda, q \in \mathbb{C})$$

are defined by means of the following generating function:

$$F_{\lambda,q}(t; u, a, b) = \left(1 - \frac{a^t}{u}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n b^{[n]t} = \sum_{n=0}^{\infty} \mathcal{H}_n(u; a, b; \lambda; q) \frac{t^n}{n!}.$$

(ii) The q-Apostol type Frobenius-Euler polynomials

$$\mathcal{H}_n(x; u; a, b; \lambda; q) \quad (\lambda \in \mathbb{C})$$

are defined by means of the following generating function:

$$F_{\lambda,q}(x, t; u, a, b) = \left(1 - \frac{a^t}{u}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n b^{[n+x]t} = \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b; \lambda; q) \frac{t^n}{n!}, \tag{1}$$

where

$$\left|\frac{\lambda}{u} b^t\right| < 1.$$

2 q-Eulerian polynomials and numbers attached to any character

In this section, we define q-Eulerian polynomials and numbers attached to any character of the finite cyclic group G. Our new generating functions are related to nonnegative real parameters.

Definition 2.1 Let $a, b \in \mathbb{R}^+$ ($a \neq b$ and $a \geq 1$), $u \in \mathbb{C} \setminus \{1\}$, $\lambda, q \in \mathbb{C}$ ($|q| < 1$). Let χ be a character of a finite cyclic group G with the conductor f.

(i) The q-Eulerian numbers attached to the character χ

$$\mathcal{H}_{n,\chi}(u; a, b; \lambda; q)$$

are defined by means of the following generating function:

$$F_{\lambda,q,\chi}(t, u, a, b) = \left(1 - \frac{a^{[f]t}}{u^f}\right) \sum_{m=0}^{\infty} \left(\frac{\lambda}{u}\right)^m b^{[m]t} \chi(m) = \sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(u; a, b; \lambda; q) \frac{t^n}{n!}. \tag{2}$$

(ii) The q-Eulerian polynomials attached to the character χ

$$\mathcal{H}_{n,\chi}(x; u; a, b; \lambda; q)$$

are defined by means of the following generating function:

$$F_{\lambda,q,\chi}(t, x, u, a, b) = \left(1 - \frac{a^{[f]t}}{u^f}\right) \sum_{m=0}^{\infty} \left(\frac{\lambda}{u}\right)^m b^{[m+x]t} \chi(m) = \sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(x; u; a, b; \lambda; q) \frac{t^n}{n!}, \tag{3}$$

where

$$\left| \frac{\lambda}{u} b^t \right| < 1.$$

It is observed that

$$\mathcal{H}_{n,\chi}(0; u; a, b; \lambda; q) = \mathcal{H}_{n,\chi}(u; a, b; \lambda; q).$$

Upon setting $t = 0$ in (3), we compute a q-Eulerian number $\mathcal{H}_{0,\chi}(u, a, b, \lambda, q)$ attached to the character χ as follows:

$$\mathcal{H}_{0,\chi}(u, a, b, \lambda, q) = \left(\frac{u^{1-f}(u^f - 1)}{u - \lambda \chi(1)} \right).$$

By using the conductor f of the character χ and combining with $F_{\lambda,q}(t, x, u, a, b)$, we modify Equation (2) and Equation (3), respectively, as follows:

$$F_{\lambda,q,\chi}(t, u, a, b) = \sum_{i=0}^{f-1} \left(\frac{\lambda \chi(1)}{u} \right)^i F_{\lambda^f, q^f} \left(t[f], \frac{i}{f}, u^f, a, b \right) \tag{4}$$

and

$$F_{\lambda,q,\chi}(t, x, u, a, b) = \sum_{i=1}^{f} \left(\frac{\lambda \chi(1)}{u} \right)^i F_{\lambda^f, q^f} \left([f]t, \frac{i+x}{f}, u^f, a, b \right). \tag{5}$$

Therefore, we provide the following relationships between q-Eulerian numbers and q-Eulerian numbers attached to the character χ.

