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# A new application of quasi power increasing sequences. II

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**Abstract**

In this paper, we prove a general theorem dealing with absolute Cesàro summability factors of infinite series by using a quasi-$f$-power increasing sequence instead of a quasi-$\sigma$-power increasing sequence. This theorem also includes several new results.

**MSC:** 26D15; 40D15; 40F05; 40G99; 46A45

**Keywords:** absolute summability; increasing sequences; sequence spaces; Hölder inequality; Minkowski inequality; infinite series

## 1 Introduction

A positive sequence $(b_n)$ is said to be almost increasing if there exists a positive increasing sequence $c_n$ and two positive constants $A$ and $B$ such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A sequence $(\lambda_n)$ is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. A positive sequence $X = (X_n)$ is said to be a quasi-$\sigma$-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ holds for all $n \geq m \geq 1$ (see [2]). It should be noted that every almost increasing sequence is a quasi-$\sigma$-power increasing sequence for any nonnegative $\sigma$, but the converse may not be true as can be seen by taking an example, say $X_n = n^{-\sigma}$ for $\sigma > 0$. Let $(\varphi_n)$ be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote by $z_n^\alpha$ and $t_n^\alpha$ the $n$th Cesàro means of order $\alpha$, with $\alpha > -1$, of the sequences $(s_n)$ and $(na_n)$, respectively, that is,

$$z_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for } n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $\varphi - |C, \alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [3, 4])

$$\sum_{n=1}^{\infty} \bigl|\varphi_n\bigl(z_n^\alpha - z_{n-1}^\alpha\bigr)\bigr|^k = \sum_{n=1}^{\infty} n^{-k} \bigl|\varphi_n t_n^\alpha\bigr|^k < \infty. \tag{4}$$

In the special case if we take $\varphi_n = n^{1-\frac{1}{k}}$, then $\varphi - |C,\alpha|_k$ summability is the same as $|C,\alpha|_k$ summability (see [5]). Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then $\varphi - |C,\alpha|_k$ summability reduces to $|C,\alpha;\delta|_k$ summability (see [6]).

## 2 The known results

**Theorem A** ([7]) *Let $(\lambda_n) \in \mathcal{BV}$ and let $(X_n)$ be a quasi-$\sigma$-power increasing sequence for some $\sigma$ $(0 < \sigma < 1)$. Suppose also that there exist sequences $(\beta_n)$ and $(\lambda_n)$ such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{5}$$

$$\beta_n \to 0 \quad \text{as } n \to \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{7}$$

$$|\lambda_n|X_n = O(1) \quad \text{as } n \to \infty. \tag{8}$$

*If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}|\varphi_n|^k)$ is nonincreasing and if the sequence $(w_n^\alpha)$ defined by (see [8])*

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1\leq v\leq n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases} \tag{9}$$

*satisfies the condition*

$$\sum_{n=1}^{m} \frac{(|\varphi_n|w_n^\alpha)^k}{n^k} = O(X_m) \quad \text{as } m \to \infty, \tag{10}$$

*then the series $\sum a_n\lambda_n$ is summable $\varphi - |C,\alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$ and $k\alpha + \epsilon > 1$.*

**Remark 1** Here, in the hypothesis of Theorem A, we have added the condition '$(\lambda_n) \in \mathcal{BV}$' because it is necessary.

**Theorem B** ([9]) *Let $(X_n)$ be a quasi-$\sigma$-power increasing sequence for some $\sigma$ $(0 < \sigma < 1)$. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}|\varphi_n|^k)$ is nonincreasing and if the conditions from* (5) *to* (8) *are satisfied and if the condition*

$$\sum_{n=1}^{m} \frac{(|\varphi_n|w_n^\alpha)^k}{n^k X_n^{k-1}} = O(X_m) \quad \text{as } m \to \infty, \tag{11}$$

*is satisfied, then the series $\sum a_n\lambda_n$ is summable $\varphi - |C,\alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$ and $k(\alpha - 1) + \epsilon > 1$.*

