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Best proximity point theorems for generalized contractions in partially ordered metric spaces

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Abstract
The purpose of this paper is to obtain four best proximity point theorems for generalized contractions in partially ordered metric spaces. Further, our *P*-operator technique, which changes a non-self mapping to a self-mapping, plays an important role. Some recent results in this area have been improved.
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1 Introduction and preliminaries

Let A and B be nonempty subsets of a metric space (X,d). An operator $T:A\to B$ is said to be contractive if there exists $k\in[0,1)$ such that $d(Tx,Ty)\leq kd(x,y)$ for any $x,y\in A$. The well-known Banach contraction principle is as follows: Let (X,d) be a complete metric space, and let $T:X\to X$ be a contraction of X into itself. Then T has a unique fixed point in X.

In the sequel, we denote by Γ the functions $\beta:[0,\infty)\to[0,1)$ satisfying the following condition:

$$\beta(t_n)\to 1 \quad\Rightarrow\quad t_n\to 0.$$

In 1973, Geraghty introduced the Geraghty-contraction and obtained Theorem 1.2 as follows.

Definition 1.1 [1] Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be a *Geraghty-contraction* if there exists $\beta\in\Gamma$ such that for any $x,y\in X$,

$$d(Tx,Ty)\leq\beta\bigl(d(x,y)\bigr)d(x,y).$$

Theorem 1.2 [1] *Let (X,d) be a complete metric space, and let $T:X\to X$ be an operator. Suppose that there exists $\beta\in\Gamma$ such that for any $x,y\in X$,*

$$d(Tx,Ty)\leq\beta\bigl(d(x,y)\bigr)d(x,y).$$

Then T has a unique fixed point.

Obviously, Theorem 1.2 is an extensive version of the Banach contraction principle. Recently, the generalized contraction principle has been studied by many authors in metric spaces or more generalized metric spaces. Some results have been got in partially ordered metric spaces as follows.

Theorem 1.3 [2] *Let $(X,\leq)$ be a partially ordered set, and suppose that there exists a metric d such that (X,d) is a complete metric space. Let $f:X\to X$ be an increasing mapping such that there exists an element $x_0\in X$ with $x_0\leq f(x_0)$. Suppose that there exists $\beta\in\Gamma$ such that*

$$d\big(f(x),f(y)\big)\leq\beta\big(d(x,y)\big)d(x,y),\quad\forall y\geq x.$$

Assume that either f is continuous or X is such that if an increasing sequence $x_n\to x\in X$, then $x_n\leq x$, $\forall n$. Besides, if for each $x,y\in X$, there exists $z\in X$ which is comparable to x and y, then f has a unique fixed point.

Theorem 1.4 [3] *Let $(X,\leq)$ be a partially ordered set, and suppose that there exists a metric $d\in X$ such that (X,d) is a complete metric space. Let $f:X\to X$ be a continuous and nondecreasing mapping such that*

$$d\big(f(x),f(y)\big)\leq d(x,y)-\psi\big(d(x,y)\big)\quad\text{for } y\geq x,$$

where $\psi:[0,\infty)\to[0,\infty)$ is a continuous and nondecreasing function such that ψ is positive in $(0,\infty)$, $\psi(0)=0$ and $\lim_{t\to\infty}\psi(t)=\infty$. If there exists $x_0\in X$ with $x_0\leq f(x_0)$, then f has a fixed point.

Definition 1.5 [4] An altering distance function is a function $\psi:[0,\infty)\to[0,\infty)$ which satisfies

(i) ψ is continuous and nondecreasing.
(ii) $\psi(t)=0$ if and only if $t=0$.

Theorem 1.6 [4] *Let X be a partially ordered set, and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T:X\to X$ be a continuous and nondecreasing mapping such that*

$$\psi\big(d(Tx,Ty)\big)\leq\phi\big(d(x,y)\big),\quad\forall y\geq x,$$

where ψ is an altering distance function and $\phi:[0,\infty)\to[0,\infty)$ is a continuous function with the condition $\psi(t)>\phi(t)$ for all $t>0$. If there exists $x_0\in X$ such that $x_0\leq Tx_0$, then T has a fixed point.

Theorem 1.7 [5] *Let $(X,\leq)$ be a partially ordered set, and suppose that there exists a metric $d\in X$ such that (X,d) is a complete metric space. Let $f:X\to X$ be a continuous and nondecreasing mapping such that*

$$\psi\Big(d\big(f(x),f(y)\big)\Big)\leq\psi\big(d(x,y)\big)-\phi\big(d(x,y)\big)\quad\text{for } y\geq x,$$

where ψ and ϕ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$, then f has a fixed point.

In 2012, Caballero *et al.* introduced a generalized Geraghty-contraction as follows.