Theorem 2.2 *Let $n \in \mathbb{N}_0$. Then we have*

(i)

$$\mathcal{H}_{n,\chi}(u; a, b; \lambda, u) = [f]^n \sum_{i=0}^{f-1} \left(\frac{\lambda}{u} \right)^i \chi(i) \mathcal{H}_n \left(\frac{i}{f}, u^f, a, b; \lambda^f, q^f \right),$$

(ii)

$$\mathcal{H}_{n,\chi}(u; a, b; \lambda, u) = [f]^n \sum_{i=0}^{f-1} \chi(i) \left(\frac{\lambda}{u} \right)^i \mathcal{H}_n \left(\frac{x+i}{f}, u^f, a, b; \lambda^f, q^f \right).$$

Proof By using (4), we deduce that

$$\sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(u; a, b; \lambda, u) \frac{t^n}{n!} = \sum_{i=0}^{f-1} \chi(i) \left(\frac{\lambda}{u} \right)^i \sum_{n=0}^{\infty} [f]^n \mathcal{H}_n \left(\frac{i}{f}, u^f, a, b; \lambda^f, q^f \right) \frac{t^n}{n!},$$

which, by comparing the coefficient $\frac{t^n}{n!}$ on the both sides of the above equations, yields the first assertion of Theorem 2.2.

The second assertion (ii) is proved with the same argument. $\qquad\square$

By Theorem 2.2, we also compute a q-Eulerian number attached to the character χ as follows:

$$\mathcal{H}_{0,\chi}(u,a,b,\lambda,q) = \sum_{i=0}^{f-1} \chi(i) \left(\frac{\lambda}{u}\right)^i \mathcal{H}_0\left(\frac{i}{f}, u^f, a, b; \lambda^f, q^f\right).$$

Now, we turn our attention to the following generation function defined in [1] since we need this generating function frequently to give some functional equations for a q-Eulerian number and polynomials attached to the character χ.

Let $a \geq 1$. The number $Y_n(u,a)$ is defined by means of the following generating function:

$$G(t,u,a) = \frac{1}{a^t - u} = \sum_{n=0}^{\infty} Y_n(u,a) \frac{t^n}{n!}.$$

The polynomials $Y_n(x;u,a)$ are defined by means of the following generating function:

$$\mathcal{G}(t,x,u,a) = G(t,u,a) a^{xt} = \sum_{n=0}^{\infty} Y_n(x;u,a) \frac{t^n}{n!}. \tag{6}$$

Since we need this generating function frequently in this paper, we use the notation

$$\begin{aligned}
D(k) &= \frac{a^{[f]t} - u^f}{a^{[f]q^k t} - u^f} = \mathcal{G}\left([f]q^k t, \frac{1}{q^k}, u^f, a\right) - u^f G\left([f]q^k t, u^f, a\right) \\
&= \sum_{n=0}^{\infty} [f]^n q^{kn} \left(Y_n\left(\frac{1}{q^k}, u^f, a\right) - u^f Y_n(u^f, a)\right) \frac{t^n}{n!},
\end{aligned}$$

and so it follows that $D_n(k) = [f]^n q^n (Y_n(\frac{1}{q}, u^f, a) - Y_n(u^f, a))$.

By using the following well-known identity:

$$[n + x] = [x] + q^x[n]$$

in (3), we verify the following functional equation:

$$F_{\lambda,q,\chi}(t, x+y, u, a, b) = b^{[y]t} D(y) F_{\lambda,q,\chi}(q^y t, x, u, a, b). \tag{7}$$

Hence we have the following theorem.

Theorem 2.3 *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}
\mathcal{H}_{n,\chi}(x+y; u; a, b; \lambda; q) &= \sum_{i=0}^{m} \sum_{j=0}^{i} \binom{m}{i}\binom{i}{j} ([y] \ln b)^{i-j} D_j(y) q^{(m-i)y} \\
&\quad \times \mathcal{H}_{(m-i)i,\chi}(x, u; a, b; \lambda; q).
\end{aligned}$$

Proof By applying the Cauchy product to (7), we deduce that

$$F_{\lambda,q,\chi}(t, x+y, u, a, b) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} ([y] \ln b)^{n-j} D_j(y) \frac{t^n}{n!} F_{\lambda,q,\chi}(tq^y, x, u, a, b).$$

Let $K(n) = \sum_{j=0}^{n} \binom{n}{j}([y]\ln b)^{n-j} D_j(y)$. Then it follows that

$$F_{\lambda,q,\chi}(t,x+y,u,a,b) = \sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}K(i)q^{(m-i)y}\mathcal{H}_{m-i,\chi}(x,u;a,b;\lambda;q)\frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{m}{i}\binom{i}{j}([y]\ln b)^{i-j}D_j(y)q^{(m-i)y}$$

$$\times \mathcal{H}_{(m-i)i,\chi}(x,u;a,b;\lambda;q)\frac{t^m}{m!}.$$

By comparing the coefficient of $\frac{t^m}{m!}$ on both sides of the above equation, we obtain our desired result. □