**Remark 2** It should be noted that condition (11) is the same as condition (10) when $k = 1$. When $k > 1$, condition (11) is weaker than condition (10) but the converse is not true. As in [10], we can show that if (10) is satisfied, then we get

$$\sum_{n=1}^{m} \frac{(|\varphi_n|w_n^\alpha)^k}{n^k X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{(|\varphi_n|w_n^\alpha)^k}{n^k} = O(X_m).$$

If (11) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} \frac{(|\varphi_n| w_n^\alpha)^k}{n^k} = \sum_{n=1}^{m} X_n^{k-1} \frac{(|\varphi_n| w_n^\alpha)^k}{n^k X_n^{k-1}} = O(X_m^{k-1}) \sum_{n=1}^{m} \frac{(|\varphi_n| w_n^\alpha)^k}{n^k X_n^{k-1}} = O(X_m^k) \neq O(X_m).$$

Also, it should be noted that the condition '$(\lambda_n) \in \mathcal{BV}$' has been removed.

## 3 The main result

The aim of this paper is to extend Theorem B by using a general class of quasi power increasing sequence instead of a quasi-$\sigma$-power increasing sequences. For this purpose, we need the concept of quasi-$f$-power increasing sequence. A positive sequence $X = (X_n)$ is said to be a quasi-$f$-power increasing sequence, if there exists a constant $K = K(X,f)$ such that $K f_n X_n \geq f_m X_m$, holds for $n \geq m \geq 1$, where $f = (f_n) = [n^\sigma (\log n)^\eta, \eta \geq 0, 0 < \sigma < 1]$ (see [11]). It should be noted that if we take $\eta = 0$, then we get a quasi-$\sigma$-power increasing sequence. Now, we will prove the following theorem.

**Theorem** *Let $(X_n)$ be a quasi-$f$-power increasing sequence. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}|\varphi_n|^k)$ is non-increasing and if the conditions from (5) to (8) and (11) are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |C,\alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$ and $k(\alpha - 1) + \epsilon > 1$.*

We need the following lemmas for the proof of our theorem.

**Lemma 1** ([12]) *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right|. \tag{12}$$

**Lemma 2** ([11]) *Under the conditions on $(X_n)$, $(\beta_n)$, and $(\lambda_n)$ as expressed in the statement of the theorem, we have the following*:

$$n \beta_n X_n = O(1) \quad \text{as } n \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{14}$$

## 4 Proof of the theorem

Let $(T_n^\alpha)$ be the $n$th $(C,\alpha)$, with $0 < \alpha \leq 1$, mean of the sequence $(n a_n \lambda_n)$. Then, by (2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{15}$$

First, applying Abel's transformation and then using Lemma 1, we get that

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

$$\begin{aligned}
\left|T_n^\alpha\right| &\le \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n-1}|\Delta\lambda_\nu|\left|\sum_{p=1}^{\nu}A_{n-p}^{\alpha-1}pa_p\right| + \frac{|\lambda_n|}{A_n^\alpha}\left|\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}\nu a_\nu\right| \\
&\le \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n-1}A_\nu^\alpha w_\nu^\alpha|\Delta\lambda_\nu| + |\lambda_n|w_n^\alpha \\
&= T_{n,1}^\alpha + T_{n,2}^\alpha.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}\left|\varphi_n T_{n,r}^\alpha\right|^k < \infty \quad \text{for } r = 1, 2.$$