Definition 1.8 [6] Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T: A \to B$ is said to be a *Geraghty-contraction* if there exists $\beta \in \Gamma$ such that for any $x, y \in A$,

$$d(Tx,Ty) \le \beta\bigl(d(x,y)\bigr)d(x,y).$$

Now we need the following notations and basic facts. Let A and B be two nonempty subsets of a metric space (X,d). We denote by A_0 and B_0 the following sets:

$$A_0 = \bigl\{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\bigr\},$$
$$B_0 = \bigl\{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\bigr\},$$

where $d(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\}$.

In [7], the authors give sufficient conditions for when the sets A_0 and B_0 are nonempty. In [8], the authors prove that any pair (A,B) of nonempty, closed convex subsets of a uniformly convex Banach space satisfies the P-property.

Definition 1.9 [9] Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the *P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) = d(y_1,y_2).$$

Let A, B be two nonempty subsets of a complete metric space, and consider a mapping $T: A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that $d(x_0,Tx_0) = \min\{d(x,Tx) : x \in A\}$. Since $d(x,Tx) \ge d(A,B)$ for any $x \in A$, in fact, the optimal solution to this problem is the one for which the value $d(A,B)$ is attained.

In [6], the authors give a generalization of Theorem 1.2 by considering a non-self mapping, and they get the following theorem.

Theorem 1.10 [6] *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T: A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A,B) has the P-property. Then there exists a unique x^* in A such that $d(x^*,Tx^*) = d(A,B)$.*

Inspired by [6], the purpose of this paper is to obtain four best proximity point theorems for generalized contractions in partially ordered metric spaces. Further, a series of best proximity point problems can be solved by our P-operator technique, which changes a non-self mapping to a self-mapping. Some recent results in this area have been improved.

2 Main results

Before giving our main results, we first introduce the weak P-monotone property.

Weak P-monotone property Let (A,B) be a pair of nonempty subsets of a partially ordered metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the *weak P-monotone property* if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) \leq d(y_1,y_2);$$

furthermore, $y_1 \geq y_2$ implies $x_1 \geq x_2$.

Now we are in a position to give our main results.

Theorem 2.1 *Let $(X,\leq)$ be a partially ordered set, and let (X,d) be a complete metric space. Let (A,B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $f: A \to B$ be an increasing mapping with $f(A_0) \subseteq B_0$, and let there exist $\beta \in \Gamma$ such that*

$$d\bigl(f(x),f(y)\bigr) \leq \beta\bigl(d(x,y)\bigr)d(x,y), \quad \forall y \geq x.$$

Assume that either f is continuous or that $\overline{A}_0$ is such that if an increasing sequence $x_n \to x \in \overline{A}_0$, then $x_n \leq x$, $\forall n$. Suppose that the pair (A,B) has the weak P-monotone property. And for some $x_0 \in A_0$, there exists $\hat{x}_0 \in B_0$ such that $d(x_0,\hat{x}_0) = d(A,B)$ and $\hat{x}_0 \leq f(x_0)$. Besides, if for each $x,y \in \overline{A}_0$, there exists $z \in \overline{A}_0$ which is comparable to x and y, then there exists an x^* in A such that $d(x^*,fx^*) = d(A,B)$.

Proof We first prove that B_0 is closed. Let $\{y_n\} \subseteq B_0$ be a sequence such that $y_n \to q \in B$. It follows from the weak P-monotone property that

$$d(y_n,y_m) \to 0 \quad \Rightarrow \quad d(x_n,x_m) \to 0,$$

as $n,m \to \infty$, where $x_n, x_m \in A_0$ and $d(x_n,y_n) = d(A,B)$, $d(x_m,y_m) = d(A,B)$. Then $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. By the continuity of a metric d, we have $d(p,q) = d(A,B)$, that is, $q \in B_0$, and hence B_0 is closed.

Let $\overline{A}_0$ be the closure of A_0, we claim that $f(\overline{A}_0) \subseteq B_0$. In fact, if $x \in \overline{A}_0 \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By the continuity of f and the closeness of B_0, we have $fx = \lim_{n\to\infty} fx_n \in B_0$. That is, $f(\overline{A}_0) \subseteq B_0$.