Upon setting $x = 0$ in (7), we get the following functional equation:

$$F_{\lambda,q,\chi}(t,y,u,a,b) = b^{[y]t}D(y)F_{\lambda,q,\chi}\left(q^y t,0,u,a,b\right)$$

$$= b^{[y]t}D(y)F_{\lambda,q,\chi}\left(q^y t,u,a,b\right).$$

By substituting $x = 0$ in Theorem 2.3, the following theorem is easily proved.

Theorem 2.4 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_{n,\chi}(y;u;a,b;\lambda;q) = \sum_{i=0}^{m}\sum_{j=0}^{n-i}\binom{m}{i}\binom{i}{j}([y]\ln b)^{i-j}D_j(y)q^{(n-i)y}\mathcal{H}_{(m-i)i,\chi}(u;a,b;\lambda;q).$$

So that we obtain a q-difference equation for q-Eulerian polynomials attached to the character χ, we study the following equations:

$$F_{\lambda,q,\chi}(t,x,u,a,b) = \left(1 - \frac{a^{[f]t}}{u^f}\right) + \frac{\lambda\chi(1)}{u}F_{\lambda,q,\chi}(t,x+1,u,a,b)$$

$$= \left(1 - \frac{a^{[f]t}}{u^f}\right) + \frac{\lambda\chi(1)b^t}{u}D(1)F_{\lambda,q,\chi}(qt,x,u,a,b),$$

which, in light of the Cauchy product of the three series b^t, $F_{\lambda,q,\chi}(qt,x,u,a,b)$ and

$$D(1) = \mathcal{G}\left(q[f]t,\frac{1}{q},u^f,a\right) - u^f G(q[f]t,u^f,a),$$

yield the following theorem.

Theorem 2.5 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_{0,\chi}(x;u;a,b;\lambda;q) = 1 - \frac{([f]\ln a)^n}{u^f} + \frac{\lambda\chi(1)}{u}C_0\mathcal{H}_{0,\chi}(x;u;a,b;\lambda;q)$$

and

$$\mathcal{H}_{n,\chi}(x;u;a,b;\lambda;q) = -\frac{([f]\ln a)^n}{u^f} + \frac{\lambda\chi(1)}{u}\sum_{i=0}^{n}\binom{n}{i}C_{n-i}q^i\mathcal{H}_{i,\chi}(x;u;a,b;\lambda;q),$$

where

$$C_n = \sum_{i=0}^{n} \binom{n}{i} (\ln b)^{n-i} [f]^i q^i \left(Y_i \left(\frac{1}{q}, u^f, a \right) - u^f Y_i(0, u, a) \right).$$

Now, we turn our attention to studying the derivative of the polynomials

$$\frac{\partial}{\partial t} F_{\lambda, q, \chi}(x, t, u, a, b) = \frac{-[f] a^{[f]t} \ln a}{u^f} \sum_{n=0}^{\infty} \left(\frac{\lambda}{u} \right)^n \chi(n) b^{[n+x]t}$$

$$+ \left(1 - \frac{a^{[f]t}}{u^f} \right) \sum_{n=0}^{\infty} \left(\frac{\lambda}{u} \right)^n \chi(n)[n+x] b^{[n+x]t} \ln b. \qquad (8)$$

By

$$\sum_{i=0}^{\infty} y^n [n] = \frac{1}{(1-y)(1-q)} - \frac{y}{(1-qy)(1-q)}$$

and substituting $t = 0$ in (8), we get that

$$\mathcal{H}_{1,\chi}(x; u; a, b; \lambda; q) = \frac{-[f] \ln a}{u^f - u^{f-1} \lambda \chi(1)} + \left(\frac{u^f - 1}{u^f} \right) \left(\frac{u}{u - \lambda \chi(1)} \right) [x] \ln b$$

$$+ \left(\frac{u^f - 1}{u^f(1-q)} \right) q^x \left(\frac{u}{u - \lambda \chi(1)} + \frac{u}{u - q \lambda \chi(1)} \right).$$

Hence, using the induction method, we arrive at the following result.