Now, when $k > 1$, applying Hölder's inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-k}\left|\varphi_n T_{n,1}^\alpha\right|^k &\le \sum_{n=2}^{m+1} n^{-k}\left(A_n^\alpha\right)^{-k}|\varphi_n|^k\left\{\sum_{\nu=1}^{n-1}A_\nu^\alpha w_\nu^\alpha|\Delta\lambda_\nu|\right\}^k \\
&\le \sum_{n=2}^{m+1} n^{-k}n^{-\alpha k}|\varphi_n|^k\sum_{\nu=1}^{n-1}\nu^{\alpha k}\left(w_\nu^\alpha\right)^k|\Delta\lambda_\nu|^k \times \left\{\sum_{\nu=1}^{n-1}1\right\}^{k-1} \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\alpha k}\left(w_\nu^\alpha\right)^k(\beta_\nu)^k\sum_{n=\nu+1}^{m+1}\frac{n^{\epsilon-k}|\varphi_n|^k}{n^{k(\alpha-1)+\epsilon+1}} \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\alpha k}\left(w_\nu^\alpha\right)^k(\beta_\nu)^k\nu^{\epsilon-k}|\varphi_\nu|^k\sum_{n=\nu+1}^{m+1}\frac{1}{n^{k(\alpha-1)+\epsilon+1}} \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\alpha k}\left(w_\nu^\alpha\right)^k(\beta_\nu)^k\nu^{\epsilon-k}|\varphi_\nu|^k\int_\nu^\infty\frac{dx}{x^{k(\alpha-1)+\epsilon+1}} \\
&= O(1)\sum_{\nu=1}^{m}\beta_\nu(\beta_\nu)^{k-1}\left(w_\nu^\alpha|\varphi_\nu|\right)^k \\
&= O(1)\sum_{\nu=1}^{m}\beta_\nu\left(\frac{1}{\nu X_\nu}\right)^{k-1}\left(w_\nu^\alpha|\varphi_\nu|\right)^k \\
&= O(1)\sum_{\nu=1}^{m-1}\Delta(\nu\beta_\nu)\sum_{r=1}^{\nu}\frac{(|\varphi_r|w_r^\alpha)^k}{r^kX_r^{k-1}} + O(1)m\beta_m\sum_{\nu=1}^{m}\frac{(|\varphi_\nu|w_\nu^\alpha)^k}{\nu^kX_\nu^{k-1}} \\
&= O(1)\sum_{\nu=1}^{m-1}\left|\Delta(\nu\beta_\nu)\right|X_\nu + O(1)m\beta_mX_m \\
&= O(1)\sum_{\nu=1}^{m-1}\left|(\nu+1)\Delta\beta_\nu - \beta_\nu\right|X_\nu + O(1)m\beta_mX_m \\
&= O(1)\sum_{\nu=1}^{m-1}\nu|\Delta\beta_\nu|X_\nu + O(1)\sum_{\nu=1}^{m-1}\beta_\nu X_\nu + O(1)m\beta_mX_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2. Finally, we have that

$$\begin{aligned}\sum_{n=1}^{m} n^{-k}\left|\varphi_n T_{n,2}^{\alpha}\right|^k &= \sum_{n=1}^{m} |\lambda_n||\lambda_n|^{k-1} n^{-k}\left(w_n^{\alpha}|\varphi_n|\right)^k\\ &= O(1)\sum_{n=1}^{m}|\lambda_n|\left(\frac{1}{X_n}\right)^{k-1} n^{-k}\left(w_n^{\alpha}|\varphi_n|\right)^k\\ &= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{(|\varphi_v|w_v^{\alpha})^k}{v^k X_v^{k-1}} + O(1)|\lambda_m|\sum_{n=1}^{m}\frac{(|\varphi_n|w_n^{\alpha})^k}{n^k X_n^{k-1}}\\ &= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m\\ &= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m = O(1) \quad \text{as } m\to\infty,\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\epsilon = 1$, $\alpha = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$), then we get a new result dealing with $|C,\alpha|_k$ (resp. $|C,1|_k$) summability factors of infinite series. Also, if we take $\epsilon = 1$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we get another new result concerning the $|C,\alpha;\delta|_k$ summability factors of infinite series. Furthermore, if we take $(X_n)$ as an almost increasing sequence, then we get the result of Bor and Seyhan under weaker conditions (see [13]). Finally, if we take $\eta = 0$, then we obtain Theorem B.

**Competing interests**

The author declares that he has no competing interests.

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