Define an operator $P_{A_0}: f(\overline{A}_0) \to A_0$, by $P_{A_0}y = \{x \in A_0 : d(x,y) = d(A,B)\}$. Since the pair (A,B) has the weak P-monotone property and f is increasing, we have

$$d(P_{A_0}fx, P_{A_0}fy) \leq d(fx,fy) \leq \beta\bigl(d(x,y)\bigr)d(x,y), \quad P_{A_0}fy \geq P_{A_0}fx,$$

for any $y \geq x \in \overline{A}_0$. Obviously, $P_{A_0}f$ is increasing. Let $x_n, x \in \overline{A}_0$, $x_n \to x$, when f is continuous, then we have

$$d(P_{A_0}fx_n, P_{A_0}fx) \leq d(fx_n,fx) \to 0 \quad \Rightarrow \quad P_{A_0}fx_n \quad \Rightarrow \quad P_{A_0}fx, \quad \text{as } n \to \infty.$$

Then $P_{A_0}f$ is continuous. When $\overline{A}_0$ is such that if an increasing sequence $x_n \to x \in \overline{A}_0$, then $x_n \leq x$ $(\forall n)$, we need not prove the continuity of $P_{A_0}f$. For some $x_0 \in A_0$, there exists $\hat{x}_0 \in B_0$ such that $d(x_0, \hat{x}_0) = d(A, B)$ and $\hat{x}_0 \leq f(x_0)$. That is,

$$d(x_0, \hat{x}_0) = d(A, B), \qquad d(P_{A_0}fx_0, fx_0) = d(A, B), \quad \hat{x}_0 \leq f(x_0).$$

By the weak P-monotone property, we have $P_{A_0}fx_0 \geq x_0$.

This shows that $P_{A_0}f : \overline{A}_0 \to \overline{A}_0$ is a contraction satisfying all the conditions in Theorem 1.3. Using Theorem 1.3, we can get $P_{A_0}f$ has a unique fixed point x^*. That is, $P_{A_0}fx^* = x^* \in A_0$, which implies that

$$d\left(x^*, fx^*\right) = d(A, B).$$

Therefore, x^* is the unique one in A_0 such that $d(x^*, fx^*) = d(A, B)$. □

Theorem 2.2 *Let $(X, \leq)$ be a partially ordered set, and let (X, d) be a complete metric space. Let (A, B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $f : A \to B$ be a continuous and nondecreasing mapping with $f(A_0) \subseteq B_0$, and let f satisfy*

$$d\big(f(x), f(y)\big) \leq d(x, y) - \psi\big(d(x, y)\big) \quad \text{for } y \geq x,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Suppose that the pair (A, B) has the weak P-monotone property. If for some $x_0 \in A_0$, there exists $\hat{x}_0 \in B_0$ such that $d(x_0, \hat{x}_0) = d(A, B)$ and $\hat{x}_0 \leq f(x_0)$, then there exists an x^* in A such that $d(x^*, fx^*) = d(A, B)$.

Proof In Theorem 2.1, we have proved that B_0 is closed and $f(\overline{A}_0) \subseteq B_0$. Now we define an operator $P_{A_0} : f(\overline{A}_0) \to A_0$ by $P_{A_0}y = \{x \in A_0 : d(x, y) = d(A, B)\}$. Since the pair (A, B) has the weak P-monotone property, by the definition of f, we have

$$d(P_{A_0}fx, P_{A_0}fy) \leq d(fx, fy) \leq d(x, y) - \psi\left(d(x, y)\right), \quad P_{A_0}fy \geq P_{A_0}fx,$$

for any $y \geq x \in \overline{A}_0$. Obviously, $P_{A_0}f$ is continuous and nondecreasing. For some $x_0 \in A_0$, there exists $\hat{x}_0 \in B_0$ such that $d(x_0, \hat{x}_0) = d(A, B)$ and $\hat{x}_0 \leq f(x_0)$. That is,

$$d(x_0, \hat{x}_0) = d(A, B), \qquad d(P_{A_0}fx_0, fx_0) = d(A, B), \quad \hat{x}_0 \leq f(x_0).$$

By the weak P-monotone property, we have $P_{A_0}fx_0 \geq x_0$.

This shows that $P_{A_0}f : \overline{A}_0 \to \overline{A}_0$ is a contraction satisfying all the conditions in Theorem 1.4. Using Theorem 1.4, we can get $P_{A_0}f$ has a fixed point x^*. That is, $P_{A_0}fx^* = x^* \in A_0$, which implies that

$$d\left(x^*, fx^*\right) = d(A, B).$$

Therefore, x^* is the one in A_0 such that $d(x^*, fx^*) = d(A, B)$. □

Theorem 2.3 *Let X be a partially ordered set, and let (X,d) be a complete metric space. Let (A,B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $T: A \to B$ be a continuous and nondecreasing mapping with $T(A_0) \subseteq B_0$, and let T satisfy*

$$\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(x,y)\big), \quad \forall y \geq x,$$

where ψ is an altering distance function and $\phi: [0,\infty) \to [0,\infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$. Suppose that the pair (A,B) has the weak P-monotone property. If for some $x_0 \in A_0$, there exists $y_0 \in B_0$ such that $d(x_0,y_0) = d(A,B)$ and $y_0 \leq Tx_0$, then there exists an x^* in A such that $d(x^*,Tx^*) = d(A,B)$.