Theorem 2.6 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_{n,\chi}(x; u; a, b; \lambda; q) \frac{t^n}{n!} = \frac{\partial^n}{\partial t^n} F_{\lambda, q, \chi}(x, t, u, a, b) \bigg|_{t=0}.$$

Now we give a generalization of the Raabe formula by the following theorem.

Theorem 2.7 *Let $c, n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_{n,\chi}(x; u; a, b; \lambda; q) = \sum_{n=0}^{c-1} \sum_{j=0}^{n} \binom{n}{j} \frac{\lambda \chi(1)}{u} \mathcal{R}_j(c)[c]^{n-j}$$

$$\times \mathcal{H}_{n-j, \chi_c} \left(\frac{x+l}{c}; u^c; a, b; \lambda^c; q^c \right),$$

where

$$\mathcal{R}_j(c) = [cf_c]^j \left(Y_n \left(\frac{[f]}{[cf_c]}; u^f, a \right) - u^f Y_n \left(u^f, a \right) \right)$$

and

$$\chi_c(x) = \chi(cx).$$

Proof We start the proof with defining the character

$$\chi_c(x) = \chi(cx)$$

with the conductor $f_c = \frac{f}{\gcd(f,c)}$. On the other hand, we derive that

$$\left(\frac{u^f - a^{[f]t}}{u^f} \right) = \frac{u^f - a^{[f]t}}{u^f - a^{[f_c]_{q^c}[c]t}} \frac{u^f - a^{[f_c]_{q^c}[c]t}}{u^f}$$

$$= \mathcal{R}(c) \frac{u^f - a^{[f_c]_{q^c}[c]t}}{u^f},$$

where

$$\mathcal{R}(c) = \frac{a^{[f]t} - u^f}{a^{[cf_c]t} - u^f}.$$

Then by using (6), it follows that

$$\mathcal{R}(c) = \mathcal{G}\left([cf_c]t, \frac{[f]}{[cf_c]}, u^f, a \right) - u^f \mathcal{G}\left([cf_c]t, 0, u^f, a \right)$$

$$= \sum_{n=0}^{\infty} [cf_c]^n \left(Y_n\left(\frac{[f]}{[cf_c]}; u^f, a \right) - u^f Y_n(u^f, a) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \mathcal{R}_n(c) \frac{t^n}{n!}.$$

Now, we are ready to prove our result.

$$\sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(x; u; a, b; \lambda; q) \frac{t^n}{n!}$$

$$= \left(1 - \frac{a^{[f]t}}{u^f} \right) \sum_{m=0}^{\infty} \sum_{l=0}^{c-1} \left(\frac{\lambda}{u} \right)^{l+mc} b^{[l+mc+x]t} \chi(l+mc)$$

$$= \left(1 - \frac{a^{[f]t}}{u^f} \right) \sum_{l=0}^{c-1} \left(\frac{\lambda\chi(1)}{u} \right)^l \sum_{m=0}^{\infty} \left(\frac{\lambda^c}{u^c} \right) b^{[m+\frac{l+x}{c}]_{q^c}[c]t} \chi(cm)$$

$$= \sum_{l=0}^{c-1} \left(\frac{\lambda\chi(1)}{u} \right)^l \sum_{n=0}^{\infty} \mathcal{R}_n(c) \frac{t^n}{n!} \sum_{n=0}^{\infty} [c]^n \mathcal{H}_{n,\chi_c}\left(\frac{x+l}{c}; u^c; a, b; \lambda^c; q^c \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{c-1} \left(\frac{\lambda\chi(1)}{u} \right)^l \sum_{j=0}^{n} \binom{n}{j} \mathcal{R}_j(c)[c]^{n-j} \mathcal{H}_{n-j,\chi_c}\left(\frac{x+l}{c}; u^c; a, b; \lambda^c; q^c \right) \frac{t^n}{n!}.$$

Hence, comparing the coefficient of $\frac{t^n}{n!}$ on both sides yields the assertion of this theorem. □

3 Decomposition of the generating function

In this section, using the generating element of the finite cyclic group G and the generating element of the subgroups of G, we show that the generating function with a conductor f can be written as a sum of the generating function with conductors which are less than f.