Proof In Theorem 2.1, we have proved that B_0 is closed and $f(\overline{A}_0) \subseteq B_0$. Define an operator $P_{A_0}: T(\overline{A}_0) \to A_0$ by $P_{A_0}y = \{x \in A_0 : d(x,y) = d(A,B)\}$. Since the pair (A,B) has the weak P-monotone property and ψ is nondecreasing, by the definition of T, we have

$$\psi\big(d(P_{A_0}Tx,P_{A_0}Ty)\big) \leq \psi\big(d(Tx,Ty)\big) \leq \phi\big(d(x,y)\big), \quad P_{A_0}Ty \geq P_{A_0}Tx,$$

for any $y \geq x \in \overline{A}_0$. Since

$$\begin{aligned} &\phi\big(d(x,y)\big) \to 0 \quad \Leftrightarrow \quad d(x,y) \to 0 \\ &\quad \Rightarrow \quad \psi\big(d(P_{A_0}Tx,P_{A_0}Ty)\big) \to 0 \quad \Leftrightarrow \quad d(P_{A_0}Tx,P_{A_0}Ty) \to 0, \end{aligned}$$

this shows that $P_{A_0}T$ is continuous and nondecreasing. Because there exist $x_0 \in A_0$ and $y_0 \in B_0$ such that $d(x_0,y_0) = d(A,B)$ and $y_0 \leq Tx_0$, by the weak P-monotone property, we have $x_0 \leq P_{A_0}Tx_0$.

This shows that $P_{A_0}T: \overline{A}_0 \to \overline{A}_0$ is a contraction from a complete metric subspace $\overline{A}_0$ into itself and satisfies all the conditions in Theorem 1.6. Using Theorem 1.6, we can get $P_{A_0}T$ has a fixed point x^*. That is, $P_{A_0}Tx^* = x^* \in A_0$, which implies that

$$d\big(x^*,Tx^*\big) = d(A,B).$$

Therefore, x^* is the one in A_0 such that $d(x^*,Tx^*) = d(A,B)$. □

Theorem 2.4 *Let $(X,\leq)$ be a partially ordered set, and let (X,d) be a complete metric space. Let (A,B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $T: A \to B$ be a continuous and nondecreasing mapping with $T(A_0) \subseteq B_0$, and let T satisfy*

$$\psi\big(d\big(T(x),T(y)\big)\big) \leq \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big) \quad \text{for } y \geq x,$$

where ψ and ϕ are altering distance functions. Suppose that the pair (A,B) has the weak P-monotone property. If for some $x_0 \in A_0$, there exists $y_0 \in B_0$ such that $d(x_0,y_0) = d(A,B)$ and $y_0 \leq Tx_0$, then there exists an x^* in A such that $d(x^*,Tx^*) = d(A,B)$.

Proof In Theorem 2.1, we have proved that B_0 is closed and $f(\overline{A}_0) \subseteq B_0$. Define an operator $P_{A_0}: T(\overline{A}_0) \to A_0$ by $P_{A_0}y = \{x \in A_0 : d(x,y) = d(A,B)\}$. Since the pair (A,B) has the weak P-monotone property and ψ is nondecreasing, by the definition of T, we have

$$\psi\big(d(P_{A_0}Tx,P_{A_0}Ty)\big) \leq \psi\big(d(Tx,Ty)\big) \leq \psi\big(d(Tx,Ty)\big) - \phi\big(d(x,y)\big), \quad P_{A_0}Ty \geq P_{A_0}Tx,$$

for any $y \geq x \in \overline{A}_0$. Since

$$\begin{aligned} &d(x,y) \to 0 \quad \Rightarrow \quad \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big) \to 0 \\ &\quad \Rightarrow \quad \psi\big(d(P_{A_0}Tx, P_{A_0}Ty)\big) \to 0 \quad \Leftrightarrow \quad d(P_{A_0}Tx, P_{A_0}Ty) \to 0. \end{aligned}$$

This shows that $P_{A_0}T$ is continuous and nondecreasing. Because there exist $x_0 \in A_0$ and $y_0 \in B_0$ such that $d(x_0, y_0) = d(A, B)$ and $y_0 \leq Tx_0$, by the weak P-monotone property, we have $x_0 \leq P_{A_0}Tx_0$.

This shows that $P_{A_0}T : \overline{A}_0 \to \overline{A}_0$ is a contraction from a complete metric subspace $\overline{A}_0$ into itself and satisfies all the conditions in Theorem 1.7. Using Theorem 1.7, we can get $P_{A_0}T$ has a fixed point x^*. That is, $P_{A_0}Tx^* = x^* \in A_0$, which implies that

$$d\big(x^*, Tx^*\big) = d(A, B).$$

Therefore, x^* is the one in A_0 such that $d(x^*, Tx^*) = d(A, B)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to the writing of the present article. All authors read and approved the final manuscript.

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