In this section, we use the following notations otherwise stated:

(i) $x\mathbb{N}_0 = \{0, x, 2x, 3x, 4x, \ldots\}$,

(ii) p_i denotes the integer such that $\gcd(p_i, p_j) = 1$ for all $i, j \in \{1, \ldots, n\}$,

(iii) $S = \bigcup_{i=1}^{n}(p_i\mathbb{N}_0)$,

(iv) $S_0 = \{p_1, \ldots, p_n\}$,

(v) $S_i = \{\operatorname{lcd}(a, b) : a, b \in S_{i-1}\}$.

We start to recall the fact that

$$x\mathbb{N}_0 \cap y\mathbb{N}_0 = \operatorname{lcd}(x, y)\mathbb{N}_0$$

for positive integers x and y. In particular, we get that

$$x^k\mathbb{N}_0 \cap y^t\mathbb{N}_0 = xy\mathbb{N}_0$$

whenever x and y are distinct prime numbers.

Theorem 3.1 *With the above notations, we get*

$$\sum_{i \in S} F(i) = \sum_{j=0}^{n-1}(-1)^j \sum_{t \in S_j} \sum_{i \in t\mathbb{N}_0} F(i).$$

Proof We start to recall the fact that for sets A, B,

$$\sum_{i \in A \cup B} F(i) = \sum_{i \in A} F(i) + \sum_{i \in B} F(i) - \sum_{i \in A \cap B} F(i).$$

Then by using De-Morgan's law of sets, we get the following equality:

$$\sum_{i \in (p_1\mathbb{N}_0) \cup (\bigcup_{i=2}^{n}(p_i\mathbb{N}_0))} F(i) = \sum_{i \in (p_1\mathbb{N}_0)} F(i) + \sum_{i \in (\bigcup_{i=2}^{n}(p_i\mathbb{N}_0))} F(i) - \sum_{i \in (p_1\mathbb{N}_0) \cap (\bigcup_{i=2}^{n}(p_i\mathbb{N}_0))} F(i)$$

$$= \sum_{i \in (p_1\mathbb{N}_0)} F(i) + \sum_{i \in (\bigcup_{i=2}^{n}(p_i\mathbb{N}_0))} F(i) - \sum_{i \in (\bigcup_{i=2}^{n} \operatorname{lcd}(p_1, p_i)\mathbb{N}_0)} F(i).$$

Now we use induction on n.

If $n = 2$, then $S_0 = \{p_1, p_2\}$ and $S_1 = \{\operatorname{lcd}(p_1, p_2)\}$. We have

$$\sum_{i \in (\bigcup_{i=1}^{2}(p_i\mathbb{N}_0))} F(i) = \sum_{i \in (p_1\mathbb{N}_0)} F(i) + \sum_{i \in (p_2\mathbb{N}_0)} F(i) - \sum_{i \in \operatorname{lcd}(p_1, p_2)\mathbb{N}_0} F(i)$$

$$= \sum_{j=0}^{1}(-1)^j \sum_{t \in S_j} \sum_{i \in t\mathbb{N}_0} F(i).$$

We construct the following sets:

$$S = \bigcup_{i=1}^{n}(p_i\mathbb{N}_0),$$

$$S' = \bigcup_{i=2}^{n}(p_i\mathbb{N}_0)$$

and

$$S = (p_1 \mathbb{N}_0) \cup S'.$$

By using De-Morgan's law of sets, we verify the following equalities:

$$S'' = (p_1 \mathbb{N}_0) \cap S' = \bigcup_{i=2}^{n} (\mathrm{lcd}(p_1, p_i)\mathbb{N}_0),$$

$$S_0 = S'_0 \cup \{p_1\},$$

$$S_1 = \{\mathrm{lcd}(a, b) : a, b \in S_0\} = S'_1 \cup S''_0,$$

$$S_2 = \{\mathrm{lcd}(a, b) : a, b \in S_1\} = S'_2 \cup S''_1,$$

$$S_i = \{\mathrm{lcd}(a, b) : a, b \in S_{i-1}\} = S'_i \cup S''_{i-1}$$

and for all $i \geq 1$, it is clear that $S'_i \cap S''_{i-1} = \varnothing$.

Now assume that it is true for the set with $n - 1$ elements. Hence, the hypothesis for the sets S'' and S' is true, and we get that

$$\sum_{i \in S''} F(i) = \sum_{j=0}^{n-2} (-1)^j \sum_{t \in S''_j} \sum_{i \in t\mathbb{N}_0} F(i)$$

and

$$\sum_{i \in S'} F(i) = \sum_{j=0}^{n-2} (-1)^j \sum_{t \in S'_j} \sum_{i \in t\mathbb{N}_0} F(i).$$

Hence it follows

$$\sum_{i \in S} F(i) = \sum_{i \in p_1 \mathbb{N}_0} F(i) + \sum_{i \in S'} F(i) - \sum_{i \in S''} F(i)$$

$$= \sum_{i \in p_1 \mathbb{N}_0} F(i) + \sum_{j=0}^{n-2} (-1)^j \sum_{t \in S'_j} \sum_{i \in t\mathbb{N}_0} F(i) - \sum_{j=0}^{n-2} (-1)^j \sum_{t \in S''_j} \sum_{i \in t\mathbb{N}_0} F(i)$$

$$= \sum_{t \in S_0} \sum_{i \in t\mathbb{N}_0} F(i) - \sum_{t \in S'_1 \cup S''_0} \sum_{i \in t\mathbb{N}_0} F(i) + \sum_{j=2}^{n-2} (-1)^j \sum_{t \in S'_j \cup S''_{j-1}} \sum_{i \in t\mathbb{N}_0} F(i)$$

$$= \sum_{j=0}^{n-1} (-1)^j \sum_{t \in S_j} \sum_{i \in t\mathbb{N}_0} F(i). \qquad \square$$

Example 3.2 Let p_1, p_2, p_3 be distinct prime integers. Then we have

$$\sum_{i \in (\bigcup_{i=1}^{3} (p_i \mathbb{N}_0))} F(i) = \sum_{i \in (p_1 \mathbb{N}_0)} F(i) + \sum_{i \in (p_2 \mathbb{N}_0)} F(i) + \sum_{i \in (p_3 \mathbb{N}_0)} F(i) - \sum_{i \in (p_2 p_3 \mathbb{N}_0)} F(i)$$

$$- \sum_{i \in (p_1 p_2 \mathbb{N}_0)} F(i) - \sum_{i \in (p_1 p_3 \mathbb{N}_0)} F(i) + \sum_{i \in (p_1 p_2 p_3 \mathbb{N}_0)} F(i).$$

Lemma 3.3 *Let p_i be a prime number for all i and*

$$C = \left\{ l \in \mathbb{N}_0 : \gcd(f, l) = 1 \right\}.$$

Then $\mathbb{N}_0 = C \cup (\bigcup_{i=1}^n (p_i \mathbb{N}_0))$ and $C \cap (\bigcup_{i=1}^n (p_i \mathbb{N}_0))$ is an empty set.

Proof By using the fact $(l, f) = 1$ if and only if $(l, \prod_{i=1}^n p_i) = 1$ if and only if $(l, p_i) = 1$ for all $i \in \{1, \dots, n\}$ for $n \in \mathbb{N}_0$, we get that

$$\begin{aligned} C &= \left\{ l \in \mathbb{N}_0 : \gcd(f, l) = 1 \right\} \\ &= \left\{ l \in \mathbb{N}_0 : \text{ for all } i \in \{1, \dots, n\}, \gcd(p_i, l) = 1 \right\}. \end{aligned}$$

Let $x \notin \bigcup_{i=1}^n (p_i \mathbb{N}_0)$. Then we get that $\gcd(x, p_i) = 1$ for all i and so $x \in C$.
If $x \notin C$, then there is $i \in \{1, \dots, n\}$ such that $\gcd(x, p_i)$ is p_i. Therefore

$$\mathbb{N}_0 = C \cup \left(\bigcup_{i=1}^n (p_i \mathbb{N}_0) \right).$$

Let $x \in C \cap (\bigcup_{i=1}^n (p_i \mathbb{N}_0))$. Then there is $i \in \{1, \dots, n\}$ such that $x \in p_i \mathbb{N}_0$. This means that $\gcd(x, p_i) = p_i$ and so $x \notin C$, a contradiction. Thus the proof is completed. □

By using the Lemma 3.3, we have one of the main results in this section.

Theorem 3.4 *With the above notations, we get that*

$$\sum_{i \in \mathbb{N}_0} F(i) = \sum_{i \in C} F(i) + \sum_{i \in (\bigcup_{j=1}^n (p_j \mathbb{N}_0))} F(i).$$

As stated before, to decompose the generating function of q-Eulerian polynomials attached to the character χ, now we need to compute the following relation:

$$\sum_{n \in (dZ)} \left(\frac{\lambda}{u} \right)^n b^{[n]t} \chi(n),$$

where

$$f_d = \frac{f}{d} \in \mathbb{N}_0.$$

Also, we define the character χ_g with the conductor h such as

$$\chi_g(i) := \chi(gi),$$

where $g, h \in \mathbb{N}_0$ and $\gcd(g, h) = 1$. Then

$$\sum_{n \in (dZ)} \left(\frac{\lambda}{u} \right)^n b^{[n]t} \chi(n) = \sum_{n \in \mathbb{N}_0} \left(\frac{\lambda}{u} \right)^{dn} b^{[dn]t} \chi(dn)$$

$$= \sum_{n \in \mathbb{N}_0} \sum_{i=0}^{f_d-1} \left(\frac{\lambda}{u} \right)^{d(f_d n + i)} b^{[d(f_d n + i)]t} \chi(d(f_d n + i))$$

$$= \sum_{n\in\mathbb{N}_0}\sum_{i=0}^{f_d-1}\left(\frac{\lambda^d}{u^d}\right)^{f_d n+i} b^{t[d][f_d n+i]_{q^d}}\chi_d(i)$$

$$= \sum_{i\in\mathbb{N}_0}\left(\frac{\lambda^d}{u^d}\right)^{i} b^{t[d][i]_{q^d}}\chi_d(i).$$

Hence, we combine the above equation with the generating function

$$\left(1-\frac{a^{[f]t}}{u^f}\right)\sum_{n\in(dZ)}\left(\frac{\lambda}{u}\right)^n b^{[n]t}\chi(n) = \left(1-\frac{a^{[df_d]t}}{u^{df}}\right)\sum_{i\in\mathbb{N}_0}\left(\frac{\lambda}{u}\right)^i b^{[i]_{q^d}t}\chi_d(i)$$

$$= \left(1-\frac{a^{[d][f_d]_{q^d}t}}{(u^d)^{f_d}}\right)\sum_{i\in\mathbb{N}_0}\left(\frac{\lambda}{u}\right)^i b_d^{[i]_{q^d}t}\chi_d(i)$$

$$= F_{\lambda^d,q^d,\chi_d}\left(t,u^d,a^{[d]},b^{[d]}\right).$$

Now we are ready to state the main result without the proof in this section.

Theorem 3.5 *Let* $f=\prod_{i=1}^{n}p_i^{t_t}$, *where* p_i *is a prime integer. Then*

$$F_{\lambda,q,\chi}(t,u,a,b) = \sum_{j=0}^{n-1}(-1)^j\sum_{t\in S_j}\sum_{i\in tZ}F_{\lambda^i,q^i,\chi_i}\left(t,u^i,a^{[i]},b^{[i]}\right) + F_{\lambda,q,\chi_D}(t,u,a,b),$$

where χ_D *is a Dirichlet character corresponding to the character* χ.

Example 3.6 *Let* $f=p_1^{t_1}p_2^{t_2}p_3^{t_3}$, *where* p_i *is a prime integer. Then*

$$F_{\lambda,q,\chi}(t,u,a,b) = F_{\lambda,q,\chi_D}(t,u,a,b) + \sum_{i=1}^{3}F_{\lambda^{p_i},q^{p_i},\chi_{p_i}}\left(t,u^{p_i},a^{[p_i]},b^{\overline{[f_{p_i}]_{q^{p_i}}}}\right)$$

$$+ F_{\lambda^{p_1 p_2 p_3},q^{p_1 p_2 p_3},\chi_{p_1 p_2 p_3}}\left(t,u^{p_1 p_2 p_3},a^{[p_1 p_2 p_3]},b^{\overline{[f_{p_1 p_2 p_3}]_{q^{p_1 p_2 p_3}}}}\right)$$

$$- \sum_{i=1}^{3}F_{\lambda^{p_1 p_i},q^{p_1 p_i},\chi_{p_1 p_i}}\left(t,u^{p_1 p_i},a^{[p_1 p_i]},b^{\overline{[f_{p_1 p_i}]_{q^{p_1 p_i}}}}\right)$$

$$- F_{\lambda^{p_2 p_3},q^{p_2 p_3},\chi_{p_2 p_3}}\left(t,u^{p_2 p_3},a^{[p_2 p_3]},b^{\overline{[f_{p_2 p_3}]_{q^{p_2 p_3}}}}\right).$$

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